On Two-Dimensional Markov Chains in the Positive Quadrant with Partial Spatial Homogeneity

S. Zachary

Department of Actuarial Mathematics and Statistics, Heriot-Watt University, Riccarton, Edinburgh, EH14 4AS, UK [1]

Abstract

We consider the problem of classifying Markov chains on the quarter plane $\mathbf{Z}_+^2$ which possess a property of partial spatial homogeneity. Such chains arise frequently in the study of queueing and loss networks and have been previously studied, notably by Malyshev and Menshikov and by Fayolle. However existing results are either given under restrictive conditions, or lack complete proofs. We take a new approach to the construction of Lyapounov functions for such chains to give proofs which require weaker conditions, are complete in all cases and additionally are considerably simpler.

We also describe the application of these results to the study of the dynamic and equilibrium behaviour of large loss networks.

Keywords: Markov chains; random walks; ergodicity; transience; Lyapounov functions; loss networks; queueing networks.

1 Introduction

In this paper we study discrete time irreducible Markov chains on the quarter plane $\mathbf{Z}_+^2$ (where $\mathbf{Z}_+$ is the set of nonnegative integers) whose transition structures possess a property of partial spatial homogeneity. This property, made precise below, may be described informally as requiring that such a chain feels the influence of either of the two plane 'boundaries' only when sufficiently close to it. Our interest is in describing further the behaviour of such chains and in particular in classifying them as positive recurrent, null recurrent or transient.

Markov chains such as these have been much studied, both in two and higher dimensions—see in particular Malyshev [12, 13], Malyshev and Menshikov [14], Borovkov [4], Fayolle [6], Rosenkrantz [16] and also the recent book by Fayolle,

[1] Email: stan@ma.hw.ac.uk

Malyshev and Menshikov [8]. However, even in two dimensions, there appears to be no complete treatment in the published literature of the partially homogeneous case, where the constants K_1, K_2 defined below may be non-zero. It is this case which is important for many applications, in particular those described below for loss networks. The partially homogeneous case is considered by Malyshev and Menshikov [14] under the restriction that the jumps of the chains be bounded—which is too restrictive for some applications. It is further considered by Fayolle [6] under the weaker assumption that the jumps of the chains have bounded second moments. However, for partial homogeneity, Fayolle only gives a proof in the positive recurrent case, and then only when the drifts α_1, α_2 defined below are both negative. Further, this proof involves the construction of a Lyapounov function for a Markov chain embedded in the original chain. This is in order to deal with the anomalous behaviour of the original process close to the boundaries of the plane.

In the present work we take a different approach to the partially homogeneous case by constructing functions which are Lyapounov directly with respect to the original chain. This new approach, which is the major feature of the paper, both permits a substantially simpler derivation of existing results and makes it possible to give a complete and comprehensive treatment of the problem under weaker moment conditions than those for which proofs have previously been published. (In the case of bounded jumps, Lemma 2 below is well known—see Fayolle, Malyshev and Menshikov [8], while, under the weaker assumption that jumps have bounded second moments, Lemma 3 below is trivial. In either of these cases, our proofs become particularly simple.) We also suspect that the present approach extends to provide corresponding simplification and generalisation in higher dimensions, and for the case of *asymptotic* partial homogeneity.

Our results are presented in Section 2 below. In Section 3 we describe the importance of these results—and their potential generalizations—for the theory of large loss networks, as developed by Kelly [11] and Hunt and Kurtz [10]. It is these applications in particular which motivate the present work. However, Markov chains with partial spatial homogeneity also arise naturally elsewhere in the study of queueing and loss networks—see, for example, Malyshev [13], Fayolle, Malyshev and Menshikov [8] and the references therein.

Let $P = \{p(x,y),\ x,y \in \mathbf{Z}_+^2\}$ be the transition matrix of a discrete time Markov chain $\{X_n,\ n \geq 0\}$ on $\mathbf{Z}_+^2$. We suppose that the chain is irreducible and that P has the following property of (partial) spatial homogeneity: for each $j = 1, 2$, there exists $K_j \geq 0$ such that, for each $z \in \mathbf{Z}^2$ and for each $\xi \in \mathbf{Z}_+$, $p(x, x+z)$ is constant on the set $\{x\colon x_j > K_j, x_{j'} = \xi\}$, where, here and elsewhere, $x = (x_1, x_2)$ and j' denotes the complementary element of j in the set $\{1, 2\}$.

Note that in particular this implies that there exists a function p^* on $\mathbf{Z}^2$ such that $p(x, x+z) = p^*(z)$ for all x in the set $\mathcal{X}_1 \cap \mathcal{X}_2$, where, for $j = 1, 2$, $\mathcal{X}_j = \{x : x_j > K_j\}$ (though the above spatial homogeneity property is of course considerably stronger than this).

We further assume that, for some $d > 1$, the jumps of the chain are bounded (and hence by the above homogeneity property uniformly bounded) in L^d, that is, that there exists $\overline{M} > 0$ such that

$$\sum_{y \in \mathbf{Z}_+^2} p(x, y)|y_j - x_j|^d < \overline{M} \quad \text{for all } x \in \mathbf{Z}_+^2,\ j = 1, 2. \tag{1}$$

For each $j = 1, 2$ and each $x \in \mathbf{Z}_+^2$, define the drift $\Delta_j(x) = \sum_y p(x, y)(y_j - x_j)$. It then follows from the homogeneity property that, for each $i = 1, 2$, there exists a function $\Delta_j^{(i)}$ on $\mathbf{Z}_+$ such that

$$\Delta_j(x) = \Delta_j^{(i)}(x_i) \quad \text{for all } x \in \mathcal{X}_{i'}.$$

Define also α_j to be the common value of $\Delta_j(x)$ on the set $\mathcal{X}_1 \cap \mathcal{X}_2$, so that $\alpha_j = \sum_{z \in \mathbf{Z}^2} p^*(z) z_j$.

It also follows from the spatial homogeneity property that we may define a one-dimensional Markov chain $\{X_n^{(j)}\}$ on $\mathbf{Z}_+$ with transition matrix $\{p^{(j)}(\xi, \eta),\ \xi, \eta \in \mathbf{Z}_+\}$ which is also that of the jth coordinate process of the chain $\{X_n\}$ while it remains within the set $\mathcal{X}_{j'}$, that is,

$$p^{(j)}(\xi, \eta) = \sum_{y : y_j = \eta} p(x, y)$$

for any $x \in \mathcal{X}_{j'}$ such that $x_j = \xi$. We assume that this chain is also irreducible.

Note that $\Delta_j^{(j)}(\xi) = \sum_{\eta \geq 0} p^{(j)}(\xi, \eta)(\eta - \xi)$ and so in particular the chain $\{X_n^{(j)}\}$ has drift α_j on the set $\{\xi : \xi > K_j\}$. Hence, from well-known standard results on random walks (see, for example, Durrett [5]) or by elementary Lyapounov function techniques (see Fayolle [6] or Asmussen [1]), the chain $\{X_n^{(j)}\}$ is positive recurrent, null recurrent or transient according as $\alpha_j < 0$, $\alpha_j = 0$ or $\alpha_j > 0$ respectively. In the case $\alpha_j < 0$ let $\pi^{(j)}$ denote the stationary distribution on $\mathbf{Z}_+$ of this chain. Finally, for each j, define

$$\beta_j = \begin{cases} \sum_{\xi \geq 0} \pi^{(j')}(\xi) \Delta_j^{(j')}(\xi), & \text{if } \alpha_{j'} < 0 \\ \alpha_j\ , & \text{if } \alpha_{j'} \geq 0. \end{cases} \tag{2}$$

Our main result, Theorem 4 below (which is well-known under other conditions), is that the two-dimensional chain $(X_n,\ n \geq 0)$ is positive recurrent if $\beta_1 \vee \beta_2 < 0$ and transient if $\beta_1 \vee \beta_2 > 0$; we also say something about the case $\beta_1 \vee \beta_2 = 0$—though it is known that in this case the summary statistics α_j and β_j, $j = 1, 2$, are not always sufficient to classify the chain.

For details on how to determine the one-dimensional equilibrium distributions $\pi^{(j)}$ and hence the parameters β_j, in the case where the corresponding chains have bounded jumps, see Bean *et al* [2].

2 Classification of the Markov chain $\{X_n\}$

Most of the work required for the proof of Theorem 4 is done by the following lemma, which relates to a one-dimensional Markov chain on $\mathbf{Z}_+$, and enables the construction of the appropriate Lyapounov functions for our two-dimensional chain $\{X_n\}$.

Lemma 1 *Let $\hat{P} = \{\hat{p}(\xi, \eta),\ \xi, \eta \in \mathbf{Z}_+\}$ be the transition matrix of an irreducible Markov chain $\{\hat{X}_n\}$ on $\mathbf{Z}_+$ and suppose that, for some $K \geq 0$ and some function $\bar{p}$ on $\mathbf{Z}$, $\hat{p}(\xi, \xi + \zeta) = \bar{p}(\zeta)$ for all $\xi > K$. Suppose further that $\sum_{\zeta \in \mathbf{Z}} \hat{p}(\xi, \xi + \zeta)|\zeta| < \infty$ for all $\xi \in \mathbf{Z}_+$. Define $\alpha = \sum_{\zeta \in \mathbf{Z}} \bar{p}(\zeta)\zeta$. Then the chain is positive recurrent, null recurrent or transient according as $\alpha < 0$, $\alpha = 0$, or $\alpha > 0$. In the case $\alpha < 0$ let π denote its stationary distribution.*

Let the function δ on $\mathbf{Z}_+$ be such that, for some constant $\bar{\delta}$, $\delta(\xi) = \bar{\delta}$ for all $\xi > K$. Define

$$\beta = \begin{cases} \sum_{\xi \geq 0} \pi(\xi)\delta(\xi), & \text{if } \alpha < 0 \\ \bar{\delta}, & \text{if } \alpha \geq 0. \end{cases}$$

(a) Suppose that $\bar{\delta} > 0$. Then, for any $\epsilon < \beta \wedge \bar{\delta}$, there exists a bounded negative function g on $\mathbf{Z}_+$ such that

$$\sum_{\eta \geq 0} \hat{p}(\xi, \eta)(g(\eta) - g(\xi)) + \delta(\xi) \geq \epsilon \quad \text{for all } \xi. \tag{3}$$

(b) Suppose that $\bar{\delta} \leq 0$, that $\alpha < 0$ and that $\beta \geq 0$. Then, for any ϵ such that $0 \leq \epsilon \leq \beta$, there exists a negative function g on $\mathbf{Z}_+$ such that

$$g(\xi) = \lambda\xi + O(1) \quad \text{as } \xi \to \infty, \text{ for some } \lambda \leq 0 \tag{4}$$

and g again satisfies the condition (3).

Proof: As previously remarked, the asserted classification of the chain according to the value of α is well-known.

It is convenient to consider the cases (a) and (b) together. Since the properties (3) and (4) are invariant under adjustment of the function g by an additive constant, it will be sufficient to construct a function g which is bounded above rather than negative.

For $\alpha \leq 0$, let $\hat{\pi}$ denote the stationary measure associated with the chain—unique up to a multiplicative constant, so that in particular we may take $\hat{\pi} = \pi$ for $\alpha < 0$. In the case (a) define a function h on $\mathbf{Z}_+$ by

$$h(\xi) = \begin{cases} \delta(\xi) - \epsilon, & \text{if } \xi \leq k \\ 0, & \text{if } \xi > k \end{cases} \tag{5}$$

where $k \geq K$ and, in the case $\alpha \leq 0$, is additionally such that

$$\sum_{\xi \in \mathbf{Z}_+} \hat{\pi}(\xi) h(\xi) \geq 0 \tag{6}$$

(note that this is possible since $\epsilon < \beta$ if $\alpha < 0$ and $\epsilon < \bar{\delta}$ if $\alpha = 0$). Note also that, for all ξ,

$$h(\xi) \leq \delta(\xi) - \epsilon, \tag{7}$$

since $\epsilon < \bar{\delta}$. In the case (b) define the function h by

$$h(\xi) = \delta(\xi) - \epsilon. \tag{8}$$

Note that in this case the condition (6) is again satisfied, since $\epsilon \leq \beta$.

Associate with the chain $\{\hat{X}_n\}$ a 'reward' process $\{R_n,\ n \geq 0\}$ such that $R_0 = h(\hat{X}_0)$, and, for $n > 0$, $R_n = R_{n-1} + h(\hat{X}_n)$.

In the case (a) where also $\alpha > 0$ define, for all ξ, $g(\xi) = \mathbf{E}(\lim_{n\to\infty} R_n | \hat{X}_0 = \xi)$, where, here and elsewhere, $\mathbf{E}$ denotes expectation; note that it follows easily, from the transience of the chain in this case and the definition (5) of h here, that the function g is well-defined and bounded. Furthermore

$$\sum_{\eta \geq 0} \hat{p}(\xi, \eta)(g(\eta) - g(\xi)) + h(\xi) = 0$$

for all ξ. Thus, using (7), it follows that the function g satisfies the condition (3).

In the case (a) where $\alpha \leq 0$ the chain $\{\hat{X}_n\}$ is recurrent and in the case (b) it is positive recurrent; let $T = \inf\{n \geq 0 \colon \hat{X}_n = 0\}$ and, for all ξ, define $g(\xi) = \mathbf{E}(R_T | \hat{X}_0 = \xi)$. It again follows easily, from the definition (5) or (8) of h, that in the case (a) the function g is bounded (since here a non-zero 'reward' is only earned in states $\xi \leq k$) and that in the case (b) the function g satisfies the condition (4)—with $\lambda = -\alpha^{-1}(\bar{\delta} - \epsilon)$. (This latter result follows by Wald's equation for random walks, together with the finiteness of the first moments of the jumps of the chain.) Define now (in either case)

$$a(\xi) = \sum_{\eta \geq 0} \hat{p}(\xi, \eta)(g(\eta) - g(\xi)) + h(\xi).$$

Then, from the definition of g, $a(\xi) = 0$ for all $\xi > 0$. Define also $T' = \inf\{n \geq 1 \colon \hat{X}_n = 0\}$ and, for each ξ, define $N(\xi)$ to be the number of hits of the chain on the state ξ in the interval $[1, T']$. Then, since $g(0) = h(0)$,

$$\begin{aligned} a(0) &= \sum_{\eta \geq 0} \hat{p}(0,\eta) g(\eta) \\ &= \mathbf{E}(R_{T'} - R_0 | \hat{X}_0 = 0) \\ &= \sum_{\eta \geq 0} \mathbf{E}(N(\eta) | \hat{X}_0 = 0) h(\eta) \\ &= \hat{\pi}(0)^{-1} \sum_{\eta \geq 0} \hat{\pi}(\eta) h(\eta) \\ &\geq 0, \end{aligned}$$

where the third equality above is justified from the definition (5) or (8) of h, and, in the case (b), by the finiteness of $\mathbf{E}(T' | \hat{X}_0 = 0)$ and the dominated convergence theorem, and where the final inequality follows using the condition (6). We thus have that $a(\xi) \geq 0$ for all ξ and so, again using the condition (7) or (8) as appropriate, it follows that the function g satisfies the condition (3). ■

Lemma 2 below is well-known in the case of Markov chains with bounded jumps—see Theorem 2.1.9 of Fayolle, Malyshev and Menshikov [8].

Lemma 2 *Let $\tilde{P} = \{\tilde{p}(x,y),\ x, y \in S\}$ be the transition matrix of a Markov chain on some countable state space S. Suppose that there exists a function f on S and constants K, $M > 0$, $\epsilon > 0$, $d > 1$ such that if $E = \{x \colon f(x) > K\}$ then*

$$E \neq \emptyset, \tag{9}$$

$$\sum_{y \in S} \tilde{p}(x,y)(f(y) - f(x)) \geq \epsilon, \qquad x \in E, \tag{10}$$

$$\sum_{y \in S} \tilde{p}(x,y) |f(y) - f(x)|^d \leq M, \qquad x \in E. \tag{11}$$

Then the chain is transient.

Proof: Observe that, from (9) and (10),

$$\sup_{x \in S} f(x) = \infty. \tag{12}$$

Choose k, a such that $0 < k < d-1$ and $(k+1)/d < a < 1$, and define the function ϕ on $\mathbf{R}$ (the real numbers) by

$$\phi(z) = \begin{cases} 1, & \text{if } z < 1 \\ z^{-k}, & \text{if } z \geq 1. \end{cases}$$

Now suppose $x \in S$ is such that $f(x) > \max(K, 1 + f(x)^a)$. For each $y \in S$ define

$$\theta(x,y) = f(x)^{k+1}[\phi(f(y)) - \phi(f(x))] + k(f(y) - f(x)).$$

Define also $S_x = \{y \in S : |f(y) - f(x)| \le f(x)^a\}$. For z such that $|z - f(x)| \le f(x)^a$, we have that $\phi(z) = z^{-k}$, $\phi'(z) = -kz^{-(k+1)}$ and that $\phi''(z)$ is positive, decreasing and $O(z^{-(k+2)})$ as $z \to \infty$. Hence, by the mean value theorem, for $y \in S_x$,

$$\begin{aligned} |\theta(x,y)| &\le \frac{1}{2} f(x)^{k+1} \phi''(f(x) - f(x)^a)(f(y) - f(x))^2 \\ &\le \frac{1}{2} f(x)^{k+a+1} \phi''(f(x) - f(x)^a)|f(y) - f(x)|, \end{aligned}$$

and hence, since $a < 1$,

$$\begin{aligned} |\sum_{y \in S_x} \tilde{p}(x,y)\theta(x,y)| &\le \frac{1}{2} f(x)^{k+a+1} \phi''(f(x) - f(x)^a) M^{1/d} \\ &\to 0 \end{aligned} \tag{13}$$

as $f(x) \to \infty$.

Further, since $0 < \phi(z) \le 1$ for all z, it follows from the definition of S_x and the condition (11) that

$$\begin{aligned} |\sum_{y \in S \setminus S_x} \tilde{p}(x,y)\theta(x,y)| &\le f(x)^{k+1} \sum_{y \in S \setminus S_x} \tilde{p}(x,y) + k \sum_{y \in S \setminus S_x} \tilde{p}(x,y)|f(y) - f(x)| \\ &\le f(x)^{k+1-ad} M + k f(x)^{-a(d-1)} M \\ &\to 0 \end{aligned} \tag{14}$$

as $f(x) \to \infty$, by the conditions on a and d.

It thus follows from (10), (12), (13) and (14) that there exists K' such that if $E' = \{x : f(x) > K'\}$ then both E' and $S \setminus E'$ are nonempty and, for $x \in E'$,

$$\sum_{y \in S} \tilde{p}(x,y)[\phi(f(y)) - \phi(f(x))] \le 0.$$

The required result now follows from, for example, Theorem 2.3 of Fayolle [6], or Proposition 5.4 of Chapter 1 of Asmussen [1] (there being no necessity for the requirement that the set E_0 of that proposition be finite). ■

We now return to consideration of our original Markov chain $\{X_n\}$. Define, for any function h on $\mathbf{Z}_+^2$, the function Δh on $\mathbf{Z}_+^2$ by

$$\Delta h(x) = \sum_{y \in \mathbf{Z}_+^2} p(x,y)(h(y) - h(x)).$$

We also write Δh^a for Δf where $f = h^a$. Similarly, for $j = 1, 2$ and any function g on $\mathbf{Z}_+$, we define the function $\Delta^{(j)} g$ on $\mathbf{Z}_+$ by

$$\Delta^{(j)} g(\xi) = \sum_{\eta \geq 0} p^{(j)}(\xi, \eta)(g(\eta) - g(\xi)).$$

Lemma 3 below is necessary to enable us to prove Theorem 4, (b), in the case where the condition (1) is only satisfied for some d such that $1 < d < 2$.

Lemma 3 *Suppose that the positive function h on $\mathbf{Z}_+^2$ is such that*

$$h(x) = \lambda_1 x_1 + \lambda_2 x_2 + O(1) \qquad \textit{as } x_1 + x_2 \to \infty,$$

where $\lambda_1 > 0$, $\lambda_2 \geq 0$, and let a be such that $1 < a < d$ —or in the case $d = 2$ we may take $a = 2$. Define

$$\mu(x) = h(x)^{-(a-1)}(\Delta h^a)(x) - a\Delta h(x).$$

Then

$$\sup_{x_2 \geq 0} |\mu(x)| \to 0 \qquad \textit{as } x_1 \to \infty$$

(where, for any function θ on $\mathbf{Z}_+^2$, $\sup_{x_2 \geq 0} \theta(x) = \sup_{x_2 \geq 0} \theta(x_1, x_2)$).

Proof: The result is trivially verified in the case $d \geq 2$, $a = 2$, using the condition (1). In the case of general $d > 1$ and $1 < a < d$, we have $\mu(x) = \sum_{z \in \mathbf{Z}^2} p(x, x+z)\nu(x, z)$ where

$$\nu(x, z) = h(x)\left\{\left(1 + \frac{h(x+z) - h(x)}{h(x)}\right)^a - 1\right\} - a(h(x+z) - h(x)).$$

Hence, by the spatial homogeneity property, for $x_1 > K_1$,

$$\sup_{x_2 > K_2} |\mu(x)| \leq \sum_z p^*(z) \sup_{x_2 > K_2} |\nu(x, z)|, \tag{15}$$

(where the function p^* is as defined in the Introduction). Now $\inf_{x_2 > K_2} h(x) \to \infty$ as $x_1 \to \infty$ and, for fixed z, $h(x+z) - h(x)$ is bounded in x. Thus from, for example, the elementary result that $\lim_{u \to \infty}[u\{(1 + u^{-1})^a - 1\} - a] = 0$, it follows easily that, for fixed z, $\sup_{x_2 > K_2} |\nu(x, z)| \to 0$ as $x_1 \to \infty$. It now follows from (15) and the dominated convergence theorem that

$$\sup_{x_2 > K_2} |\mu(x)| \to 0 \qquad \text{as } x_1 \to \infty,$$

on using the elementary inequality

$$|u\{(1 + u^{-1}t)^a - 1\}| \leq t_0^{d-1}|t| + |t|^d \qquad \text{for } u \geq 1,\ t \geq -u,$$

where t_0 is the strictly positive solution of $(1+t)^a - 1 = t^d$, together with the boundedness condition (1).

Similarly by the spatial homogeneity property, we have that, for x such that x_2 is fixed (in particular $x_2 \le K_2$), $\mu(x) \to 0$ as $x_1 \to \infty$, and so the required result follows. ■

We are now in a position to prove Theorem 4, which classifies the chain $\{X_n\}$.

Theorem 4

(a) If $\beta_1 \vee \beta_2 > 0$, then the Markov chain $\{X_n\}$ is transient.

(b) If $\beta_1 \vee \beta_2 < 0$, then the chain $\{X_n\}$ is positive recurrent.

(c) In the case $\beta_1 \vee \beta_2 = 0$, suppose, without loss of generality, that $\beta_1 = 0$. Then necessarily $\alpha_2 \le 0$. If $\alpha_2 < 0$ and $\alpha_1 \le 0$, then the chain $\{X_n\}$ is not positive recurrent.

Remark: In the remaining cases of (c) (with $\beta_1 = 0$), that is, (i) $\alpha_2 < 0$ and $\alpha_1 > 0$ and (ii) $\alpha_2 = 0$ (in which case, by (2), $\alpha_1 = \beta_1 = 0$ and so also, again by (2), $\beta_2 = 0$), we make no deduction. Indeed in the case (i) it is possible to construct (somewhat pathological) examples of both recurrent and transient chains. Case (ii) is the 'zero drift' case studied by, for example, Fayolle *et al* [7].

Proof of Theorem 4: To prove (a) (transience) suppose, without loss of generality, that $\beta_1 > 0$. Note that if $\alpha_1 \le 0$ then necessarily $\alpha_2 < 0$, for otherwise we would have, from (2), the contradiction that $\alpha_1 = \beta_1$. Hence, for all α_1, we may apply Lemma 1, with $\hat{P} = P^{(2)}$, $\delta = \Delta_1^{(2)}$, and $K = K_2$ (so that the constants α, $\bar{\delta}$ and β of that lemma are given by α_2, α_1 and β_1 respectively), to deduce that there exists $\epsilon > 0$ and a negative function g on $\mathbf{Z}_+$ such that

$$g(\xi) + \lambda\xi = O(1) \quad \text{as } \xi \to \infty, \text{ for some } \lambda \ge 0, \tag{16}$$

(where we may take $\lambda = 0$ in the case $\alpha_1 > 0$) and

$$\Delta^{(2)} g(\xi) + \Delta_1^{(2)}(\xi) \ge \epsilon \quad \text{for all } \xi \ge 0. \tag{17}$$

Now define the function f on $\mathbf{Z}_+^2$ by $f(x) = x_1 + g(x_2)$. Then, from (17), for all $x \in \mathcal{X}_1$,

$$\Delta f(x) = \Delta^{(2)} g(x_2) + \Delta_1^{(2)}(x_2) \ge \epsilon.$$

Further, from (16) and the condition (1), $\sum_y p(x,y)|f(y) - f(x)|^d$ is bounded above in x. Since also g is negative, we may apply Lemma 2, with the parameter K of that lemma equal to K_1, to deduce that the chain is transient.

It is now convenient to deal with the case (c). Suppose that $\beta_1 = 0$, $\beta_2 \leq 0$. That $\alpha_2 \leq 0$ follows since otherwise we would have, by (2), that $\alpha_1 = \beta_1 = 0$ and so, again by (2), $\alpha_2 = \beta_2 \leq 0$, which would again be a contradiction. Suppose now that, additionally, $\alpha_2 < 0$, $\alpha_1 \leq 0$. Then by Lemma 1, case (b), we may choose as above a negative function g on $\mathbf{Z}_+$ which again satisfies the condition (16) and is such that, if the function f is also defined as above, then $\Delta f(x) \geq 0$ for all $x \in \mathcal{X}_1$. (Indeed an examination of the proof of Lemma 2 shows that necessarily $\Delta f(x) = 0$ for all $x \in \mathcal{X}_1$, though we do not require this more careful conclusion here.) Since also $\{x : f(x) > K_1\} \subseteq \mathcal{X}_1$, the required non-positivity now follows by Theorem 11.5.1 of Meyn and Tweedie [15] (on taking the function V of that theorem to be given by $V(x) = 0 \vee f(x)$).

We now suppose that $\beta_1 \vee \beta_2 < 0$ and prove the case (b) (positive recurrence). Note first that this implies that

$$\alpha_1 \wedge \alpha_2 < 0 \tag{18}$$

(for otherwise the definition (2) would imply that $\beta_j = \alpha_j$ for $j = 1, 2$, which is once more a contradiction). It follows that, for each j, either $\alpha_j < 0$ or $\alpha_j \geq 0$, $\alpha_{j'} < 0$. In either case, since also $\beta_j < 0$, we may apply Lemma 1, with $\hat{P} = P^{(j')}$, $\delta = -\Delta_j^{(j')}$, and $K = K_{j'}$ (so that the constants α, $\bar{\delta}$ and β of that lemma are given by $\alpha_{j'}$, $-\alpha_j$ and $-\beta_j$ respectively), to deduce that there exists $\epsilon_j > 0$ and a *positive* function g_j (the negative of the function g of Lemma 1) on $\mathbf{Z}_+$ such that

$$g_j(\xi) = \lambda\xi + O(1) \quad \text{as } \xi \to \infty, \text{ for some } \lambda \geq 0, \tag{19}$$

with $\lambda = 0$ (that is, g_j bounded) in the case $\alpha_j < 0$, and

$$\Delta^{(j')} g_j(\xi) + \Delta_j^{(j')}(\xi) \leq -\epsilon_j \quad \text{for all } \xi \geq 0. \tag{20}$$

Now define the function f_j on $\mathbf{Z}_+^2$ by $f_j(x) = x_j + g_j(x_{j'})$, and let $\epsilon = \epsilon_1 \wedge \epsilon_2$. Then, from (20), for all $x \in \mathcal{X}_j$,

$$\Delta f_j(x) = \Delta^{(j')} g_j(x_{j'}) + \Delta_j^{(j')}(x_{j'}) \leq -\epsilon. \tag{21}$$

Choose a such that $1 < a < d$ (or in the case $d = 2$ we may take $a = 2$). It follows from Lemma 3 and the result (21) that, for each j, there exists $K_j^* \geq K_j \vee 1$ such that, for all x in the set $\mathcal{X}_j^* = \{x \in \mathbf{Z}_+^2 : x_j \geq K_j^*\}$,

$$\begin{aligned}(\Delta f_j^a)(x) &\leq -f_j(x)^{a-1}\epsilon \\ &\leq -x_j^{a-1}\epsilon. \end{aligned} \tag{22}$$

Now, from (18), we may suppose without loss of generality that $\alpha_1 < 0$. Define the function f on $\mathbf{Z}_+^2$ by

$$f(x) = f_1(x)^a + bf_2(x)^a, \qquad \text{where } b > 0.$$

It follows from (22) that

$$\Delta f(x) \le -\epsilon \qquad \text{for all } x \in \mathcal{X}_1^* \cap \mathcal{X}_2^*. \tag{23}$$

Further, if $\alpha_2 < 0$ then the function f_2 is bounded on the set $\mathbf{Z}_+^2 \setminus \mathcal{X}_2^*$, whereas if $\alpha_2 \ge 0$ it follows from the condition (19) and Lemma 3 that there exists M^* such that $|(\Delta f_2^a)(x)| \le M^* x_1^{a-1}$ for all $x \in \mathbf{Z}_+^2 \setminus \mathcal{X}_2^*$. Thus, in either case and using (22), we may choose b sufficiently small so that

$$\Delta f(x) \le -\epsilon \qquad \text{for all } x \in \mathbf{Z}_+^2 \setminus \mathcal{X}_2^* \text{ with } x_1 \text{ sufficiently large.} \tag{24}$$

Finally the function f_1 is bounded on the set $\mathbf{Z}_+^2 \setminus \mathcal{X}_1^*$, and so

$$\Delta f(x) \le -\epsilon \qquad \text{for all } x \in \mathbf{Z}_+^2 \setminus \mathcal{X}_1^* \text{ with } x_2 \text{ sufficiently large.} \tag{25}$$

It follows from the results (23), (24) and (25) that we may choose f_0 sufficiently large so that $\Delta f(x) \le -\epsilon$ for all x with $f(x) > f_0$. Since $\{x \colon f(x) \le f_0\}$ is finite the required positive recurrence now follows by, for example, Proposition 5.3 of Chapter 1 of Asmussen [1] or Theorem 2.2 of Fayolle [6]. ■

3 Application to large loss networks

Consider a loss network, as defined, for example, by Kelly [11], with a finite set J of resources (or links), each member j of which has integer capacity NC_j for some large scale parameter N. Calls are indexed in a finite set R and calls of each type $r \in R$ arrive as a Poisson process of rate $N\kappa_r$. Each such call, if accepted, simultaneously requires an integer A_{jr} units of the capacity of each resource j for the duration of its holding time, which is exponentially distributed with mean $1/\mu_r$. All arrival streams and holding times are independent. We also assume the irreducibility of the Markov process $(\hat{n}_r(\cdot),\ r \in R)$, where $\hat{n}_r(t)$ is the number of calls of type r in progress at time t.

Let $\hat{m}(t) = (\hat{m}_j(t),\ j \in J)$ where $\hat{m}_j(t) = NC_j - \sum_{r \in R} A_{jr}\hat{n}_r(t)$ is the free capacity of resource j at time t. A call of type r arriving at time t is accepted if and only if $\hat{m}(t)$ (the vector of free capacities immediately prior to its arrival) belongs to some acceptance region $\mathcal{A}_r$, which we formally regard as a subset of the compactified space $E = (\mathbf{Z}_+ \cup \{\infty\})^J$. (Of course, the process $\hat{m}(\cdot)$ only takes values in $\mathbf{Z}_+^J$.) We further require that each set $\mathcal{A}_r$ is 'nice' in the sense that its indicator function $I_{\mathcal{A}_r}$ is continuous with respect to the topology on E which is the product of the one-point compactification of $\mathbf{Z}_+$.

For future use, for each $x \in \mathbf{R}_+^R$, define a Markov process $m_x(\cdot)$ with state space E and transitions corresponding to each $r \in R$ given by

$$m_x \to \begin{cases} m_x - A_r & \text{at rate } \kappa_r I_{\mathcal{A}_r}(m_x) \\ m_x + A_r & \text{at rate } \mu_r x_r, \end{cases}$$

where A_r denotes the vector $(A_{jr}, j \in J)$ and $\infty \pm a = \infty$ for any $a \in \mathbf{Z}_+$. The process $m_x(\cdot)$ is reducible and so may possess multiple stationary distributions on E. For each subset $S \subseteq J$, let π_x^S be the stationary distribution, where it exists, which assigns probability one to the set $\{m \in E \colon m_j < \infty \text{ if and only if } j \in S\}$. In particular π_x^J may be regarded as the stationary distribution, if it exists, of the restriction of the process $m_x(\cdot)$ to $\mathbf{Z}_+^J$. It is not difficult to show that the condition on the sets $\mathcal{A}_r$ is equivalent to the requirement that this restricted process possess the property of partial spatial homogeneity which we have considered in detail in this paper for the case $|J| = 2$. Similarly, for each $S \subset J$, π_x^S may be regarded as the stationary distribution of the obvious 'projected' process on $\mathbf{Z}_+^S$. This process again possesses partial spatial homogeneity.

Now let $\hat{x}(t) = (\hat{x}_r(t),\ r \in R)$ where $\hat{x}_r(t) = \hat{n}_r(t)/N$. The process $\hat{m}(\cdot)$ is a function of the process $\hat{x}(\cdot)$ with transitions corresponding to each $r \in R$ given by

$$\hat{m} \to \begin{cases} \hat{m} - A_r & \text{at rate } N\kappa_r I_{\mathcal{A}_r}(\hat{m}) \\ \hat{m} + A_r & \text{at rate } N\mu_r \hat{x}_r. \end{cases}$$

Since the scale parameter N is large, the dynamics of the process $\hat{x}(\cdot)$ might reasonably be expected to be close to that of a deterministic 'fluid' limit. For example, suppose that, at some time t, $\hat{x}(t) = x$ where $\sum_{r \in R} A_{jr} x_r = C_j$ for all j (so that all resources are being fully utilized) and that the restriction to $\mathbf{Z}_+^J$ of the Markov process $m_x(\cdot)$ defined above is positive recurrent. Then over any very short time period $[t, t']$ the process $\hat{x}(\cdot)$ will remain close to x and the process $\hat{m}(\cdot)$ will behave approximately as a fast version of the process $m_x(\cdot)$ restricted to $\mathbf{Z}_+^J$ as above. Thus, for sufficiently large N, the proportion of calls of each type r accepted over this time period should be close to $\pi_x^J(\mathcal{A}_r)$ and the process $\hat{x}_r(\cdot)$ should increase at an approximate rate $\kappa_r \pi_x^J(\mathcal{A}_r) - \mu_r x_r$.

This idea is generalized and made rigorous by Hunt and Kurtz [10], who show that in the limit, as $N \to \infty$ with all else fixed, the dynamics of the process $\hat{x}(\cdot)$ are given by

$$\hat{x}_r(t) = \hat{x}_r(0) + \int_0^t (\kappa_r \pi_u(\mathcal{A}_r) - \mu_r \hat{x}_r(u)) du$$

where, for each t, π_t is *some* stationary distribution on E of the process $m_{\hat{x}(t)}(\cdot)$ and further is such that, for all j,

$$\pi_t\{m\colon m_j = \infty\} = 1 \text{ if } \sum_{r \in R} A_{jr}\hat{x}_r(t) < C_j.$$

Thus the probability measure π_t is some convex combination of those extreme stationary measures $\pi^S_{\hat{x}(t)}$ which exist and are associated with subsets S of J such that $\sum_{r \in R} A_{jr}\hat{x}_r(t) = C_j$ for all $j \in S$.

However, the theory of Hunt and Kurtz does not go very much further in identifying the appropriate convex combination. Yet this is necessary in order to determine the limiting dynamics of the process $\hat{x}(\cdot)$. In particular suppose that, under appropriate conditions, π_t can be shown to depend on t only through $\hat{x}(t)$. Then, for each x, there exists a stationary distribution π'_x of the process $m_x(\cdot)$ such that, for all t, $\pi_t = \pi'_{\hat{x}(t)}$. Suppose further that the *fixed point* equations

$$\kappa_r \pi'_x(\mathcal{A}_r) = \mu_r x_r, \quad r \in R,$$

have a unique solution $\bar{x}$ to which all trajectories of the limiting process $\hat{x}(\cdot)$ converge. Then, for each r, $\pi'_{\bar{x}}(\mathcal{A}_r)$ can be shown to be the limit, as $N \to \infty$, of the equilibrium acceptance probability for calls of type r (see Bean *et all* [3]).

Various, rather ad-hoc, techniques exist for identifying π_t in particular networks. Sometimes the theory of Hunt and Kurt may be refined to exclude further subsets S of J from contributing to the above convex combination, and the simple observation that the process $\hat{x}(\cdot)$ must remain positive is often sufficient to exclude yet further subsets. However, it is in general important to know whether, for any given S and x, the stationary distribution π^S_x exists. As explained above, this involves the determination of whether an associated (and partially spatially homogeneous) Markov process on $\mathbf{Z}^S_+$ is positive recurrent.

Using the results contained in Section 2 of this paper, Bean *et all* [3] show how to identify π_t completely for two-resource networks. However, work remains to be done on identifying π_t in the more general case. Higher-dimensional generalizations of the techniques of this paper may be expected to play a part here.

Acknowledgements

The author is grateful to the following for comments, suggestions, helpful conversations and corrections: Alexandre Borovkov, Terence Chan, Richard Gibbens, Jennie Hansen, Paul Hulse, Vadim Malyshev Hanspeter Schmidli and the referees.

References

[1] S. Asmussen, *Applied Probability and Queues* (Wiley, Chichester, 1987).

[2] N.G. Bean, R.J. Gibbens and S. Zachary, Analysis of large single resource loss systems under heavy traffic, with applications to integrated networks, Adv. Appl. Prob. 27 (1995) 273-292.

[3] N.G. Bean, R.J. Gibbens and S. Zachary, Dynamic and equilibrium behaviour of controlled loss networks, Research Report #94-31, Statistical Laboratory, The University of Cambridge, 16 Mill Lane, Cambridge, CB2 1SB, U.K (1994).

[4] A.A. Borovkov, Lyapunov functions and ergodicity of multidimensional Markov chains, Theory of Probability and its Applications 36 (1991) 1–18.

[5] R. Durrett, *Probability: Theory and Examples* (Wadsworth, California, 1991).

[6] G. Fayolle, On random walks arising in queueing systems: ergodicity and transience via quadratic forms as Lyapounov functions—part I, Queueing Systems 5 (1989) 167–184.

[7] G. Fayolle, V.A. Malyshev and M.V. Menshikov, Random walks in a quarter plane with zero drifts. I. ergodicity and null recurrence, Ann. Inst. Henri Poincaré Probabilites et Statistiques 28 (1992) 179–194.

[8] G. Fayolle, V.A. Malyshev and M.V. Menshikov, *Topics in the Constructive Theory of Countable Markov Chains* (Cambridge University Press, 1995).

[9] P.J. Hunt, Pathological behaviour in loss networks, Research Report #92-17, Statistical Laboratory, The University of Cambridge, 16 Mill Lane, Cambridge, CB2 1SB, U.K (1992).

[10] P.J. Hunt and T.G. Kurtz, Large loss networks, Stochastic Processes and their Applications 53 (1994) 363–378.

[11] F.P. Kelly, Loss networks, Ann. Appl. Prob. 1 (1991) 319–378.

[12] V.A. Malyshev, Classification of two-dimensional positive random walks and almost linear semi-martingales, Soviet Math. Dokl. 13 (1972) 136–139.

[13] V.A. Malyshev, Networks and dynamical systems, Adv. Appl. Prob. 25 (1993) 140–175.

[14] V.A. Malyshev and M.V. Menshikov, Ergodicity, continuity and analyticity of countable Markov chains, Trans. Moscow Math. Soc. 39 (1979) 1–48

[15] S.P. Meyn and R.L. Tweedie, *Markov chains and stochastic stability* (Springer-Verlag, London, 1993).

[16] W.A. Rosenkrantz, Ergodicity conditions for two-dimensional Markov chains on the positive quadrant, Probab. Th. Rel. Fields 83 (1989) 309–319.