# Answers to the Exercises for Statistics 4 (Elements of Probability) from the file [E1] Autumn 2006

1. The probability that the match can go ahead is 0.55.
2. 0.35.
3. (a) $P(A) = 1/36, P(B) = 11/36,\ P(C) = 3/4$.

   (b)
   $$P(N_1 = n \mid N = 5) = \begin{cases} 1/4 & \text{if } n = 1,2,3,4 \\ 0 & \text{if } n = 5,6. \end{cases}$$
4. (a) $P(A) = p^4$
   (b) $P(B) = p^3$
   (c) $P(C) = 4p^3(1-p) + p^4$
   (d) $P(D) = 1 - P(A) = 1 - p^4$
   (e) $P(A \mid B) = P(A \cap B)/P(B) = P(A)/P(B) = p$
   (f) $P(A \mid C) = P(A \cap C)/P(C) = P(A)/P(C) = p/(4-3p)$.
5. (a) 0.902.
   (b) 0.006.
6. Let $E_k$ be the event that component $k$ works ($k = 1,2,3,4,5$).

   (a) 0.1512.
   (b) 0.9988.
   (c) 0.7788.
   (d) 0.7952.
   (e) 0.5208.
7. $P(A_1 \cap A_2) = 3/10,\ P(A_2) = 2/5$.
8. (a) 5/28.
   (b) 5/28.
   (c) 4/7, 4/7.
9. (a) 0.705.
   (b) $P(A \mid E) = \frac{0.4\times1.0}{0.705} = 0.567,\ P(B \mid E) = \frac{0.35\times0.8}{0.705} = 0.397,\ P(C \mid E) = \frac{0.25\times0.1}{0.705} = 0.035$.
10. 2/3.
11. 3/128.
12. The probability it is possible to travel by road from $A$ to $D$ is
    $$(1-p)^2(1+p)(1+p-p^2) = 1 - 3p^2 + p^3 + 2p^4 - p^5.$$
13. Removed.
14. (a) $p_H(n) = \binom{5}{n} p^n (1-p)^{5-n}, \qquad n = 0,1,2,3,4,5,$
    $EH = 5p, \qquad \operatorname{var} H = 5p(1-p)$.

    (b) $p_T(n) = \binom{5}{n}(1-p)^n p^{5-n}, \qquad n = 0,1,2,3,4,5,$
    $ET = 5(1-p), \qquad \operatorname{var} T = 5p(1-p)$.

    (c)
    $$p_{H+T}(n) = \begin{cases} 1, & n = 5 \\ 0, & \text{otherwise.} \end{cases}$$
    $E(H+T) = 5, \qquad \operatorname{var}(H+T) = 0.$

(d)

$$p_{H-T}(n) = \binom{5}{\frac{n+5}{2}} p^{\frac{n+5}{2}} (1-p)^{\frac{5-n}{2}}, \qquad n = -5, -3, -1, 1, 3, 5,$$

$E(H-T) = EH - ET = 5(2p-1)$, $\qquad \mathrm{var}(H-T) = \mathrm{var}(2H-5) = \mathrm{var}(2H) = 4\,\mathrm{var}\,H = 20p(1-p)$.

15. (a) i.

| $k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $p_R(k)$ | $\frac{1}{42}$ | $\frac{10}{42}$ | $\frac{20}{42}$ | $\frac{10}{42}$ | $\frac{1}{42}$ |

$ER = 2$, $\mathrm{var}\,R = 2/3$.

ii.

| $k$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $p_{R+2}(k)$ | $\frac{1}{42}$ | $\frac{10}{42}$ | $\frac{20}{42}$ | $\frac{10}{42}$ | $\frac{1}{42}$ |

$E(R+2) = ER + 2 = 4$, $\mathrm{var}(R+2) = \mathrm{var}\,R = 2/3$.

iii.

| $k$ | 0 | 2 | 6 | 12 |
|---|---|---|---|---|
| $p_{R^2-R}(k)$ | $\frac{11}{42}$ | $\frac{20}{42}$ | $\frac{10}{42}$ | $\frac{1}{42}$ |

$E(R^2 - R) = ER^2 - ER = 8/3$, $\mathrm{var}(R^2 - R) = 428/63$.

iv.

| $k$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $p_{R+B}(k)$ | $\frac{1}{30}$ | $\frac{9}{30}$ | $\frac{15}{30}$ | $\frac{5}{30}$ |

$E(R+B) = 14/5$, $\mathrm{var}(R+B) = 14/25$.

16. 7/24.

17.

| $n$ | 0 | 1 | 2 |
|---|---|---|---|
| $p_N(n)$ | $\frac{4}{15}$ | $\frac{10}{15}$ | $\frac{1}{15}$ |

$EN = 4/5$, $\mathrm{var}\,N = 22/75$.

18. (a)

| $n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $p_N(n)$ | $\frac{9}{24}$ | $\frac{8}{24}$ | $\frac{6}{24}$ | $\frac{0}{24}$ | $\frac{1}{24}$ |

$EN = 1$, $\mathrm{var}\,N = 1$.

19.

20. (a) $X_{\min}$ has a geometric distribution with parameter $p_1 + p_2 - p_1 p_2$.

(b)

$$P(X_{\max} \le k) = P(X_1 \le k,\, X_2 \le k)[1 - (1-p_1)^k][1 - (1-p_2)^k].$$

$$P(X_{\max} = k) = (1-p_1)^{k-1} p_1 + (1-p_2)^{k-1} p_2 - (1-p_1)^{k-1}(1-p_2)^{k-1}(p_1 + p_2 - p_1 p_2).$$

In the case $p_1 = p_2 = 1/2$, we have

$$P(X_{\max} = k) = \left(\frac{1}{2}\right)^{k-1} - 3\left(\frac{1}{2}\right)^{2k}$$

| $k$ | 1 | 2 | 3 | 4 | 5 | $\ge 6$ |
|---|---|---|---|---|---|---|
| $P(X_{\max} = k)$ | 0.2500 | 0.3125 | 0.2031 | 0.1133 | 0.0596 | 0.0615. |

$$EX_{\max} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_1 + p_2 - p_1 p_2}.$$

21. (a)

$$ET = \frac{3}{p}, \qquad \operatorname{var} T = \frac{3(1-p)}{p^2}.$$

(b)

$$P(T = k) = \binom{k-1}{2} p^3 (1-p)^{k-3}, \quad k \geq 3.$$

22. (a) i. 0.00034,
    ii. 0.135,
    iii. 0.000045.

(b) $N_m + N_f \sim \text{Pois}(10)$.

23.

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                    k     0      1      2      3      4      5      6
B(200,0.005)          0.3670 0.3688 0.1844 0.0612 0.0151 0.0030 0.0005
Pois(1)               0.3679 0.3679 0.1839 0.0613 0.0153 0.0031 0.0005
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24. (a) $P(X = 1) = 0$

(b) $P(X \geq 2) = 1 - \frac{n-1}{n}\frac{n-2}{n} \cdots \frac{n-k+1}{n}$. For $n = 365$ and $k = 22, 23, 24$, these probabilities are respectively 0.4757, 0.5073, 0.5383.

(c) $EX = k(1 - \left(\frac{n-1}{n}\right)^{k-1})$.

25. (a) Yes.

(b) No.

(c) No.

(d) No.

(e) Yes.

26. (a) No.

(b) Yes.

(c) No.

27. (a)

(b) $F_X(x) = (1 - e^{-\lambda x})\mathbf{1}(x \geq 0)$.

(c) For $\lambda = 3$, we have $P(X > 2) = 1 - F_X(2) = e^{-6}$. Also $P(X > 11 \,|\, X > 9) = e^{-6}$.

28. (a) $f_X(x) = \frac{1}{60}\mathbf{1}(0 < x < 60)$.

(b) Same.

(c) $\frac{1}{3}$.

29. (a)

$$P(Y \leq y) = \begin{cases} 0, & y < 0, \\ \frac{y^2}{100^2}, & 0 \leq y < 100, \\ 1, & y \geq 100. \end{cases}$$

(b)

$$P(X \leq x) \begin{cases} 0, & x < 0, \\ 1 - \frac{(100-x)^2}{100^2}, & 0 \leq x < 100, \\ 1, & x \geq 100. \end{cases}$$

(c)

$$\begin{cases} 0, & x < 0, \\ \frac{2(100-x)}{100^2}, & 0 \leq x < 100, \\ 0, & x \geq 100. \end{cases}$$

No.

(d) i. 0.
 ii. 0.48.

(e)

$$P(X \le x) = \begin{cases} 0, & x < 0, \\ \frac{1}{2} + \frac{1}{2}\left(1 - \frac{(100-x)^2}{100^2}\right), & 0 \le x < 100, \\ 1, & x \ge 100. \end{cases}$$

$P(X = 20) = 0$, $P(30 < X < 90) = 0.24$.

30. (a) $a = 4/3$.

(b)

$$P(X \le x) = \begin{cases} 0, & x < 0, \\ \frac{4}{3}\left(x - \frac{1}{4}x^4\right), & 0 \le x < 1, \\ 1, & x \ge 1. \end{cases}$$

(c) $EX = \frac{2}{5}$, $\operatorname{var} X = \frac{14}{225}$.

(d) $EY = 3/5$, $\operatorname{var} Y = \operatorname{var} X = 14/225$.

31. (a)

$$P(Y \le y) = (1 - \frac{1}{y^3})\mathbf{1}(y \ge 1), \qquad \text{density} : (\frac{3}{y^4})\mathbf{1}(y > 1).$$

$EY = 3/2$, $\operatorname{var} Y = 3/4$.

(b)

$$P(Z \le z) = (1 - e^{-z/\lambda})\mathbf{1}(z \ge 0), \qquad \text{density} : \frac{1}{\lambda}e^{-z/\lambda}\mathbf{1}(z > 0).$$

$EZ = \lambda$, $\operatorname{var} Z = \lambda^2$.

(c)

$$F_{W'}(w) = P\left(X \le w^{\frac{1}{2}}\right) = \begin{cases} 0, & w < 0, \\ w^{\frac{1}{2}}, & 0 \le w < 1, \\ 1, & w \ge 1. \end{cases}$$

$$P(W \le w) = \begin{cases} 0, & w < 0, \\ w^{\frac{1}{2}}, & 0 \le w < \frac{1}{2}, \\ 1, & w \ge \frac{1}{2}. \end{cases}$$

$$EW = \frac{3 - \sqrt{2}}{6}, \qquad \operatorname{var} W = \frac{6\sqrt{2} - 5}{90}.$$

32. $EX = 0$, $\operatorname{var} X = \frac{1}{6}$.

33.

$$P(Y \le y) = \begin{cases} 0, & y < 1, \\ \ln y, & 1 \le y < e, \\ 1, & y \ge e. \end{cases}$$

density: $(1/y)\mathbf{1}(1 < y < e)$, $EY = e - 1$.

34.

$$P(X \le x) = \begin{cases} 0, & x < 0, \\ 1 - (1-x)^n, & 0 \le x < 1, \\ 1, & x \ge 1. \end{cases}$$

Density: $n(1-x)^{n-1}\mathbf{1}(0 < x < 1)$, $EX = \frac{1}{n+1}$, $\operatorname{var} X = \frac{n}{(n+1)^2(n+2)}$.

35. (a) 15.
    (b) 11.25.
    (c) $20, 60, 100$.

36. (a) 0.3446
    (b) 0.3446
    (c) 0.6730

37. $T_{\min} \sim \text{Exp}(\lambda_1 + \lambda_2)$.

$$P(T_{\max} \leq t) = (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t}),$$

density of $T_{\max}$: $\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t} - (\lambda_1 + \lambda_2) e^{-(\lambda_1+\lambda_2)t}$.

$$ET_{\max} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}, \qquad \operatorname{var} T_{\max} = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} - \frac{3}{(\lambda_1 + \lambda_2)^2}.$$

38. 4.

39. (a) $a = 1/9$.
    (b)

$$P(Y \leq y) = \left(1 - \frac{1}{27y^3}\right) \mathbf{1}(y \geq 1/3), \qquad \text{density: } \frac{1}{9y^4} \mathbf{1}(y \geq \frac{1}{3}), \quad EY = \frac{1}{2}.$$

40. (a) $F_{X^+}(x) = F_X(x) \mathbf{1}(x \geq 0)$.
    (b) $F_{X^-}(x) = (1 - F_X(-x)) \mathbf{1}(x \geq 0)$.

41.

42.

$$\begin{array}{cc|cccc|cc}
\multicolumn{8}{c}{p_{B,Y}(b,y)} \\
 & & \multicolumn{4}{c}{b} & & \\
 & & 0 & 1 & 2 & 3 & & \\
\hline
 & 0 & 0 & \frac{4}{84} & \frac{12}{84} & \frac{4}{84} & \frac{20}{84} & \\
y & 1 & \frac{3}{84} & \frac{24}{84} & \frac{18}{84} & 0 & \frac{45}{84} & p_Y(y) \\
 & 2 & \frac{6}{84} & \frac{12}{84} & 0 & 0 & \frac{18}{84} & \\
 & 3 & \frac{1}{84} & 0 & 0 & 0 & \frac{1}{84} & \\
\hline
 & & \frac{10}{84} & \frac{40}{84} & \frac{30}{84} & \frac{4}{84} & & \\
 & & \multicolumn{4}{c}{p_B(b)} & &
\end{array}
\qquad
\begin{array}{cc|cccc|cc}
\multicolumn{8}{c}{F_{B,Y}(b,y)} \\
 & & \multicolumn{4}{c}{b} & & \\
 & & 0 & 1 & 2 & 3 & & \\
\hline
 & 0 & 0 & \frac{4}{84} & \frac{16}{84} & \frac{20}{84} & \frac{20}{84} & \\
y & 1 & \frac{3}{84} & \frac{31}{84} & \frac{61}{84} & \frac{65}{84} & \frac{65}{84} & F_Y(y) \\
 & 2 & \frac{9}{84} & \frac{49}{84} & \frac{79}{84} & \frac{83}{84} & \frac{83}{84} & \\
 & 3 & \frac{10}{84} & \frac{50}{84} & \frac{80}{84} & 1 & 1 & \\
\hline
 & & \frac{10}{84} & \frac{50}{84} & \frac{80}{84} & 1 & & \\
 & & \multicolumn{4}{c}{F_B(b)} & &
\end{array}$$

$$\begin{array}{cc|cccc|}
\multicolumn{6}{c}{p_{Y\,|\,B}(y\,|\,b)} \\
 & & \multicolumn{4}{c}{b} \\
 & & 0 & 1 & 2 & 3 \\
\hline
 & 0 & 0 & \frac{4}{40} & \frac{12}{30} & 1 \\
y & 1 & \frac{3}{10} & \frac{24}{40} & \frac{18}{30} & 0 \\
 & 2 & \frac{6}{10} & \frac{12}{40} & 0 & 0 \\
 & 3 & \frac{1}{10} & 0 & 0 & 0 \\
\hline
\end{array}$$

| $m$ | 0 | 1 | 2 |
|---|---|---|---|
| $p_M(m)$ | $\frac{35}{84}$ | $\frac{42}{84}$ | $\frac{7}{84}$ |

43. (a) 6/5.
    (b) $f_X(x) = a\left(x + \frac{1}{3}\right)$, $F_X(x) = a\left(\frac{x^2}{2} + \frac{x}{3}\right)$, for $0 \leq x \leq 1$ *only.*
    $f_Y(y) = a\left(\frac{1}{2} + y^2\right)$, $F_Y(y) = a\left(\frac{y}{2} + \frac{y^3}{3}\right)$, for $0 \leq y \leq 1$ *only.*
    (c) $F_{X,Y}(x,y) = a\left(\frac{1}{2}x^2 y + \frac{1}{3}xy^3\right)$, for $0 \leq x \leq 1$, $0 \leq y \leq 1$ *only.*

(d) $P(X > Y) = 0.5$.

(e) $P(X^2 > Y) = 5/14$.

44. (a) $a = \lambda\mu$.

(b) $X \sim \mathrm{Exp}(\lambda)$, $Y \sim \mathrm{Exp}(\mu)$.

(c) $f_{X\,|\,Y}(x\,|\,y) = \lambda e^{-\lambda x}$, $f_{Y\,|\,X}(y\,|\,x) = \mu e^{-\mu y}$, for $x \geq 0$, $y \geq 0$ *only.*

(d) $\frac{\lambda}{\lambda+\mu}$.

45. (a) $1/4$.

(b) $f_X(x) = 2ae^{-\frac{x}{2}}\mathbf{1}(x > 0)$, $X \sim \mathrm{Exp}(\frac{1}{2})$,
$f_Y(y) = \frac{1}{4}ye^{-\frac{y}{2}}\mathbf{1}(y > 0)$, $Y \sim \Gamma(2, \frac{1}{2})$.

46. $F_{\min}(x) = 1 - (1 - F(x))^n$.

47. (a) $p_1 + p_2 - p_1 p_2$.

(b) $[1 - (1 - p_1)^n][1 - (1 - p_2)^n]$.

(c) For $n \geq 2$:

$$\begin{cases} p_1 p_2 \dfrac{(1-p_2)^{n-1} - (1-p_1)^{n-1}}{p_1 - p_2}, & p_1 \neq p_2 \\ (n-1)p_1^2(1-p_1)^{n-2} & p_1 = p_2. \end{cases}$$

(d) $\frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1-p_1}{p_1^2} + \frac{1-p_2}{p_2^2}$.

48. (a) Density: $z\mathbf{1}(0 \leq z < 1) + (2 - z)\mathbf{1}(1 \leq z < 2)$.

(b) Density: $-\ln z\mathbf{1}(0 < z \leq 1)$.

49.

$$P(T \leq t) = \left(t - \frac{1}{\lambda}\left(1 - e^{-\lambda t}\right)\right)\mathbf{1}(0 \leq t < 1) + \left(1 - \frac{e^\lambda - 1}{\lambda}e^{-\lambda t}\right)\mathbf{1}(t \geq 1).$$

$$\text{Density: } \begin{cases} 1 - e^{-\lambda t}, & 0 \leq t < 1, \\ (e^\lambda - 1)e^{-\lambda t}, & t \geq 1, \\ 0, & \text{otherwise} \end{cases}.$$

50. Look at your lecture notes.

51. (a)

| $b$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $E(Y\,|\,B = b)$ | $\frac{9}{5}$ | $\frac{6}{5}$ | $\frac{3}{5}$ | 0 |

| $y$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $E(B\,|\,Y = y)$ | $\frac{6}{3}$ | $\frac{4}{3}$ | $\frac{2}{3}$ | 0 |

(b)

(c) $1, -1/3, 7/18$.

(d)

52. (a) $3/5, 3/5, 7/20$.

(b) $11/150, 2/25, -1/100, 2/15$.

(c) Are they?

53. (a)

(b)

(c) $2(1 - x)\mathbf{1}(0 < x < 1)$, $2y\mathbf{1}(0 < y < 1)$.

(d) $1/3, 2/3, 1/4$.

(e) $1/36$.

(f) Are they?

54. (a)

$$P(X = k\,|\,N = n) = \binom{n}{k}p^k(1-p)^{n-k}, \qquad k = 0, 1, \ldots, n.$$

(b)

$$P(X = k) = e^{-\lambda p} \frac{(\lambda p)^k}{k!}.$$