Exercises for Statistics 4 (Elements of Probability) Autumn 2006

1. The probability of the event A that a cricket match is cancelled because it is too wet is 0.35, and the probability of the event B that it is cancelled because it is too windy is 0.3. The probability (of the event) that it is simultaneously both too wet and too windy is 0.2. Find the probability that it is either too wet or too windy, and hence the probability that the match can go ahead.

2. A physical device only works if the ambient temperature lies within the range $[T_1, T_2]$ (where $T_1 < T_2$). Suppose that the probability of the event A that the temperature is below T_2 is 0.6, and that the probability of the event B that the temperature is above T_1 is 0.75. Find the probability that the device works.

3. A fair die (one for which each face is likely to show) is given two *independent* throws.

   (a) Rather than looking at the probabilities of individual outcomes, use the given independence to find the probability of each of the following events:
   the event A that both throws show a 'six';
   the event B that at least one throw shows a 'six';
   the event C that the first throw shows an even number and the second throw shows at most two;
   the event D that at least one of the two throws shows an even number.

   (b) Let N_1 and N_2 be the successive numbers obtained on the two throws, and let $N = N_1 + N_2$. For $n = 1, 2, \dots, 6$ find the conditional probability $P(N_1 = n \,|\, N = 5)$.

4. A coin is given 4 *independent* tosses. On each toss it lands heads with probability p, where $0 < p < 1$. Let A be the event that all 4 tosses are heads, let B be the event that the first 3 tosses are heads, let C be the event that at least 3 tosses are heads, and let D be the event that at least 1 toss is a tail. Find

   (a) $P(A)$
   (b) $P(B)$
   (c) $P(C)$
   (d) $P(D)$
   (e) $P(A \,|\, B)$
   (f) $P(A \,|\, C)$.

   Evaluate the above probabilities in the case $p = 1/2$, and understand why the probability obtained in (f) is different from that obtained in (e).

5. An aircraft contains three on-board computers. These work with probabilities 0.7, 0.8 and 0.9, independently of each other.

   (a) The aircraft can be flown automatically if at least two of the computers are working. Find the probability of this event.

   (b) The aircraft will crash if none of the computers is working. Find the probability of this event.

6. A system contains components numbered 1 to 5 which work with probabilities 0.5, 0.6, 0.7, 0.8 and 0.9 respectively. The components work, or fail to work, independently of each other. Find the probability that the system works under each of the following sets of conditions:

   (a) the system works if and only if all five components work;
   (b) the system works if and only if at least one component works;
   (c) the system works if and only if components 1, 2 and 3 all work *or* components 4 and 5 both work;

(d) the system works if and only if at least one of components 1, 2 *and* at least one of components 3, 4, 5 work;

(e) the system works if and only if components 1, 2, 3 and 4 all work *or* components 3, 4 and 5 all work.

7. A box contains 3 black balls and 2 white balls. A ball is drawn at random and then, *without replacing the 1st ball drawn*, a second ball is drawn at random. Let A_1 = 1st ball drawn is black and A_2 = 2nd ball drawn is white. Find $P(A_1 \cap A_2)$ and also $P(A_2)$.

   (*Hint:* Use the chain rule and the partition rule.) Why is the answer to the last part also obvious?

8. An urn contains 5 red and 3 green balls. Three balls are chosen at random, in succession, and without replacement.

   (a) Find the probability that all three balls drawn are red.

   (b) Use the chain and partition rules to find the probability that the third ball drawn is red. (*Hint:* draw a tree diagram.) Can you give also a quick derivation of this result?

   (c) Find the (conditional) probability that the third ball drawn is red, given that the first ball drawn is red, and the (conditional) probability that the first ball drawn is red, given that the third ball drawn is red. Can you draw any further interesting conclusions?

9. A man discovers that he is without his wallet. There are three (disjoint) possibilities: the event A that he left it at home, the event B that he left it on a desk at the office, and the event C that he dropped it on the pavement. He estimates the respective probabilities of these possibilities to be 0.4, 0.35 and 0.25. He further estimates that given each of these possibilities his respective (conditional) probabilities of recovery of the wallet are 1.0, 0.8 and 0.1.

   (a) Based on these estimates, find the probability of the event E that he recovers the wallet.

   (b) Given the event that he does recover the wallet, find the conditional probability of each of the three possible events A, B and C above. Interpret these probabilities.

10. Of three cards, one is red on both sides, one is black on both sides, and one is red on one side and black on the other. A card is chosen at random (each choice being equally likely) and then placed flat so that either side is equally likely to show. Given that a red side shows, what is the (conditional) probability that the other side shows red?

11. Five teams play in a football tournament. Every team plays every other team. Each match is equally likely to be won by either team (a draw cannot happen) *independently* of the outcome of the other matches. Find the probability of the (one) event that, at the end of the tournament, each team has won precisely two matches.

12. Consider 4 cities A, B, C, D. The following pairs are connected by roads: A with B, B with D, D with C, C with A, and B with C. Each road is independently blocked by snow with probability p.
    (i) Find the probability that it is possible to travel by road from A to D.
    (ii) Funds are available to snow-proof just one road. Would it be better to snow-proof AB or BC?

13. You are lost on an island in the summer, when tourists are 2/3 of the population. If you ask a tourist for directions the answer is correct with probability 3/4; answers to repeated questions are independent even if the question is the same. If you ask a local for directions, the answer is always false.
(a) You ask a passer-by whether the capital city is East or West. The answer is East. What is the probability that it is correct?
(b) You ask her again, and get the same reply. Show that the probability that it is correct is 1/2.
(c) You ask her one more time, and the answer is East again. What is the probability that it is correct?
(d) You ask her for a fourth time and get the answer West. What is the probability that East is correct?
(e) What if the fourth answer were also East?

14. A coin is given 5 independent tosses. On each toss the probability of obtaining a head is p. Let the random variable H be the total number of heads obtained, and let the random variable T be the total number of tails obtained.

    (a) Write down the probability function p_H of H, and also EH, $\operatorname{var} H$. In the case $p = 1/2$ also calculate EH, $\operatorname{var} H$, directly from p_H (and verify that you obtain the correct answers).
    (b) Write down the probability function p_T of T, and also ET, $\operatorname{var} T$. Derive also the latter two quantities from the corresponding results for H.
    (c) Write down the probability function, expectation (mean) and variance of $H+T$.
    (d) Find the probability function, expectation (mean) and variance of $H - T$.

15. An urn contains 5 red, 3 green, and 2 blue balls.

    (a) Four balls are drawn at random from the urn (without replacement). Let the random variables R, G, and B denote respectively the numbers of red, green and blue balls obtained. Find the probability function, expectation (mean) and variance of each of the following random variables:

        i. R
        ii. $R + 2$
        iii. $R^2 - R$
        iv. $R + B$

    (b) Suppose again that four balls are drawn at random from the urn, but that after each ball is drawn it is replaced. Again let R be the number of red balls obtained. Write down the new expectation and variance of R. Give a simple explanation of why ER is the same as before, and an intuitive explanation of why $\operatorname{var} R$ is larger than before.

16. An urn again contains 5 red and 3 black balls. Three balls are drawn at random from the urn, but now suppose that after each ball is drawn it is *replaced, along with another ball of the same colour.* Let R be the number of red balls obtained. Find $P(R = 3)$.

17. A drawer contains 6 socks; 2 are red, 2 are brown, 1 is black, and 1 is grey. Four socks are chosen at random from the drawer (without replacement). Let the random variable N be the number of *pairs* obtained (a pair is two socks of the same colour). Find the probability function, expectation (mean) and variance of N.

18. Four letters fall out of their envelopes and are replaced at random. Let the random variable N be the number of letters which are replaced in their correct envelopes.

    (a) Find the probability function, expectation (mean) and variance of N.

(b) Now define Bernoulli (indicator) random variables I_1, I_2, I_3, I_4 as follows:

$$I_j = \begin{cases} 1 & \text{if letter } j \text{ is replaced in its correct envelope} \\ 0 & \text{otherwise.} \end{cases}$$

Then $N = I_1 + I_2 + I_3 + I_4$. Hence give an alternative derivation of EN, $\operatorname{var} N$.

19. Let X have a *geometric distribution* with parameter p.

   (a) Show that, for $k \geq 1$, $P(X \geq k) = (1-p)^{k-1}$.

   (b) Use the alternative calculation for the expectation of a positive random variable to show again that $EX = 1/p$.

20. Let X_1 and X_2 be *independent* random variables, having geometric distributions with parameters p_1 and p_2 respectively.

   (a) Define $X_{\text{min}} = \min(X_1, X_2)$. Use the given independence to find $P(X_{\text{min}} \geq k)$ for $k \geq 1$. Hence, or otherwise, deduce that X_{min} also has a geometric distribution, and identify its parameter.

   (b) Define $X_{\text{max}} = \max(X_1, X_2)$. Find $P(X_{\text{max}} \leq k)$ for $k \geq 1$. Find also $P(X_{\text{max}} = k)$ for $k \geq 1$. In the case $p_1 = p_2 = 1/2$, evaluate $P(X_{\text{max}} = k)$ for $k = 1, 2, 3, 4, 5$ and comment. Finally (for general p_1, p_2 again) use the identity $X_{\text{max}} + X_{\text{min}} = X_1 + X_2$ to determine EX_{max}.

21. In any given year I have a claim under my insurance policy with probability p, independently of all other years. However, I never have more than one claim in any given year. After a total of 3 claims the company cancels the policy. Let the random variable T be the number of years required to make a total of 3 claims.

   (a) Find ET and $\operatorname{var} T$. (*Hint:* T is a sum of independent geometric random variables.)

   (b) (*Slightly more challenging.*) Find the probability function of T.

22. In a certain town male motorcycle injuries occur as a Poisson process of rate 4 per week, while female motorcycle injuries occur as an *independent* Poisson process of rate 1 per week.

   (a) Find the probability that, in a given two-week period, there are (i) no male accidents, (ii) no female accidents, (iii) no accidents of either type.

   (b) Write down also the distribution of the total number of accidents in any given two-week period, and thus give an alternative derivation of the probability that there are no accidents of either type.

23. Let $X \sim \text{B}(200, 0.005)$. For $k = 0, 1, \ldots, 6$, compare $P(X = k)$ with the estimate of this quantity given by the Poisson approximation to the binomial distribution.

24. Suppose there are n days in the year (on Earth we have $n = 365$, but it may be different elsewhere). Suppose that, in a group of k people, each has a birthday which is equally likely to fall on any day of the year, and that the birthdays of individuals are *independent* of each other (no twins). For $j = 1, 2, \ldots, k$ define the Bernoulli (indicator) random variable I_j by

$$I_j = \begin{cases} 1 & \text{if individual } j \text{ has the same birthday as someone else in the group} \\ 0 & \text{otherwise.} \end{cases}$$

Let the random variable $X = I_1 + I_2 + \cdots + I_k$ be the total number of individuals in the group with a non-unique birthday.

(a) What is $P(X = 1)$?

(b) Find a general expression for $P(X \geq 2)$, i.e. the probability that at least two individuals share the same birthday. (*Hint:* consider instead $P(X = 0)$.) Evaluate it for $n = 365$ and each of $k = 22, 23, 24$.

(c) Find EX. (*Hint:* use the indicator random variables defined above.)

25. Are the following functions distribution functions? If not, why not? *Hint:* Sketch the graph of each function.

(a)

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$$

(b)

$$F(x) = \begin{cases} 0, & x \leq -1, \\ 1 - |x|, & -1 \leq x < 1, \\ 1, & x \geq 1. \end{cases}$$

(c)

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{4}x^4, & x \geq 0. \end{cases}$$

(d)

$$F(x) = \begin{cases} 0, & x \leq 0, \\ \frac{1}{2}, & 0 < x \leq 1, \\ 1, & x > 1. \end{cases}$$

(e)

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & 0 \leq x < 1, \\ 1, & x \geq 1. \end{cases}$$

26. Are the following functions densities (probability density functions)? If not, why not?

(a)

$$f(x) = \begin{cases} \frac{4}{15}x^3, & -1 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

(b)

$$f(x) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{2}, \quad 1 \leq x \leq \frac{3}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

(c)

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

27. Suppose that $\lambda > 0$ is a positive constant and that X is a random variable with the *exponential* density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Check that f_X is a probability density. Find also the distribution function of X and sketch it.

(b) Suppose that $\lambda = 3$. Find $P(X > 2)$ and also $P(X > 11 \,|\, X > 9)$. (This illustrates the *memoryless* property of the exponential distribution.)

28. Buses arrive at fifteen minute intervals starting at noon. A woman arrives at the bus stop X minutes after noon, where X is a random variable with distribution function

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{x}{60}, & 0 \le x < 60, \\ 1, & x \ge 60. \end{cases}$$

(a) What is the density of X? Sketch it.

(b) Is the woman more likely to arrive before 12.30 pm or after 12.30 pm?

(c) What is the probability that the woman waits less than five minutes for a bus?

29. A dart is thrown at a circular target of radius 1 metre in such a way that the probability of hitting any portion of the target is proportional to the area of the region. To begin with, suppose that the dart never misses the target. Whenever the dart hits the target, let X be the player's score, where $X = 100 - Y$ and Y is the distance of the dart from the centre of the target (measured in centimetres). Thus the score is a continuous variable with the highest possible score being 100.

(a) Find the distribution function of Y and sketch it.

(b) Find the distribution function of X and sketch it.

(c) Find the density of X. Is X uniformly distributed?

(d) Determine: (i) $P(X = 20)$, and (ii) $P(30 < X < 90)$.

(e) Now suppose that the dart misses the target completely one-half of the time, and that the score $X = 0$ is assigned whenever the dart misses the target. How do your answers to b), c), and d) change?

30. Suppose that a random variable X has density f_X given by

$$f_X(x) = \begin{cases} a(1 - x^3), & 0 \le x \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

for some constant a.

(a) Find a.

(b) Find the distribution function F_X of X.

(c) Find EX and $\operatorname{var} X$. Verify the answer for EX by considering also the alternative expression for the expectation of a positive random variable,

(d) Find also the density, distribution function, expectation and variance of $Y = 1 - X$.

31. Suppose that a random variable X has a uniform $\mathrm{U}(0, 1)$ distribution. Find the density, distribution function, expectation and variance of each of the following random variables.

(a) $Y = \dfrac{1}{X^{\frac{1}{3}}}$.

(b) $Z = -\lambda \ln X$.

(c) $W = \min\left(\frac{1}{2}, X^2\right)$.

32. Let X be a random variable with density function given by

$$f_X(x) = \begin{cases} 1 - |x|, & |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Sketch the graph of f_X. Find EX and $\operatorname{var} X$. Is your value for EX intuitively reasonable?

33. Suppose that a random variable X has a uniform $\mathrm{U}(0,1)$ distribution. Find the distribution function, density and expectation of $Y = e^X$.

34. Let $U_1, \dots, U_n$ be *independent* random variables each with a uniform $\mathrm{U}(0,1)$ distribution. Let $X = \min(U_1, \dots, U_n)$.

Find the distribution function, density, expectation and variance of X. Comment on the dependence on n. Sketch the density in the case $n = 2$ and comment.

Independence: recall that independent random variables define independent events; thus, for example,

$$P(U_1 \le x, \dots, U_n \le x) = P(U_1 \le x) \dots P(U_n \le x).$$

35. A bus travels between two cities, A and B, which are 120 miles apart. There is a bus service station in city A, in city B, and a further service station located halfway between cities A and B. If the bus has a breakdown, the distance in miles from the breakdown to city A has a uniform distribution over the interval $[0, 120]$, and the bus then travels to the nearest service station.

   (a) In the event of a breakdown, find the expected distance to the nearest service station.
   (b) It is suggested that it would be more efficient (in terms of minimising the expected distance to the nearest service station) to have the three stations located 30, 60 and 90 miles from city A. Do you agree?
   (c) What are the optimal locations?

36. Let $X \sim \mathrm{N}(4, 5^2)$. Use tables to find

   (a) $P(X > 6)$
   (b) $P(X < 2)$
   (c) $P(0 < X < 10)$.

37. I am waiting for my two friends. The times T_1 and T_2 until they arrive are *independent* exponentially distributed random variables with parameters λ_1 and λ_2, i.e.

$$P(T_i \le t) = 1 - e^{-\lambda_i t}.$$

Find the distribution function, expectation and variance of the time $T_{\min}$ until the first of my friends arrives, and of the time $T_{\max}$ until the second of my friends arrives.

38. Let $X \sim \Gamma(2, 0.5)$. Then X has density

$$f_X(x) = \begin{cases} \frac{x}{4} \exp\left(-\frac{x}{2}\right), & \text{if } x \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

Use this to find EX and verify that it agrees with the standard result.

39. Let X be a random variable with density

$$f_X(x) = \begin{cases} ax^2, & \text{if } 0 \le x \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find a.

(b) Let $Y = 1/X$. Find the distribution function and density of Y. Find also EY.

40. Let X be a random variable with a continuous distribution function F_X.

(a) Express the distribution function F_{X^+} of $X^+ = \max(0, X)$ in terms of F_X.

(b) Similarly, express the distribution function F_{X^-} of $X^- = \max(0, -X)$ in terms of F_X.

41. Let F be a continuous distribution function, and let $U \sim \mathrm{U}(0,1)$. Show that the random variable $F^{-1}(U)$ has distribution function F.

42. An urn contains 4 blue, 3 yellow, and 2 magenta balls. Three balls are chosen at random from the urn (without replacement). Let the random variables B, Y and M denote respectively the numbers of blue, yellow, and magenta balls obtained.

(a) Find the joint probability function $p_{B,Y}$ of (the distribution of) B and Y—you may find it helpful later if you lay out the values in a table.

(b) Find also the individual (marginal) probability functions p_B and p_Y of B and Y.

(c) Find the joint distribution function $F_{B,Y}$ of B and Y.

(d) Find the conditional probability function $p_{Y\,|\,B}$ of Y given B (as a function $p_{Y\,|\,B}(y\,|\,b)$ of y for each b).

(e) Find the probability function p_M of M *directly* from $p_{B,Y}$.

43. Suppose that the random variables X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} a(x+y^2), & 0 \le x \le 1, \quad 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the constant a.

(b) Find the marginal densities f_X and f_Y and the marginal distribution functions F_X and F_Y of X and Y.

(c) Find the joint distribution function $F_{X,Y}$ of X and Y.

(d) Find $P(X > Y)$.

(e) Find $P(X^2 > Y)$.

44. Suppose that the random variables X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} ae^{-\lambda x - \mu y}, & x \ge 0, \quad y \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the constant a.

(b) Find the marginal densities f_X and f_Y of X and Y.

(c) Find the conditional densities $f_{X\,|\,Y}$ of X given Y and $f_{Y\,|\,X}$ of Y given X (*comment*).

(d) Find $P(X < Y)$.

45. Suppose that the random variables X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} ae^{-y/2}, & 0 \le x \le y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the constant a.

(b) Find the marginal densities f_X and f_Y of X and Y and identify the distributions of these two random variables.

46. Let $X_1, \ldots, X_n$ be independent identically distributed random variables, with common distribution function F. Let

$$X_{\max} = \max(X_1, \ldots, X_n), \qquad X_{\min} = \min(X_1, \ldots, X_n),$$

and let $F_{\max}$ and $F_{\min}$ be the distribution functions of $X_{\max}$ and $X_{\min}$ respectively. Show that

$$F_{\max}(x) = F(x)^n \qquad \text{for all } x.$$

Find an expression for $F_{\min}$.

47. Every week I win the UK lottery with probability p_1, independently of all other weeks. Also each week I win the Irish lottery with probability p_2, independently of all other weeks. The number of weeks X_1 until I first win the UK lottery and the number of weeks X_2 until I first win the Irish lottery are thus geometric random variables with parameters p_1 and p_2 respectively. Assume these two random variables to be independent.

(a) Show that the distribution of the random variable $X_{\min} = \min(X_1, X_2)$ (the number of weeks until I first win either lottery) is also geometric, and identify its parameter.
[*Hint: either* (a) find an expression for $P(X_{\min} > n)$, *or* (b) find the probability of a win in either lottery in any given week.]

(b) Find an expression for the *distribution function* of $X_{\max} = \max(X_1, X_2)$ (the number of weeks until I have achieved a win in both lotteries—not necessarily in the same week).

(c) Find also an expression for the *probability function* of $X_1 + X_2$ (my total stake until I achieve wins in both lotteries if I stake one unit per week in each).

(d) Find $E(X_1 + X_2)$ and $\mathrm{var}(X_1 + X_2)$.

48. Suppose that the random variables X and Y are independent identically distributed, each with a $\mathrm{U}(0,1)$ distribution.

(a) Find the distribution function and density of $Z = X + Y$.
[*Hint:* draw a picture on the unit square: for $0 \le z \le 1$, we have

$$P(Z \le z) = \int_0^z \left[\int_0^{z-x} 1\, dy \right] dx;$$

for $1 \le z \le 2$, we have

$$P(Z \le z) = \int_0^{z-1} \left[\int_0^1 1\, dy \right] dx + \int_{z-1}^1 \left[\int_0^{z-x} 1\, dy \right] dx.$$

]

(b) Find the distribution function and density of $Z = XY$.
[*Hint:* draw a picture on the unit square and, for $z \le 1$, sketch the region corresponding to $Z \le z$.]

49. The time T_1 weeks until I replace my failed light bulb is uniformly distributed on $(0,1)$, while the lifetime T_2 weeks of the replacement bulb is exponentially distributed with mean λ^{-1} ($T_2 \sim \mathrm{Exp}(\lambda)$) and is independent of T_1. Find the distribution function and density of the total time $T = T_1 + T_2$ weeks between the failure of the original bulb and that of its replacement.

[*Hint:* draw a picture corresponding to $0 \le t_1 \le 1$, $t_2 \ge 0$ and sketch the region corresponding to $T \le t$. Hence find first the distribution function of T.]

50. Show that for any two random variables X and Y, and any two constants a and b,

$$\operatorname{var}(aX + bY) = a^2 \operatorname{var} X + 2ab \operatorname{cov}(X, Y) + b^2 \operatorname{var} Y.$$

51. Consider again Question 42. An urn contains 4 blue, 3 yellow, and 2 magenta balls. Three balls are chosen at random from the urn (without replacement). Let the random variables B, Y and M denote respectively the numbers of blue, yellow, and magenta balls obtained.

 (a) Find $E(Y \mid B = b)$ (for all b) and also $E(B \mid Y = y)$ (for all y).
 (b) Verify directly that

$$EY = \sum_b P(B = b)E(Y \mid B = b), \qquad EB = \sum_y P(Y = y)E(B \mid Y = y).$$

 (c) Find $E(BY)$, $\operatorname{cov}(B, Y)$ and hence $\operatorname{var}(B + Y)$.
 (d) Verify directly that $\operatorname{var}(B + Y) = \operatorname{var}(M)$.

52. Consider again Question 43. Suppose that the random variables X and Y have joint density

$$f_{X,Y}(x, y) = \begin{cases} \frac{6}{5}(x + y^2), & 0 \le x \le 1, \quad 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

 (a) Find EX, EY, and $E(XY)$.
 (b) Find $\operatorname{var} X$, $\operatorname{var} Y$, $\operatorname{cov}(X, Y)$ and $\operatorname{var}(X + Y)$.
 (c) Are X and Y independent. Give reasons for your answer.

53. I am waiting for two important phone calls (then I can go out). Their arrival times T_1, T_2 (in hours) are *independent* $\mathrm{U}(0, 1)$ random variables. Let $T_{\min} = \min(T_1, T_2)$ be the arrival time of the first call, and let $T_{\max} = \max(T_1, T_2)$ be the arrival time of the second call.

 (a) Show that, for $0 \le x \le y \le 1$,

$$P(T_{\min} \le x, T_{\max} \le y) = 2xy - x^2.$$

 [*Hint:* for given x and y as above, identify the area within the unit square which defines the set of values of (T_1, T_2) such that $T_{\min} \le x, T_{\max} \le y$.]
 (b) Hence, or otherwise, explain why $T_{\min}$ and $T_{\max}$ have joint density

$$f_{T_{\min},T_{\max}}(x, y) = \begin{cases} 2, & 0 \le x \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

 (c) Find the marginal densities of $T_{\min}$ and $T_{\max}$.
 (d) Find $ET_{\min}$, $ET_{\max}$, and $E(T_{\min}T_{\max})$.
 (e) Find $\operatorname{cov}(T_{\min}, T_{\max})$. Interpret your answer.
 (f) Are $T_{\min}$ and $T_{\max}$ independent. Give reasons for your answer.

54. A coin which lands heads with probability p is tossed N times where N is a random variable with a $\mathrm{Pois}(\lambda)$ distribution. Let X be the total number of heads obtained.

 (a) Write down $P(X = k \mid N = n)$ for $k = 0, 1, \dots, n$.
 (b) Use the result

$$P(X = k) = \sum_{n=k}^{\infty} P(N = n)P(X = k \mid N = n)$$

 to find $P(X = k)$ for $k = 0, 1, 2, \dots$. Hence state the distribution of X.