Regularity Assumptions and Length Scales for the Navier-Stokes Equations

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by

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This Thesis is dedicated to

my parents

John F. and Jeanne A. Malham

and also

Dr Mike J. Sutcliffe

Abstract

This Thesis presents an analytic examination of the classical, incompressible Navier-Stokes equations using the techniques of elementary functional analysis to establish new results in the theory of attractors and length scales. This approach, initiated by Leray (1934) and expounded in the last decade by Constantin, Foias, Temam and others, provides us with the necessary tools in order to fully understand and appreciate these equations from the mathematical point of view.

Firstly, the more abstract ideas/notions related to a general evolution equation are outlined (including absorbing sets and attractors) and our objectives are established within this general framework. In addition, a more detailed (and rigorous) discussion specific to the structure and properties of Sobolev spaces as well as a full account of the Gagliardo-Nirenberg inequality are given. Both of these, with some new extended results, are essential to the work which follows.

Secondly, the theory outlined above is now specialized to the Navier-Stokes equations and also corresponding well-known results/properties are shown – with a few alternative proofs provided. Importantly, the 'Ladder Theorem' is introduced.

Thirdly, we generalize the analytic structure of the ladder to a 'Lattice Theorem' in order to look for the minimum assumptions necessary to show the existence of an attractor for the $d = 3$ Navier-Stokes equations consisting of C^∞ functions. The lattice approach reproduces the classical $L^{3+\epsilon}$ ($\epsilon > 0$) result due to Serrin, and also provides us with an alternative assumption for C^∞ regularity of the attractor: that of assuming $\|\mathcal{P}\|_{2(1+\delta)}$ (for any δ such that $1/5 < \delta \leq 1$) is uniformly bounded, where $\mathcal{P}$ is the pressure field.

Fourthly, we define a natural set of length scales which are more sensitive to intermittent fluctuations (and so much shorter) than the Kolmogorov length and are therefore a more realistic minimum length scale. We determine how these new scales compare with those that can be derived from the attractor dimension or the number of determining modes.

Lastly, a similiar analytic approach is initiated for the MHD equations.

Contents

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Chapter 1

Introduction

The study and research of this thesis concerns the aspects of regularity (smoothness) of the solution of the incompressible Navier-Stokes equations and also a natural definition for a minimum length scale for the fluid flow. The incompressible Navier-Stokes equations are a set of nonlinear Partial Differential Equations (PDEs) which describe the (Eulerian formulation) evolution of the velocity field $u = u(x,t)$ at a point $x = (x_1, \ldots, x_d) \in \Omega \subseteq \mathbb{R}^d$ and time $t \in \mathbb{R}$, where $d = 2$ or 3 is the space dimension. For a Newtonian (i.e. with 'stress' linearly proportional to 'rate of strain') and isotropic fluid, with $\mathcal{P} = \mathcal{P}(x,t)$ as the pressure, we normally write them as

$$\partial_t u + u \cdot \nabla u = \nu \, \Delta u - \nabla \mathcal{P} + f$$
$$\text{div } u = 0$$

where f is the 'external forcing' and ν is the 'viscosity' (dissipation coefficient). For their simple derivation see for instance, Batchelor [8], Tritton [89] or any other elementary fluid mechanics textbook.

The examination of these equations provided here is purely analytical and at all times from a strictly mathematical (as opposed to physical) point of view.

1.1 Existence, Uniqueness, Regularity and Attractors

A solution to the general set of equations, under realistic initial and boundary conditions, has not been found in $d = 2$ or 3. Solutions which have been found are highly specialized meaning that they are only valid under severe restrictions to the general problem – for example, we can solve the 'Stokes problem', but this is a stationary, linear version/simplification of the Navier-Stokes equations.

So, for the evolutionary, nonlinear incompressible Navier-Stokes equations ($d = 2, 3$) we must address the questions:

- Does a solution exist and on what interval of time?
- Is that solution unique?

In order to (at least partially) answer these questions, it is necessary to apply the theory and ideas of elementary functional analysis and in particular, determine the various function spaces in which the solution can be proved to lie. Subsequently, a third important question is very relevant:

- Is that solution regular (i.e. smooth) ?

Leray [57] (1934) proved that "weak" solutions exist for the incompressible Navier-Stokes equations ($d = 2, 3$). Such solutions allow singularities in the velocity or vorticity (curl of the velocity) fields. The functional-analytic approach he initiated has in recent years been extended by Peter Constantin, Ciprian Foiaš, Roger Temam and many others.

In two dimensions, we can prove the existence of "strong" solutions (which we can subsequently show as being regular and smooth) for the incompressible Navier-Stokes initial value problem on all finite intervals of time $[0, T]$, provided the initial data are sufficiently smooth.

In three dimensions, such "strong" solutions are only known to exist on some finite interval $[0, T^*]$, T^* depending on the initial conditions. For this case, we might ask ourselves the questions:

- What are the assumptions we must make in order to prove the existence of strong solutions for all time? Our aim would be to try to reduce these assumptions (eventually) to some information which is already known to be true for the Navier-Stokes equations.

- Having shown, under these assumptions, the existence of strong solutions for all finite-time intervals $[0, T]$, and so consequently (as we will see later) these solutions will be regular, how does the solution behave as $t \to \infty$? Does the solution tend to some complicated attracting set? We might expect, by analogy with evolutionary, nonlinear Ordinary Differential Equations (ODEs), the solution (in certain parameter regimes) to be sensitive to initial conditions and so the long-time behaviour of the solution could appear chaotic.

The questions of finite-time as well as long-time ($t \to \infty$) regularity are obviously related to giving a qualitative as well as quantative description of the solution (which we are unable to solve analytically).

Chapters 2 and 3 of this thesis outline the functional-analytic techniques and framework we need in order to tackle the questions posed above, which we try to answer in Chapters 4 and 5.

In Chapter 2, we introduce the semi-group of operators, the concepts of absorbing sets and attractors, the theory of linear operators and discuss techniques for showing the existence of a 'Global Attractor'.

The essentials of functional analysis are provided in the first part of Chapter 3. However, the main result of Chapter 3 is the proof of the Gagliardo-Nirenberg inequality in terms of semi-norms for mean-zero and space-periodic functions which we will need in the following chapters. The proof is non-trivial, and some new results are presented.

In Chapter 4, we provide a detailed account of the exact setting in which we are to study the incompressible Navier-Stokes equations and outline the main results proved for them so far. We also introduce the 'Ladder Theorem' (an infinite set of a-priori estimates) and we examine the assumptions necessary to prove (for $d = 3$) regular solutions on all finite intervals of time as well as in the attractor. The concept of a ladder theorem was first introduced by Bartuccelli, Constantin, Doering, Gibbon and Gisselfält [3] for the Complex Ginzburg-Landau equation, and later applied to the Navier-Stokes equations by Bartuccelli, Doering and Gibbon [4].

A generalization of the 'Ladder Theorem' which we shall call the 'Lattice Theorem', is introduced in Chapter 5. We attempt to relax the minimum assumptions we must make in order to prove regular solutions in three dimensions for all time, but instead, we reproduce the well-known result of Serrin [79] – which essentially says that provided we assume the velocity field to be uniformly bounded in a particular function space, then we can prove the existence of strong, regular solutions. However, we are able to provide 'alternative' assumptions via the pressure field and open a new avenue of attack via some further sharper attractor estimates.

At this point the examination of the existence of regular solutions ends.

1.2 Turbulence, Relevant Modes and Minimum Length Scales

In the second part of this thesis, we address the notion of fluid turbulence.

The incompressible Navier-Stokes equations are the simplest general equations describing incompressible fluid flow such as water or oil flow. With appropriate initial and boundary conditions, they apply to a large variety of flow situations which are extended further when we couple them with the field equations describing other naturally occurring physical phenomena – for example convection systems, magneto-hydrodynamic flows, thermohydraulic situations, in meterological applications and of course aeronautical flows.

Turbulent behaviour (highly irregular variations in the velocity field) is obvious in a large number of such flows – for example in fluid mixing, convection and flows around ships.

Such behaviour is more prevalent in low viscosity as opposed to high viscosity (respectively, high and low Reynolds number) flows. Low Reynolds number flows are said to be laminar i.e. they exhibit regular and predictable variations in space and time.

Chapter 6 addresses the problem of defining minimum length scales for turbulent flow. Firstly, we review the existing theories on turbulence and in particular the scaling arguments of Kolmogorov which give rise to the Kolmogorov dissipation scale. After discussing the limitations of this scaling theory and some more recent alternative theories that have been suggested, we give a definition for a length scale via the number of relevant (Fourier) modes, which are determined via a 'Fourier splitting' argument. We show how the classical Kolmogorov dissipation scale is only relevant for laminar (or quiescent) flows while the length scale we have defined includes a term which might account for observed excursions from Kolmogorov scaling theory for strongly turbulent flows. We then compare our length scale estimates with estimates for length scales provided via the attractor dimension and the number of determining modes. Such a minimum scale is very important to numerical analysts who are trying to simulate 'real' flows computationally and are therefore interested in 'resolving' minimum relevant scales for accurate numerical schemes.

1.3 Magneto-Hydrodynamics

In the last chapter we derive the general form of the Magneto-Hydrodynamics (MHD) equations and we show how we can apply similiar techniques to them. We derive some a-priori estimates which include an 'MHD Ladder Theorem' for smooth solutions, and set the path for a further in-depth investigation of these equations.

Chapter 2

The General Framework

In this chapter we introduce the abstract theory necessary for dealing with a general evolution equation. We assume that we have shown the existence and uniqueness of the solution, which needs to be proved for each Partial Differential Equation separately. With our ultimate goal being to determine the long-time behaviour of the solution, we outline the concepts of absorbing sets, attractors and their regularity, which we will eventually apply to the Navier-Stokes equations.

I have based a lot of the material of this chapter on Temam [88].

2.1 The General Evolution Equation

Consider the general form of an evolution equation:

$$\begin{aligned} \frac{du(t)}{dt} &= F(u(t)) \\ u(0) &= u_0 \end{aligned} \tag{2.1}$$

where the solution $u = u(t)$ belongs to a function space H (the time-like variable t varying continuously on some interval $\subseteq \mathbb{R}$) and F is a mapping from H into itself.

If the Initial Value Problem (IVP) is well-posed, then we know $u(t) \in H$, $\forall\, t > 0$ and can give a complete description of the dynamical system at any time – our interest is specific to the long-time ($t \longrightarrow \infty$) behaviour of the solution $u(t)$.

Usually, H will be a function space defined on $\Omega \subseteq \mathbb{R}^d$ (and in most instances a Banach or Hilbert space), however at this point, we need only assume H to be an arbitrary metric space (with metric $\rho(\cdot,\cdot)$).

2.2 Semi-groups of Operators

Under the assumption that we know the solution $u = u(t)$ of such a dynamical system, i.e. the initial value problem (2.1) is well-posed (see later theory for non-well-posed problems), we can then formulate the evolution of the system by $u(t) = S(t)u_0$, where $\{S(t), t \geq 0\}$ are a semi-group of operators (on H) which satisfy the usual semi-group properties:

$$S(t+s) = S(t) \cdot S(s) \quad \forall\, s, t \geq 0\ , \tag{2.2}$$

$$S(0) = I_D \quad (\text{in } H) \tag{2.3}$$

and we will also assume

$$S(t) : H \to H \text{ is a continuous operator for all } t \geq 0 \,. \tag{2.4}$$

Some Basic Definitions and Results:

1. If the $S(t)$ are injective then (2.1) have the "backward uniqueness" property.

2. The orbit (or trajectory) starting at $u_0 = \left\{ \bigcup_{t \geq 0} S(t)u_0 \right\}$ = positive orbit.

3. The ω-limit set of $u_0 \in H$ is

$$\omega(u_0) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)u_0} \,, \tag{2.5}$$

   and similarly, the ω-limit set of $\mathcal{A} \subseteq H$ is

$$\omega(\mathcal{A}) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)\mathcal{A}} \,. \tag{2.6}$$

   where the closures are in H.

4. A set $X \subseteq H$ is positively invariant for the semi-group $S(t)$ if $S(t)X \subseteq X, \quad \forall\, t \geq 0$; negatively invariant if $S(t)X \supseteq X, \quad \forall\, t \geq 0$; and a (functional) invariant set for the semi-group $S(t)$ if $S(t)X = X, \quad \forall\, t \geq 0$.

2.3 Absorbing Sets and Attractors

If $\rho(x, A) = \inf_{y \in A} \rho(x, y)$ (where $\rho(\cdot, \cdot)$ is the metric for H) we define:

Definition 2.3.1 *If $\mathcal{A} \subseteq H$ satisfies*

1. $S(t)\mathcal{A} = \mathcal{A}, \quad \forall\, t \geq 0$ (i.e. it is an invariant set)

2. There exists an open neighbourhood $\mathcal{U}$ of $\mathcal{A}$, such that:

$$\forall\, u_0 \in \mathcal{U}, \quad \rho(S(t)u_0, \mathcal{A}) \to 0 \quad as \quad t \to \infty \tag{2.7}$$

then $\mathcal{A}$ is said to be an attractor *(set) - "$\mathcal{A}$ attracts the points of $\mathcal{U}$".*

Definition 2.3.2 *We say $\mathcal{A} \subseteq H$* uniformly attracts *the points of a set $\mathcal{B} \subseteq \mathcal{U}$ if*

$$\sup_{u_0 \in \mathcal{B}} \rho(S(t)u_0, \mathcal{A}) \to 0 \quad as \quad t \to \infty \tag{2.8}$$

"$\mathcal{A}$ uniformly attracts $\mathcal{B}$" or simply, "$\mathcal{A}$ attracts $\mathcal{B}$".

Definition 2.3.3 *The semi-group* $\{S(t) : t \geq 0\}$ *possesses a global (or universal, maximal) attractor* $\mathcal{A} \subseteq H$, *if* $\mathcal{A}$ *is a compact attractor that attracts the bounded sets of* H.

A global attractor is unique, and maximal for its inclusion in the set of bounded attractors and bounded invariant sets.

Definition 2.3.4 *Suppose* $\mathcal{B} \subseteq H$ *and* $\mathcal{U} \supseteq \mathcal{B}$ *is open. Then* $\mathcal{B}$ *is an* absorbing set *in* $\mathcal{U}$ *if,*

$$\forall\ \mathcal{B}_0 \subseteq \mathcal{U},\ \mathcal{B}_0 \text{ bounded, } \exists\ T_1(\mathcal{B}_0) \text{ such that } \forall\ t > T_1(\mathcal{B}_0),\ S(t)\mathcal{B}_0 \subseteq \mathcal{B} \tag{2.9}$$

the orbit of any bounded set of $\mathcal{U}$ *enters* $\mathcal{B}$ *after a certain time ("*$\mathcal{B}$ *absorbs the bounded sets of* $\mathcal{U}$*").*

A sufficient condition for the existence of an absorbing set is the existence of a global attractor. However, we are interested in the reverse implication, and in order to prove the existence of a global attractor when the existence of an absorbing set is known, we need to make the following extra assumption about the semi-group $S(t)$:

Definition 2.3.5 *The operators* $S(t)$ *are* uniformly compact *for* t *large if*

$$\forall\ \mathcal{B} \text{ bounded, } \exists\ T_0(\mathcal{B}) \text{ such that } \bigcup_{t \geq T_0} S(t)\mathcal{B} \text{ is relatively compact in } H\ . \tag{2.10}$$

Temam [88] thus provides us with the following theorem:

Theorem 2.3.1 *Suppose* H *is a metric space, and the operators* $S(t)$ *are given and satisfy (2.2)–(2.4) and (2.10). Also, assume there exists a bounded set* $\mathcal{B} \subseteq \mathcal{U}$ *(*$\mathcal{U}$ *open) such that* $\mathcal{B}$ *is absorbing in* $\mathcal{U}$.

Then $\mathcal{A} = \omega(\mathcal{B})$ *is a compact attractor (which attracts the bounded sets of* $\mathcal{U}$*), and it is maximal for the inclusion relation among the bounded attractors in* $\mathcal{U}$.

Further, if H *is a Banach space and* $\mathcal{U}$ *convex and connected, then* $\mathcal{A}$ *is also connected.*

2.4 Operators and Imbeddings

Let f be an operator from the normed space X into the normed space Y.

Definition 2.4.1 f *is called* compact *if* $f(A)$ *is pre-compact (*$\Rightarrow$ $\overline{f(A)}$ *is compact) in* Y *whenever* A *is bounded in* X.

(A bounded set in a normed space is one which is contained in the ball $B(0, R)$ *(centre* 0, *radius* R*) for some* R*.)*

f *is said to be* bounded, *if* $f(A)$ *is bounded in* Y *whenever* A *is bounded in* X.

If f *is continuous and compact, then* f *is said to be* completely continuous.

Remark: We can immediately deduce: *Every compact operator is bounded; and since every bounded linear operator is continuous, every compact linear operator is completely continuous.*

Definition 2.4.2 *If*

1. *The normed space X is a vector subspace of the normed space Y,*
2. *The operator $I_D : X \to Y$ such that $I_D x = x \ \forall\, x \in X$ is continuous*

then X is said to be imbedded *in Y (we write $X \to Y$).*

Remark: Since the identity operator outlined in 2 is linear, then 2 is equivalent to the assertion: there exists a constant K such that

$$||I_D x||_Y \leq K\,||x||_X \qquad \forall\, x \in X. \tag{2.11}$$

If I_D (imbedding operator) is compact, X is *compactly imbedded* in Y.

Definition 2.4.3 *A subset A of a normed space X is said to be* dense *in X if each $x \in X$ is a limit point of A.*

Remark: Suppose the normed spaces X, Y and Z are such that $X \subseteq Y \subseteq Z$. Then clearly, if we know the inclusion $X \subseteq Z$ is dense, then the inclusion $Y \subseteq Z$ is also dense.

2.5 Linear Operators

2.5.1 Linear Operators and Bi-linear Forms

We assume V to be an arbitrary Hilbert space with scalar product $(\cdot,\cdot)_V \equiv ((\cdot,\cdot))$ and corresponding norm $||\cdot||_V$. Let V' signify the dual of V.

We introduce another Hilbert space H, with dual H', scalar product $(\cdot,\cdot)_H$ and norm $||\cdot||_H$, such that $V \subseteq H$, where the imbedding is dense and continuous.

i.e. $I_D : V \to H$ is a continuous injection.

From the theory of dual spaces, we can infer that the adjoint operator, $I'_D : H' \to V'$, is injective, and $I'_D(H') = W$ densely imbedded in V'.

The Riesz Representation Theorem allows us to identify H and H' so that we can write

$$V \subseteq H \equiv H' \subseteq V' \tag{2.12}$$

where the injections are continuous and each space is dense in the following.

The inclusions (2.12) imply that the scalar product of $f \in H$ and $u \in V$ in H is identical to the scalar product of f and u in the duality between V' and V (denoted $\langle\cdot,\cdot\rangle$),

$$\langle f,u\rangle = (f,u)_H \quad \forall\, f \in H,\ u \in V \tag{2.13}$$

Let us, for the moment concentrate on the space V.

Suppose we are given $a : V \times V \to \mathbb{R}$, a bi-linear continuous form (hence it is linear, continuous with respect to each of its dependents on V).

So for all $u \in V$, the mapping $v \mapsto a(u,v)$ is linear, continuous from V into $\mathbb{R}$ and identifies an element $\ell_u \in V'$.

If we denote by A the mapping (from V into V')

$$u \mapsto \ell_u \tag{2.14}$$

then $A \in \mathcal{L}(V, V')$, i.e. a linear, continuous operator from V into V'. Further, there exists $M < \infty$ such that (since a is a bi-linear continuous form)

$$|a(u,v)| \leq M\|u\|_V\|v\|_V \quad \forall\, u, v \in V \tag{2.15}$$

$$\Rightarrow \|A\|_{\mathcal{L}(V,V')} \leq M \; . \tag{2.16}$$

Hence to each bi-linear, continuous form a on $V \times V$ we can associate a linear, continuous operator $A : V \to V'$.

Also, given a linear, continuous operator $A \in \mathcal{L}(V, V')$, set

$$a(u,v) = \langle Au, v\rangle \quad \forall\, u, v \in V \tag{2.17}$$

i.e. we can associate A with a bi-linear, continuous form a on $V \times V$.

Definition 2.5.1 *The bi-linear form a is* coercive *if*

$$\exists\, \alpha > 0 \text{ such that } a(u,u) \geq \alpha\|u\|_V^2 \quad \forall\, u \in V \; . \tag{2.18}$$

Theorem 2.5.1 (Lax-Milgram) *If a is a bi-linear, continuous, coercive form on $V \times V$, then A is an isomorphism from V onto V'.*

In other words (under the assumptions outlined), for each linear continuous form $\ell_u \in V'$, there exists a unique $u \in V$ such that $a(u,v) = \langle \ell_u, v\rangle$, $\forall\, v \in V$.

Noting the inclusions (2.12) above, and assuming the form a to be continuous and coercive, it is natural to define the domain of A in H as

$$D(A) = \{u \in V : \; Au \in H\} \subseteq V \; . \tag{2.19}$$

We can now consider A as a linear, *unbounded* operator from $D(A)$ to H, which (from (2.17) and (2.18)) is also *strictly* positive:

$$\langle Au, u\rangle = a(u,u) \geq \alpha\|u\|_V^2 > 0 \quad \forall\, u \neq 0 \; . \tag{2.20}$$

$D(A)$ is a Hilbert space for the norm

$$\|u\|_{D(A)} := \|Au\|_H \tag{2.21}$$

We also see that A is an isomorphism from $D(A)$ onto H.

2.5.2 Spectral Representation of A

Consider the following two further assumptions about A:

1. A is self-adjoint: If the form a is symmetric, i.e.

$$\langle Au, v\rangle = \langle Av, u\rangle = a(u,v) \quad \forall\, u, v \in V \tag{2.22}$$

then, clearly A is self-adjoint (from V to V' and as an unbounded operator in H). From (2.22) we also see that A^{-1} is also self-adjoint (in H).

This assumption alone (which implies A is a strictly positive, self-adjoint, unbounded operator in H) means that we can apply spectral theory and so consider operators of the form A^s ($s \in \mathbb{R}$).

2. The injection of V onto H is compact: This allows us to consider A^{-1} as a self adjoint, compact operator in H, indeed:

Let $\mathcal{B} \subseteq H$ be a bounded set. We see that

$$||A^{-1}u||_{D(A)} = ||AA^{-1}u||_H = ||u||_H \quad \forall\, u \in H\ . \tag{2.23}$$

Hence $A^{-1}\mathcal{B}$ is bounded in $D(A)$.

Further we see that

$$\begin{aligned} \alpha||u||_V^2 \le a(u,u) = \langle Au, u\rangle &\le ||Au||_{V'}||u||_V \\ &\le c_1||Au||_H||u||_V \\ &= c_1||u||_{D(A)}||u||_V \end{aligned} \tag{2.24}$$

where c_1 is the (operator) norm of the injection of V into H (= norm of the injection of H into V', by duality) i.e.

$$||u||_H \le c_1\, ||u||_V \quad \forall\, u \in V\ . \tag{2.25}$$

The sequence (2.24) implies that

$$||u||_V \le \frac{c_1}{\alpha}\, ||u||_{D(A)} \tag{2.26}$$

and this combined with (2.23) and (2.25) give

$$||A^{-1}u||_V \le \frac{c_1}{\alpha}\, ||A^{-1}u||_{D(A)} = \frac{c_1}{\alpha}||u||_H \quad \forall\, u \in H \tag{2.27}$$

and so $A^{-1}\mathcal{B}$ is bounded in V. Since the injection V onto H is compact it follows that $A^{-1}\mathcal{B}$ is relatively compact in H.

Note: (2.24) and (2.25) imply $||u||_H \le c_1^2/\alpha\, ||Au||_H$ and so the norm $\{||Au||_H^2 + ||u||_H^2\}^{1/2}$ is equivalent to the norm $||Au||_H$ on $D(A)$.

Thus, with these two assumptions, A^{-1} is a self-adjoint, compact operator in H and consequently, by the spectral theory for such operators in a Hilbert space, there exists an orthonormal basis for H, $\{w_j \in D(A), j \in \mathbb{N}\}$ which are the eigenvectors of A:

$$A^{-1}w_j = (\lambda_j)^{-1}w_j \quad \forall\, j \in \mathbb{N} \tag{2.28}$$

where the $(\lambda_j)^{-1}$ are a decreasing sequence (in j) tending to zero, and therefore

$$Aw_j = \lambda_j w_j \quad \forall\, j \in \mathbb{N} \tag{2.29}$$

$$0 < \lambda_1 \leq \lambda_2 \leq \dots$$

where the $\lambda_j \to \infty$ as $j \to \infty$.

See for instance [92], [75], [21, 22], [88] and [16].

2.5.3 Powers of the Operator A

For the moment we assume the first point of the previous section – i.e. we assume a is symmetric and therefore A is a strictly positive, self-adjoint, unbounded operator in H.

For $s \in \mathbb{R}, s > 0$: We can define A^s by spectral theory (see Yosida [92]), where A^s is a strictly positive, self-adjoint, unbounded, injective operator in H whose domain is $D(A^s) \subseteq H$ (the inclusion being dense in H).

Further, $D(A^s)$ is a Hilbert space for the scalar product and norm:

$$(u, v)_{D(A^s)} = (A^s u, A^s v) \ , \tag{2.30}$$

$$\|u\|_{D(A^s)} = \{(u, u)_{D(A^s)}\}^{1/2} \ . \tag{2.31}$$

Also, $A^s : D(A^s) \to H$ is an isomorphism.

We define $D(A^{-s})$ to be the dual of $D(A^s)$, $(s > 0)$. With $(u, v)_{D(A^{-s})}$ defined in the obvious manner, we can deduce the following:

$$D(A^{s_1}) \subseteq D(A^{s_2}) \quad \forall\, s_1, s_2 \in \mathbb{R}, \quad s_1 \leq s_2 \tag{2.32}$$

where the injections are continuous, each space dense in the following one, and further, $A^{s_2 - s_1}$ is an isomorphism of $D(A^{s_1})$ onto $D(A^{s_2})$, $\forall\, s_1, s_2 \in \mathbb{R}$, with $s_1 < s_2$.

Some examples are

- $D(A^0) = H \equiv H'$
- $D(A^{1/2}) = V$
- $D(A^1) = D(A)$

If in addition, the injection from V into H is compact, we can use the spectral representation of A to characterize the operators A^s:

$$\forall\, u \in D(A^s), s \in \mathbb{R}, \qquad A^s u = \sum_{j=1}^{\infty} \lambda_j^s (u, w_j)_H w_j \tag{2.33}$$

and the scalar product and norm of $D(A^s)$ take the form

$$(u, v)_{D(A^s)} = \sum_{j=1}^{\infty} \lambda_j^{2s} (u, w_j)_H (v, w_j)_H \ , \tag{2.34}$$

$$\|u\|_{D(A^s)} = \left(\sum_{j=1}^{\infty} \lambda_j^{2s} |(u, w_j)_H|^2 \right)^{1/2} \tag{2.35}$$

and further, we can characterize $D(A^s)$ in the obvious manner,

$$D(A^s) = \begin{cases} \{u \in H : \|u\|_{D(A^s)} < \infty\} & s \geq 0 \\ \{\text{completion of } H \text{ wrt the norm } \|u\|_{D(A^s)}\} & s < 0 \end{cases}$$

In the compact case, the imbedding of $D(A^s)$ into $D(A^{s-\epsilon})$ is compact, for all $s \in \mathbb{R}$, $\epsilon > 0$.

2.6 The Existence of an Attractor and its Regularity

Suppose we have established the existence and uniqueness of a solution to the problem (2.1) and we are therefore able to define the semi-group $S(t) : u_0 \in H \mapsto u(t) \in H$ for the system.

Let us further assume that we have shown the continuity of the semi-group $S(t)$ and we can therefore appeal to some of the theory of Sections 2.2 and 2.3.

We wish to characterize the behaviour of the solutions, in particular their long-time behaviour, and so we must essentially address the following points:

1. The non-linear stability of the problem: Do the solutions remain bounded as $t \to \infty$?
2. Can we show the existence of absorbing sets in particular function spaces? This usually amounts to proving a-priori estimates for the PDE (these are time-differential inequalities for function space norms of the solution and we discuss them in Chapters 4 and 5).
3. Can we show the existence of a global attractor? This means we need to show that the semi-group is uniformly compact for large t (see (2.10)).
4. Supposing we have shown the existence of an attractor $\mathcal{A} \subseteq H$, can we determine if the attractor is actually included in a subspace of H of more regular functions?

Now, let us suppose that we can show points 1 and 2 for our PDE, and in particular, $S(t) : H \to V$ is continuous and that there exists absorbing sets $\mathcal{B}_H$ in H and $\mathcal{B}_V$ in V (where V and H are as in Sections 2.2 and 2.3 with V compactly imbedded in H).

Further, let us assume that we can show that if $\mathcal{B} \subseteq H$ is any (arbitrary) given bounded set, then $S(t)\mathcal{B}$ is included in $\mathcal{B}_V$ (in particular $\mathcal{B}_H$ is so included) after a certain time (this result will typically follow from the a-priori estimates we present in Chapters 4 and 5) i.e.

$$\exists\ T_0 \text{ such that } \forall\ t > T_0 \quad S(t)\mathcal{B} \subseteq \mathcal{B}_V \tag{2.36}$$

and so

$$\bigcup_{t \geq T_0} S(t)\mathcal{B} \subseteq \mathcal{B}_V \quad \text{is bounded in } V \tag{2.37}$$

and further, since V is compactly imbedded in H,

$$\bigcup_{t \geq T_0} S(t)\mathcal{B} \text{ is relatively compact in } H \tag{2.38}$$

and so the semi-group $\{S(t), t \geq 0\}$ is uniformly compact.

We can therefore apply Theorem 2.3.1 with $\mathcal{U} = H$.

So, in summary, *the existence of an attractor in H follows when the existence of an absorbing set in a space V (compactly imbedded in H) is known.*

We now address the last question, and that is of the regularity of the attractor (i.e. regularity as $t \to \infty$): If the data we are presented with are sufficiently regular, can we show that the attractor is included in a set of more spatially regular functions W, where $W \subset H$, or even better $W \subseteq V$?

In terms of PDEs, if we can show that (the solutions which lie in) the maximal attractor lies in the Sobolev spaces $H^m(\Omega)$, $m = 1, \ldots, M$ for some M, then we say that we have achieved "partial regularity" of the attractor. If, however, we can prove this $\forall\, M \in \mathbb{N}$ (with C^∞ initial data) then we say that we have $C^\infty(\overline{\Omega})$ regularity (or just "regularity") of the attractor.

And so, generalizing the above argument, if we can show the existence of an absorbing set in H^m for some m, the attractor $\mathcal{A}$ is bounded in H^m and (H^m compactly imbedded in H^{m-1} implies) that the global attractor lies in H^{m-1}.

We shall, in Chapters 4 and 5, show the existence of absorbing sets in all the spaces H^m, $m \in \mathbb{N}$ for the Navier-Stokes equations where,

1. For $d = 2$, we do not need to make any assumptions.
2. For $d = 3$, we will investigate the minimum assumptions necessary to show this.

2.7 The Semi-group for Non-well-posed Problems

For the case when the semi-group $S(t)$ associated with the initial value problem (2.1) cannot be defined everywhere, i.e. the initial value problem (2.1) is not well-posed, we must rely on the following more general setting. We will need this theory in Chapter 4.

Let two Hilbert spaces, $W \subseteq H$ (the injection being continuous) be given.

Suppose that a family of operators $\{S(t), t \geq 0\}$ is defined and continuous from one part $\mathcal{D}(S(t)) \subseteq W$ to W, where

$$\mathcal{D}(S(t)) = \bigcup_{\rho>0} \mathcal{D}_\rho(S(t)) \tag{2.39}$$

$$\mathcal{D}_\rho(S(t)) = \{u_0 \in W,\ \|S(\tau)u_0\|_W \leq \rho,\ 0 \leq \tau \leq t\} \tag{2.40}$$

Obviously, $\mathcal{D}(S(t)) = \mathcal{D}(S(t); W)$ is the domain of $S(t)$ in W.

Given $u_0 \in W$, $S(t)u_0$ is defined if and only if $u_0 \in \mathcal{D}(S(t))$.

By analogy with (2.2), (2.3) we require our family of operators $S(t)$ satisfy a set of semi-group properties in the sense:

$$\begin{array}{rll} S(0) = I_D & (\text{in } H) & \mathcal{D}(S(0)) = W \\ S(t+s) = S(t)\cdot S(s) & (\text{on } \mathcal{D}(S(t+s))) & \forall\, s,t \geq 0 \end{array} \tag{2.41}$$

Remark: It is apparent from the definitions above that

$$\forall\, t_1 \geq t_2 \quad \mathcal{D}(S(t_1)) \subseteq \mathcal{D}(S(t_2)) \text{ and } \mathcal{D}_\rho(S(t_1)) \subseteq \mathcal{D}(S(t_2)) \tag{2.42}$$

Definition 2.7.1 *If a set* $X \subseteq W$ *is such that*

1. $S(t)u_0$ *exists* $\forall\ u_0 \in X,\ \forall\ t \geq 0$
2. $S(t)X = X,\ \forall\ t \geq 0$

then X *is said to be a* functional invariant *set for the semi-group* $S(t)$.

Definition 2.7.2 *If a set* $\mathcal{A} \subseteq W$ *satisfies*

1. $\mathcal{A}$ *is an invariant set (as above)*
2. *there exists an open neighbourhood* $\mathcal{U} \supseteq W$ *of* $\mathcal{A}$ *such that* $\forall\ u_0 \in \mathcal{U},\ S(t)u_0 \longrightarrow \mathcal{A}$ *in* H *as* $t \longrightarrow \infty$ *(i.e. convergence in the norm of* H*)*

then $\mathcal{A}$ *is said to be an* attractor.

2.8 Summary and Additionals

Thus for the general evolution equation, we have defined the semi-group of operators, functional invariant sets and we have provided a theorem which guarantees (under the conditions outlined) the existence of a global attractor when the existence of an absorbing set is known. After defining what we mean by an 'imbedding' from one normed linear space into another, we have examined in detail the theory of linear, continuous operators (such as A) and also, we have defined the domain of A in H, namely $D(A)$. Under further restrictions, we then defined powers of these linear operators (A^s, $s \in \mathbb{R}$) and their corresponding domains $D(A^s)$.

Subsequently, we consider the question of regularity of the attractor and we outline how we intend to proceed in order to show the existence of a C^∞ attractor. Finally, in the last section, the theory of the semi-group of operators is presented in a more general setting.

All the theory of this chapter (and the significance of the order) I have given will become relevant in Chapters 4 and 5 when we specialize to the incompressible Navier-Stokes equations – the next chapter outlines the important theory of Sobolev spaces and interpolation inequalities in detail, which will also be very relevant in Chapters 4 and 5.

The main references I have used for nearly all the main theory of this chapter were Temam [88], Adams [1] and Oliver [72]. Further material can be found in Yosida [92], Courant and Hilbert [21, 22] and also Riesz and Nagy [75].

It would be natural to follow our discussion in this chapter with one on the general theory for the determination of the dimension of the attractor, however, we are not interested in this problem in this thesis, although we do make some comparisons with such calculations in Chapter 6.

Chapter 3

The Gagliardo-Nirenberg Inequality

3.1 Introduction and Aims

In this chapter, we outline the function space theory (for real valued functions) we intend to use in the following chapters, and in particular, we introduce Sobolev spaces as well as several versions of an inequality of the Gagliardo-Nirenberg type (which is an example of an interpolation inequality). We will consider this important inequality for the following distinct cases:

1. For functions with compact support in $\mathbb{R}^d$.

2. For functions defined on a domain $\Omega_L \equiv [0, L]^d \subset \mathbb{R}^d$, with no specified boundary conditions. For this case, the Gagliardo-Nirenberg inequality is given for the usual Sobolev norms (which we will also refer to as full Sobolev norms). To prove this case we will use extension theorems for full Sobolev norms combined with the last result. Extension theorems show the existence of extension operators which allow us to prove norm inequalities for functions on bounded domains when the same inequalities are known to hold for functions defined on $\mathbb{R}^d$.

3. For functions defined on a domain Ω_L which have zero mean and which are also periodic in $\mathbb{R}^d$ with period Ω_L. In this case the Gagliardo-Nirenberg inequality can be proved for semi-norms – we will combine the first result above with extension theorems and a new inequality of the Poincaré type (which holds for mean-zero and space-periodic functions).

Remarks:

1. Multiplicative constants will be calculated (as far as possible), but no claim is made that these are optimal constants – indeed, their optimization invites some interesting further work!

2. We will provide the proofs as a series of lemmas. Each lemma will be proved in as full generality as possible (to invite further work).

3. We can easily generalize the three cases above to d-vector valued functions $u \in W^{m,p}(\Omega)^d$ – see Section 3.2.4.

4. There are three primary references: Nirenberg [70], Adams [1] and Friedman [33].

3.2 Preliminaries

Let Ω be a domain in $\mathbb{R}^d$.

The *generalised derivative* operator of order N is defined as

$$D^n = \frac{\partial^{n_1+n_2+\cdots+n_d}}{\partial x_1^{n_1}\partial x_2^{n_2}\cdots\partial x_d^{n_d}} \tag{3.1}$$

where $n = (n_1, n_2, \ldots, n_d)$ is a multi-index and $|n| = n_1 + n_2 + \cdots + n_d = N$.

Recall the general theory of the 'Operators and Imbeddings' section in Chapter 2.

3.2.1 Spaces of Continuous Functions

Definition 3.2.1 *For m a non-negative integer, $C^m(\Omega)$ is the (vector) space of all functions ϕ which are continuous on Ω and whose partial derivatives of order $D^\alpha\phi$, $|\alpha| \le m$ are also continuous on Ω.*

Further, $C(\Omega) \equiv C^0(\Omega)$ and $C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega)$.

The subspaces $C_0^m(\Omega)$ consist of those functions in $C^m(\Omega)$ that also have compact support in Ω.

For functions $\phi \in C(\Omega)$ which are bounded and uniformly continuous on Ω there exists a unique, bounded, continuous extension to the closure $\overline{\Omega}$ of Ω.

Definition 3.2.2 *$C^m(\overline{\Omega})$ is the vector space of those functions $\phi \in C^m(\Omega)$ whose derivatives $D^\alpha\phi$ for $0 \le |\alpha| \le m$ are bounded and uniformly continuous on Ω.*

Remark: $C^m(\overline{\Omega})$ is a Banach space with the norm

$$||\phi; C^m(\overline{\Omega})|| = \max_{0\le|\alpha|\le m} \sup_{x\in\Omega} |D^\alpha\phi(x)| \tag{3.2}$$

Definition 3.2.3 *For $0 < \lambda \le 1$, define $C^{m,\lambda}(\overline{\Omega})$ to be the subspace of $C^m(\overline{\Omega})$ consisting of those functions ϕ for which $D^\alpha\phi$ $(0 \le |\alpha| \le m)$ satisfies a Hölder condition of exponent λ in Ω, i.e. there exists a constant K such that*

$$|D^\alpha\phi(x) - D^\alpha\phi(y)| \le K\,|x-y|^\lambda \quad \forall\ x, y \in \Omega \tag{3.3}$$

Remark: $C^{m,\lambda}(\overline{\Omega})$ is a Banach space with norm given by

$$||\phi; C^{m,\lambda}(\overline{\Omega})|| = ||\phi; C^m(\overline{\Omega})|| + \max_{0\le|\alpha|\le m} \sup_{\substack{x,y\in\Omega \\ x\ne y}} \frac{|D^\alpha\phi(x) - D^\alpha\phi(y)|}{|x-y|^\lambda} \tag{3.4}$$

For further details of various inclusions and imbeddings between these spaces, see Adams [1], Chapter 1.

Definition 3.2.4 *A function $v(x)$ defined on a set S is said to be* uniformly Hölder continuous of exponent α on S *if*

$$\sup_S[v]_\alpha \equiv \sup_{x,y\in S} \frac{|v(x)-v(y)|}{|x-y|^\alpha} \tag{3.5}$$

is finite. If a function v is uniformly Hölder continuous of exponent α on each compact subset of a domain Ω, then it is said to be Hölder continuous of exponent α in Ω.

Definition 3.2.5 *For $p < 0$, set $h = [-d/p]$, $-\alpha = h + d/p$, and define*

$$|u|_{p,\Omega} = \sup_\Omega |D^h u| \equiv \sum_{|\beta|=h} \sup_\Omega |D^\beta u| \quad \text{if } \alpha = 0\ , \tag{3.6}$$

$$|u|_{p,\Omega} = [D^h u]_{\alpha,\Omega} \equiv \sum_{|\beta|=h} \sup_\Omega [D^\beta u]_\alpha \quad \text{if } \alpha > 0\ . \tag{3.7}$$

Remark: By $[z]$ with $z \in \mathbb{R}^+$ (with no index), we mean 'take the integer part, rounding down.'

3.2.2 Lebesgue Spaces

Let Ω be domain in $\mathbb{R}^d$ and let p be any positive real number. By a.e. (= almost everywhere) we mean everywhere except on a set of measure zero.

Definition 3.2.6 *The* Lebesgue Space $L^p(\Omega)$ *is the space of equivalence classes of p-integrable functions on Ω - i.e. the class of all measurable functions u defined on Ω, for which the functional*

$$\|u\|_{p,\Omega} = \left(\int_\Omega |u|^p dx\right)^{1/p} < \infty \tag{3.8}$$

(We identify in $L^p(\Omega)$ functions that are equal a.e. on Ω.)

Definition 3.2.7 *We say that a function u, measurable on Ω, is* essentially bounded *on Ω if there exists a constant K such that $|u(x)| \leq K$ a.e. on Ω. The greatest lower bound of such constants K is called the essential supremum of $|u|$ on Ω which we write as $\operatorname{ess\,sup}_{x\in\Omega} |u(x)|$. So we call $L^\infty(\Omega)$ the vector space consisting of all functions u for which $\|u\|_{\infty,\Omega} = \operatorname{ess\,sup}_{x\in\Omega} |u(x)|$ is finite (functions being indentified if equal a.e. on Ω). Clearly, the functional $\|u\|_\infty$ is a norm on $L^\infty(\Omega)$.*

Theorem 3.2.1 (Young's Inequality) *For $a \geq 0$, $b \geq 0$ and any $\epsilon > 0$ we have*

$$ab \leq \frac{1}{p}(a\epsilon)^p + \frac{1}{q}\left(\frac{b}{\epsilon}\right)^q \tag{3.9}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ with $1 < p, q < \infty$.

See Adams [1], Chapter 8, for a more concise and at the same time much more general treatment of a range of inequalities which include the above as a special case.

Theorem 3.2.2 (Hölder's Inequality) *If* $1 \le p \le \infty$ *and* $u \in L^p(\Omega)$, $u \in L^q(\Omega)$, *then* $uv \in L^1(\Omega)$ *and*

$$\int_\Omega |u(x)\, v(x)|dx \le \|u\|_{p,\Omega}\, \|v\|_{q,\Omega} \quad \textit{where} \quad \frac{1}{p} + \frac{1}{q} = 1 \,. \tag{3.10}$$

Remark: A similar inequality holds for summations, i.e.

$$\sum_i (a_i b_i) \le \left(\sum_i |a_i|^p\right)^{1/p} \left(\sum_i |b_i|^q\right)^{1/q} \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1 \,. \tag{3.11}$$

Theorem 3.2.3 (Minkowski's Inequality) *If* $1 \le p \le \infty$, *then*

$$\|u + v\|_p \le \|u\|_p + \|v\|_p \,. \tag{3.12}$$

Note: Hölder's inequality naturally gives us: For $1 \le p \le p' \le \infty$ then

$$\|u\|_p \le (\text{vol}\ \Omega)^{1/p - 1/p'}\, \|u\|_{p'} \,. \tag{3.13}$$

(This assumes vol Ω finite.)

Theorem 3.2.4 $L^p(\Omega)$ *is a Banach space for* $1 \le p \le \infty$ *and* $L^2(\Omega)$ *is a Hilbert space with respect to the inner product*

$$(u, v) = \int_\Omega u(x)\overline{v(x)}dx \,. \tag{3.14}$$

Theorem 3.2.5 *For* $1 \le p < \infty$, $C_0(\Omega)$ *and* $C_0^\infty(\Omega)$ *are both dense in* $L^p(\Omega)$.

Remark: As an example, to prove the first result we proceed as follows: Let $u \in L^p(\Omega)$ and let $\epsilon > 0$. We show there exists a function $\phi \in C_0(\Omega)$ such that $\|u - \phi\|_{p,\Omega} < \epsilon$.

(In fact the set of simple functions in $L^p(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$.)

Remark: $C(\Omega)$, $C_0(\Omega)$ and $C_0^\infty(\Omega)$ are *not* dense in $L^\infty(\Omega)$ as they are proper closed subspaces of this space.

3.2.3 Weak and Strong Derivatives

Definition 3.2.8 *We say that a function defined a.e. on* Ω *is* locally integrable *on* Ω *provided* $u \in L^1(Q)$ *for every measurable* Q *which is such that* $\overline{Q} \subseteq \Omega$ *and* $\overline{Q}$ *is compact in* $\mathbb{R}^d$. *We write* $u \in L^1_{loc}(\Omega)$.

Definition 3.2.9 *Let* u, v *be two locally integrable functions defined on* Ω. *We call* $v = D^\alpha u$ *the* weak derivative *of* u *(of order* α*) provided* v *satisfies*

$$\int_\Omega u(x) D^\alpha \phi(x) dx = (-1)^{|\alpha|} \int_\Omega v(x)\phi(x)dx \tag{3.15}$$

for every $\phi \in C_0^\infty(\Omega)$.

Definition 3.2.10 *For u, v two locally integrable functions on Ω, $D^\alpha u = v$ in the* strong L^p *sense ($p \geq 1$) if for any compact subset K of Ω there exists a sequence of functions ϕ_j in $C^{|\alpha|}(\Omega)$ such that*

$$\int_K |\phi_j - u|^p dx \to 0, \quad \int_K |D^\alpha \phi_j - v|^p dx \to 0 \tag{3.16}$$

as $j \to \infty$.

Remark: It is clear that if $D^\alpha u = v$ in the strong L^p sense, then $D^\alpha u = v$ in the weak sense.

Theorem 3.2.6 *Suppose u, v belong to L^p locally in Ω ($p \geq 1$) and that v is the weak derivative of u (of order α), then on any compact set $K \subseteq \Omega$ there exist functions $\phi_j \in C^\infty(\Omega)$ such that (3.16) holds.*

Remark: This means that the concepts of weak and strong derivatives are consistent.

For further details, see Adams [1], Chapters 1 and 3, as well as Friedman [33], Section 6.

3.2.4 Sobolev Spaces

In this section we define Sobolev spaces of integer order for a domain $\Omega \subseteq \mathbb{R}^d$. These are vector subspaces of various $L^p(\Omega)$ spaces – we additionally demand that the weak derivatives of elements in $L^p(\Omega)$ spaces also lie in $L^p(\Omega)$.

Definition 3.2.11 *We define the functional $\|\cdot\|_{m,p,\Omega}$ (with m a non-negative integer and $1 \leq p \leq \infty$) as*

$$\|u\|_{m,p,\Omega} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_p^p \right)^{1/p} \qquad \text{when } 1 \leq p < \infty \tag{3.17}$$

and

$$\|u\|_{m,\infty,\Omega} = \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_\infty \qquad \text{when } p = \infty \tag{3.18}$$

for any function for which the right-hand side is meaningful.

We easily see that (3.17) and (3.18) define norms on vector spaces of functions for which those norms are finite (functions being identified in the space if they are equal a.e.).

Definition 3.2.12 *Consider the following three spaces:*

- *$H^{m,p}(\Omega) \equiv$ the completion of $\{u \in C^m(\Omega) : \|u\|_{m,p,\Omega} < \infty\}$ with respect to the norm $\|\cdot\|_{m,p,\Omega}$,*
- *$W^{m,p}(\Omega) \equiv \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \quad \text{for } 0 \leq |\alpha| \leq m\}$, where $D^\alpha u$ is the weak partial derivative (see above),*
- *$W_0^{m,p}(\Omega) \equiv$ the closure of $C_0^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$.*

The above spaces, when appropriately equipped with the norms (3.17) or (3.18) are called Sobolev spaces *over Ω.*

Theorem 3.2.7 *$W^{m,p}(\Omega)$ is a Banach space.*

Theorem 3.2.8 *$W^{m,2}(\Omega)$ is a separable Hilbert space with inner product*

$$(u,v)_m = \sum_{0\leq|\alpha|\leq m} (D^\alpha u, D^\alpha v). \tag{3.19}$$

Theorem 3.2.9 (Meyers and Serrin) *If $1 \leq p < \infty$, then $H^{m,p}(\Omega) = W^{m,p}(\Omega)$.*

Remark: In proving this result, it is established that *if $u \in W^{m,p}(\Omega)$ and $\epsilon > 0$, there exists $\phi \in C^\infty(\Omega)$ such that $\|u-\phi\|_{m,p} < \epsilon$; i.e. $\{\phi \in C^\infty(\Omega) : \|\phi\|_{m,p,\Omega} < \infty\}$ is dense in $W^{m,p}(\Omega)$.* Note further that this result does not extend to $p = \infty$.

Remark: Thus functions having weak derivatives can be suitably approximated by smooth functions. (See the section on weak derivatives above and also Adams [1], Chapters 1 and 3.)

Further Note: As suggested by the above theorems, it will be convenient to abbreviate $H^m(\Omega) \equiv W^{m,2}(\Omega)$.

Definition 3.2.13 *Suppose $\Omega \subseteq \mathbb{R}^d$. If for all $x \in$ bdry $\Omega \equiv \partial\Omega$ there exists an open set U_x and non-zero vector y_x such that $x \in U_x$, and if $z \in \overline{\Omega} \cap U_x$ implies $z + ty_x \in \Omega$, $0 < t < 1$, then Ω is said to have the* segment property.

Remark: If Ω has the segment property, then it must have an $(d-1)$-dimensional boundary, and the domain cannot simultaneously lie on both sides of any given part of its boundary.

Theorem 3.2.10 *Suppose Ω has the segment property. The set of restrictions to Ω of functions in $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{m,p}(\Omega)$ for $1 \leq p < \infty$.*

Remark: In particular (for Ω with the segment property):

1. $C^k(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$ for any m $(k \geq m)$,
2. $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{m,p}(\mathbb{R}^d)$.

Corollary 3.2.11 $W_0^{m,p}(\mathbb{R}^d) = W^{m,p}(\mathbb{R}^d)$.

3.2.5 Semi-norms

Definition 3.2.14 *For $1 \leq p < \infty$ and for integers j, $0 \leq j \leq m$ we introduce the functionals $|\cdot|_{j,p,\Omega}$ on $W^{m,p}(\Omega)$ as follows:*

$$|u|_{j,p} = |u|_{j,p,\Omega} = \left(\sum_{|\alpha|=j} \int_\Omega |D^\alpha u(x)|^p dx \right)^{1/p}. \tag{3.20}$$

Clearly, $|u|_{0,p} = \|u\|_{0,p} = \|u\|_p$ is the norm of u in $L^p(\Omega)$ and

$$\|u\|_{m,p,\Omega} = \left(\sum_{0 \leq j \leq m} |u|^p_{j,p,\Omega} \right)^{1/p} \tag{3.21}$$

which is the usual norm (i.e. full Sobolev norm) for $W^{m,p}(\Omega)$. If $j \geq 1$, $|\cdot|_{j,p,\Omega}$ is a *semi-norm* – it has all the properties of a norm, except that $|u|_{j,p,\Omega} = 0$ does not imply that u vanishes in $W^{m,p}(\Omega)$ e.g. u may be a non-zero constant on a domain Ω having finite volume (for example $\Omega_L \equiv [0, L]^d$).

Definition 3.2.15 *Clearly, we should define the functional (semi-norm)*

$$|u|_{m,\infty,\Omega} = \max_{|\alpha|=m} \sup_{x \in \Omega} |D^\alpha u(x)| \ . \tag{3.22}$$

So that

$$\|u\|_{m,\infty,\Omega} = \max_{0 \leq j \leq m} |u|_{j,\infty,\Omega} \tag{3.23}$$

as we would expect.

3.2.6 Some Interpolation Theorems and Equivalent Norms

Definition 3.2.16 *Two different norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space X are* equivalent *if they induce the same topology (on X), i.e. if for some constants $a, b > 0$,*

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \tag{3.24}$$

for all $x \in X$.

Remark: The results of this section (except where stated otherwise) apply for $1 \leq p < \infty$.

Lemma 3.2.12 *On an interval $\lambda \equiv [a, b]$ (length $|\lambda| = b - a > 0$), for $f \in C^2(\lambda)$, we have*

$$\int_\lambda |f'(t)|^p dt \leq c^p |\lambda|^p \int_\lambda |f''(t)|^p dt + c^p 8^p |\lambda|^{-p} \int_\lambda |f(t)|^p dt \tag{3.25}$$

where c an absolute constant independent of p and $|\lambda|$.

Remark: In fact $c^p \equiv 2^{p-1}$.

Proof: Suppose $\alpha = |\lambda|/4$. Pick $x_1 \in (a, a+\alpha)$, $x_2 \in (b-\alpha, b)$. Then by the Mean Value Theorem there exists a $y \in (x_1, x_2)$ such that

$$f'(y) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{3.26}$$

and also by the Fundamental Theorem of Calculus,

$$f'(x) = -\int_x^y f''(\xi) d\xi + f'(y) \ . \tag{3.27}$$

Combining (3.26) and (3.27) and taking the modulus of both sides we get

$$|f'(x)| \le \int_\lambda |f''(\xi)|d\xi + \frac{|f(x_2)| + |f(x_1)|}{2\alpha} . \tag{3.28}$$

Now integrate over x_2 between $(b-\alpha, b)$ and over x_1 between $(a, a+\alpha)$.

$$\Rightarrow |f'(t)| \le \int_\lambda |f''(t)|dt + \frac{8}{|\lambda|^2} \int_\lambda |f(t)|dt \tag{3.29}$$

Now use that $(A+B)^p \le c^p \left(A^p + B^p\right)$ (for $A, B \ge 0$; $1 \le p < \infty$ and where $c^p \equiv 2^{p-1}$).

$$\Rightarrow |f'(t)|^p \le c^p \left(\int_\lambda |f''(t)|dt\right)^p + c^p 8^p |\lambda|^{-2p} \left(\int_\lambda |f(t)|dt\right)^p \tag{3.30}$$

$$\Rightarrow \int_\lambda |f'(t)|^p dt \le c^p |\lambda|^p \left(\int_\lambda |f''(t)|^p dt\right) + c^p 8^p |\lambda|^{-p} \left(\int_\lambda |f(t)|^p dt\right) .$$

□

Corollary 3.2.13 *For $f \in C^2(\lambda)$ and for all $0 < \epsilon \le 1$*

$$\int_\lambda |f'(t)|^p dt \le c^p |\lambda|^p \epsilon \int_\lambda |f''(t)|^p dt + \frac{c^p 16^p}{|\lambda|^p \epsilon} \int_\lambda |f(t)|^p dt . \tag{3.31}$$

where c an absolute constant independent of p and $|\lambda|$.

Proof: For all $0 < \epsilon \le 1$, $\exists\ m \in \mathbb{N}$ such that $\frac{1}{2}\epsilon^{1/p} \le \frac{1}{m} \le \epsilon^{1/p}$. (Choose $m = [\frac{1}{\epsilon^{1/p}}] + 1$ with $[\cdot]$ defined as in Section 3.2.1.)

Set $a_j = a + (b-a)\frac{j}{m}$ for $j = 1, \dots, m$. Note that $a_j - a_{j-1} = (b-a)/m \equiv |\lambda|/m$.

$$\begin{aligned}
\int_\lambda |f'(t)|^p dt &= \sum_{j=1}^m \int_{a_{j-1}}^{a_j} |f'(t)|^p dt \\
&\le c^p \sum_{j=1}^m \left[\left(\frac{|\lambda|}{m}\right)^p \left(\int_{a_{j-1}}^{a_j} |f''(t)|^p dt\right) + 8^p \left(\frac{|\lambda|}{m}\right)^{-p} \left(\int_{a_{j-1}}^{a_j} |f(t)|^p dt\right)\right] \\
&\le c^p |\lambda|^p \epsilon \left(\int_\lambda |f''(t)|^p dt\right) + \frac{c^p 2^{4p}}{|\lambda|^p \epsilon} \left(\int_\lambda |f(t)|^p dt\right) .
\end{aligned}$$

□

Remark: A result similar to the above corollary is also true when the interval λ is not finite: Without loss of generality, assume a is finite and $b = \infty$.

Given $\epsilon > 0$, set $a_j = a + j\epsilon^{1/p}$, $j = 0, 1, 2, \dots$ ($\Rightarrow a_j - a_{j-1} = \epsilon^{1/p}$). Using the lemma,

$$\begin{aligned}
\int_a^\infty |f'(t)|^p dt &= \sum_{j=1}^\infty \int_{a_{j-1}}^{a_j} |f'(t)|^p dt \\
&\le c^p \sum_{j=1}^\infty \left[\epsilon \int_{a_{j-1}}^{a_j} |f''(t)|^p dt + 8^p \epsilon^{-1} \int_{a_{j-1}}^{a_j} |f(t)|^p dt\right] ,
\end{aligned}$$

so that the result looks like

$$\int_{-\infty}^{\infty} |f'(t)|^p dt \le c^p \left[\epsilon \int_{-\infty}^{\infty} |f''(t)|^p dt + 8^p \epsilon^{-1} \int_{-\infty}^{\infty} |f(t)|^p dt \right] . \tag{3.32}$$

The results above are the essential ingredients in the proofs of the following series of theorems outlined in Adams [1]. I include the theorems here for the sake of completeness. Adams [1] (Chapter 4) presents them in a more general setting which is not relevant here.

Note: I have included all the multiplicative constants in the results above, and therefore one should be able to calculate the appropriate multiplicative constants for the results below, by reproducing the proofs in detail. This should be quite straightforward.

Theorem 3.2.14 *For all $\epsilon > 0$, integers j, $0 \le j \le m-1$, on a domain $\Omega \subseteq \mathbb{R}^d$ and for all $u \in W_0^{m,p}(\Omega)$, there exists a constant $K(m,p,d)$ such that*

$$|u|_{j,p} \le K\epsilon |u|_{m,p} + K\epsilon^{-j/(m-j)} |u|_{0,p} . \tag{3.33}$$

Theorem 3.2.15 *$\Omega_L \equiv [0,L]^d \subset \mathbb{R}^d$. For all $0 < \epsilon \le 1$, integers j, $0 \le j \le m-1$ and for all $u \in W^{m,p}(\Omega_L)$, there exists a constant $K(\epsilon, m, p, L)$ such that*

$$|u|_{j,p} \le K\epsilon |u|_{m,p} + K\epsilon^{-j/(m-j)} |u|_{0,p} . \tag{3.34}$$

Corollary 3.2.16 *The functional $[[\cdot]]_{m,p,\Omega} := (|u|^p_{m,p,\Omega} + |u|^p_{0,p,\Omega})^{1/p}$ is an equivalent norm to the usual norm $||u||_{m,p,\Omega}$ on $W_0^{m,p}(\Omega)$ ($\Omega \subseteq \mathbb{R}^d$) and* also *on $W^{m,p}(\Omega_L)$.*

Theorem 3.2.17 *For $\Omega_L \subset \mathbb{R}^d$, $1 \le p < \infty$, integers j, $0 \le j \le m$ and for all $u \in W^{m,p}(\Omega_L)$, there exists a constant $K(m,p,L)$ such that*

$$||u||_{j,p} \le K ||u||^{j/m}_{m,p} ||u||^{1-j/m}_{0,p} . \tag{3.35}$$

Remark: This last result is also true for $u \in W_0^{m,p}(\Omega)$ with $K(m,p,d)$ independent of the domain.

Remark: Under certain circumstances $|\cdot|_{m,p,\Omega}$ is equivalent to the usual norm for the space $W_0^{m,p}(\Omega)$, $\Omega \subseteq \mathbb{R}^d$. This is particularly so if Ω is bounded, for consider the lemma:

Lemma 3.2.18 *Suppose $\Omega \subset \mathbb{R}^d$ is bounded in some direction, $u \in W_0^{1,p}(\Omega)$, then*

$$|u|_{0,p,\Omega} \le d^{-1/p} |\lambda| \, |u|_{1,p,\Omega} \tag{3.36}$$

where $|\lambda| =$ "thickness" of Ω in the bounded direction.

Proof: Assume $u \in C_0^1(\Omega)$ and that Ω is bounded in the i^{th} direction (without loss of generality, suppose $0 \le x_i \le |\lambda|$), then by the Fundamental Theorem of Calculus

$$u(x_1, \ldots, x_i, \ldots, x_d) = \int_0^{x_i} \frac{\partial u}{\partial x_i} \, dx_i . \tag{3.37}$$

Take the modulus of both sides to the p^{th} power, apply Hölder's inequality, and then integrate both sides over Ω, to get (3.36). Now use that $C_0^1(\Omega)$ is dense in $W_0^{1,p}(\Omega)$. □

Remark: This is a generalization of Poincaré's inequality for functions $u \in W_0^{1,p}(\Omega)$.

Remark: This result is true for $p = \infty$, when our assumptions are $u \in C_0^1(\Omega)$.

3.2.7 Other Spaces

When dealing with evolutionary partial differential equations it is convenient to introduce the following spaces defined on $\Omega \times (0,T) \subset \mathbb{R}^{d+1}$.

Definition 3.2.17 *Suppose X is a Banach space. For $1 \leq p < \infty$, $-\infty \leq a < b \leq \infty$, $L^p(a,b;X)$ is the space of equivalence classes of L^p functions from (a,b) into X and is a Banach space for the norm*

$$||f||_{L^p(a,b;X)} = \left(\int_a^b ||f(t)||_X^p \, dt \right)^{1/p} . \tag{3.38}$$

$L^\infty(a,b;X)$ is the space of equivalence classes of (measurable) essentially bounded functions from (a,b) into X and is also a Banach space for the norm

$$||f||_{L^\infty(a,b;X)} = ess \sup_{t \in (a,b)} ||f(t)||_X . \tag{3.39}$$

Now suppose $-\infty < a < b < \infty$. Then $C([a,b];X)$ is the space continuous functions from $[a,b]$ to X and for $k \in \mathbb{N}$, $C^k([a,b];X)$ is the space of k times continuously differentiable functions from $[a,b]$ to X. Both are Banach spaces for the respective norms

$$||f||_{C([a,b];X)} = \sup_{t \in [a,b]} ||f(t)||_X , \tag{3.40}$$

$$||f||_{C^k([a,b];X)} = \sum_{j=1}^{k} \left\| \frac{\partial^j f}{\partial t^j} \right\|_{C([a,b];X)} . \tag{3.41}$$

3.2.8 Mean-zero and Space-periodic Functions

Definition 3.2.18 *For $m \in \mathbb{N}$, $H^m_{per}(\Omega_L)$ is the space of functions in $H^m_{loc}(\mathbb{R}^d)$ which are periodic with period Ω_L: i.e. $u(x + Le_i) = u(x)$, $\forall\, x \in \mathbb{R}^d$, $i = 1, \ldots, d$ and where e_i is the canonical basis for $\mathbb{R}^d$.*

Remarks:

1. For more details on the periodicity condition, see the beginning of the next chapter.
2. $H^0_{per}(\Omega_L) \equiv L^2_{per}(\Omega_L)$.
3. H^m_{per} is a Hilbert space for the scalar product $(u,v)_{m,\Omega_L}$ giving rise to the norm

$$||u||_{m,2,\Omega_L} = \{(u,u)_{m,\Omega_L}\}^{1/2} . \tag{3.42}$$

It is apparent that

$$H^m_{per}(\Omega_L) = \{u \equiv \sum_{k \in \mathbb{Z}^d} c_k e^{2i\pi k \cdot x/L} : \bar{c}_k = c_{-k}, \sum_{k \in \mathbb{Z}^d} |k|^{2m} |c_k|^2 < \infty\} \tag{3.43}$$

and that the norm $||u||_{m,2,\Omega_L} = \{(u,u)_{m,\Omega_L}\}^{1/2}$ is equivalent to the norm

$$\left(\sum_{k \in \mathbb{Z}^d} (1 + |k|^{2m}) |c_k|^2 \right)^{1/2} . \tag{3.44}$$

Remark: In fact, $H^m_{per}(\Omega_L)$ is a Hilbert space for the norm (3.44) with $m \in \mathbb{R}$, $m \geq 0$.

Definition 3.2.19 $\dot{H}^m_{per}(\Omega_L) = \{u \in H^m_{per}(\Omega_L) : c_0 = 0\}$ *i.e. the space of periodic functions on* $H^m(\Omega_L)$ *which have mean zero:* $\int_{\Omega_L} u \, dx = 0$.

Remark: In fact, $\dot{H}^m_{per}(\Omega_L)$ is a Hilbert space for the norm $(\sum_{k \in \mathbb{Z}^n} |k|^{2m} |c_k|^2)^{1/2}$, $m \in \mathbb{R}$, with $\dot{H}^m_{per}(\Omega_L)$ and $\dot{H}^{-m}_{per}(\Omega_L)$ in duality $\forall\ m \in \mathbb{R}$.

3.2.9 Extension Theorems

Definition 3.2.20 *We call a linear operator* $E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^d)$, *a* simple (m,p)-extension operator for Ω *provided there exists a constant* $K(m,p)$ *such that for every* $u \in W^{m,p}(\Omega)$ *the conditions hold:*

1. $Eu(x) = u(x)$ *a.e. in* Ω
2. $||Eu||_{m,p,\mathbb{R}^d} \leq K ||u||_{m,p,\Omega}$.

See also the definitions for a *strong m-extension operator* and *total extension operator* for Ω in Adams [1], Chapter 4.

Adams [1] also outlines three theorems which provide the existence of all three types of extension operators for domains Ω having specific properties. Thus we know there exists a simple extension operator for the domain $\Omega_L \equiv [0,L]^d$ – this follows from the following simplified version of the Calderón extension theorem,

Theorem 3.2.19 *For* $\Omega_L \subset \mathbb{R}^d$, *there exists a simple* (m,p)*-extension operator* $E = E(m,p)$ *for any* $m \in \{1,2,\ldots\}$ *and* $1 < p < \infty$.

The existence of such a simple (m,p)-extension operator for a domain Ω means that norm inequalities which hold for functions defined on $\mathbb{R}^d$ are then known to hold for functions defined on Ω, for example:

$$||u||_{0,q,\Omega} \leq ||Eu||_{0,q,\mathbb{R}^d} \leq K_1 ||Eu||_{m,p,\mathbb{R}^d} \leq K_1 K ||u||_{m,p,\Omega} . \tag{3.45}$$

where the central inequality is known to be true (on $\mathbb{R}^d$).

Friedman [33] (Lemma 5.2) provides an extension theorem for C^m-functions:

Definition 3.2.21 *For a bounded domain* Ω *with boundary* $\partial\Omega$, *we say that* $\partial\Omega$ *is of* class C^m *if for all* $x^* \in \partial\Omega$, *there exists a ball* $B(x^*, R)$ *(some* R*) such that* $\partial\Omega \cap B(x^*, R)$ *can be represented by*

$$x_i = f(x_{j \neq i}) \quad (j = 1, \ldots, d) \tag{3.46}$$

for some i $(i = 1, \ldots, d)$ *with* f *a* C^m *regular function.*

Theorem 3.2.20 *Let $1 \le p \le \infty$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\partial\Omega$ of class C^m. For $u \in C^m(\overline{\Omega})$ there exists an extension operator $E : C^m(\overline{\Omega}) \to C_0^m(\mathbb{R}^d)$ such that*

1. *$Eu(x) = u(x)$ in Ω*
2. *$||Eu||_{m,p,\mathbb{R}^d} \le c||u||_{m,p,\Omega}$.*

where c is independent of u.

Further information can be found in the classical texts of Adams [1] and Friedman [33].

3.2.10 A Poincaré Inequality and its Implications

Theorem 3.2.21 *Suppose $f \in C^1(\Omega_L)$ has mean zero on $\Omega_L \equiv [0, L]^d$. Then*

$$|f|_{0,p,\Omega_L} \le c_1(d,p)\, L\, |f|_{1,p,\Omega_L} \tag{3.47}$$

for $1 \le p \le \infty$ and where $c_1(d,p) = d^{1-1/p}$.

Note:

1. Using that $C^1(\Omega_L)$ is dense in $W^{1,p}(\Omega_L)$ for $1 \le p < \infty$, we could relax our assumptions to $f \in W^{1,p}(\Omega_L)$ with zero mean. For the $p = \infty$ case we also only need to assume $f \in W^{1,\infty}(\Omega_L)$, although a separate proof is necessary – see page 28 of A. Ženišek, *Nonlinear Elliptic and Evolution Problems and Their Finite Element Approximations*, Academic Press, 1990.
2. For $f \in C^1(\Omega_L)$ with mean zero, $|f|_{1,p,\Omega_L}$ is equivalent to the functional $||f||_{1,p,\Omega_L}$ with $1 \le p \le \infty$.

Also see A. Kufner, O. John and S. Fučik, *Function Spaces*, Noordhoff Int. Publishing, 1977.

Proof: We will prove that

$$|f|_{0,p,[0,L]^d} \le L \sum_{i=1}^{d} \left| \frac{\partial f}{\partial x_i} \right|_{0,p,[0,L]^d} \qquad \text{for } 1 \le p \le \infty \tag{3.48}$$

by induction on d. The right-hand side of (3.48) is bounded by $c_1(d,p)\, L\, |f|_{1,p,\Omega_L}$.

Consider the case $d = 1$. Since $f(x)$ has zero mean (i. e. $\int_0^L f(x)\, dx = 0$), there exists $x^* \in [0, L]$ such that $f(x^*) = 0$. The Fundamental Theorem of Calculus then gives

$$f(x) = \int_{x^*}^{x} \frac{\partial f}{\partial x}\, dx \quad \Rightarrow \quad |f(x)| \le \int_0^L \left| \frac{\partial f}{\partial x} \right| dx\ . \tag{3.49}$$

Thus we could immediately write $|f|_{0,\infty,[0,L]} \le |f|_{1,1,[0,L]}$ and use the natural ordering of norms ($p \le p' \Rightarrow ||\phi||_p \le c(L)\, ||\phi||_{p'}$) to get the required result for $d = 1$ or we use a Hölder inequality on (3.49) to get

$$|f(x)|^p \le L^{p-1} \left(\int_0^L \left| \frac{\partial f}{\partial x} \right|^p dx \right) \tag{3.50}$$

$$\Rightarrow |f|_{0,p,[0,L]} \le L\, |f|_{1,p,[0,L]}\ . \tag{3.51}$$

Now assume the result (3.48) to be true for $d - 1$.

Consider $f(x_1, \ldots, x_{d-1}, x_d)$ and assume f has mean zero on $[0, L]^d$. Write $f = h + g$ where we define g as

$$g(x_1, \ldots, x_{d-1}) = \frac{1}{L} \int_0^L f(x_1, \ldots, x_{d-1}, x_d)\, dx_d \tag{3.52}$$

We note that g has mean zero on $[0, L]^{d-1}$.
From the definition of h it is obvious that

$$\int_0^L h(x_1, \ldots, x_d)\, dx_d = 0 \tag{3.53}$$

and so by analogy with the one dimensional case there exists $x_d^* \in [0, L]$ such that

$$h(x_1, \ldots, x_{d-1}, x_d^*) = 0 .$$

$$\begin{aligned}
\Rightarrow |h(x_1, \ldots, x_d)| &= \left| \int_{x_d^*}^{x_d} \frac{\partial h}{\partial x_d}(x_1, \ldots, x_{d-1}, x_d') dx_d' \right| \\
&\leq \int_0^L \left| \frac{\partial h}{\partial x_d} \right| dx_d \\
&= \int_0^L \left| \frac{\partial f}{\partial x_d} \right| dx_d
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \int_{[0,L]^d} |h(x)|^p\, dx &\leq L \int_{[0,L]^{d-1}} \left(\int_0^L \left| \frac{\partial f}{\partial x_d} \right| dx_d \right)^p dx_1 \cdots dx_{d-1} \\
&\leq L^p \left| \frac{\partial f}{\partial x_d} \right|_{0,p,[0,L]^d}^p
\end{aligned}$$

using Hölder's inequality.

$$\Rightarrow |h|_{0,p,[0,L]^d} \leq L \left| \frac{\partial f}{\partial x_d} \right|_{0,p,[0,L]^d} . \tag{3.54}$$

Now, since g has zero mean and we assume that the $d-1$ case (for (3.48)) is true,

$$\begin{aligned}
|g|_{0,p,[0,L]^{d-1}} &\leq L \sum_{i=1}^{d-1} \left| \frac{\partial g}{\partial x_i} \right|_{0,p,[0,L]^{d-1}} \\
&\leq L^{1-1/p} \sum_{i=1}^{d-1} \left| \frac{\partial f}{\partial x_i} \right|_{0,p,[0,L]^d}
\end{aligned} \tag{3.55}$$

where we have used (3.52) on the right-hand side above.
Now integrating with respect to x_d in (3.55) we get

$$|g|_{0,p,[0,L]^d} \leq L \sum_{i=1}^{d-1} \left| \frac{\partial f}{\partial x_i} \right|_{0,p,[0,L]^d} . \tag{3.56}$$

Since naturally $|f|_{0,p,[0,L]^d} \leq |g|_{0,p,[0,L]^d} + |h|_{0,p,[0,L]^d}$, we get

$$|f|_{0,p,[0,L]^d} \leq L \sum_{i=1}^{d-1} \left| \frac{\partial f}{\partial x_i} \right|_{0,p,[0,L]^d} + L \left| \frac{\partial f}{\partial x_d} \right|_{0,p,[0,L]^d} .$$

□

Corollary 3.2.22 *For functions which have mean zero on Ω_L, the semi-norm $|u|_{m,p,\Omega_L}$ for $1 \le p < \infty$ is equivalent to the full Sobolev norm $\|u\|_{m,p,\Omega_L}$ on $W^{m,p}(\Omega_L)$.*

Proof of corollary: This follows from the theorem above combined with the results in Section 3.2.6. □

Theorem 3.2.23 *Suppose $f \in C^{m+1}_{per}(\Omega_L)$, then for all $m \ge 1$,*

$$|f|_{m,p,\Omega_L} \le c_2(d,p,m)\, L\, |f|_{m+1,p,\Omega_L} \tag{3.57}$$

with $1 \le p \le \infty$.

Proof: Consider the case $m = 1$ in the statement of the theorem. Assume $f \in C^2_{per}(\Omega_L)$. Hence there exists $y \in [0, L]$ such that

$$\partial_i f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_d) = 0\ , \quad \forall\, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d\ .$$

Thus we can write

$$\frac{\partial f}{\partial x_i}(x_1, \ldots, x_d) = \int_y^{x_i} \frac{\partial^2 f}{\partial x_i^2}(\xi_1, \ldots, \xi_d)\, d\xi_i \tag{3.58}$$

for all $i = 1, \ldots, d$.

$$\begin{aligned}
&\Rightarrow \quad |\partial_i f(x)| \le \int_0^L |\partial_i^2 f| d\xi_i \\
&\Rightarrow \quad |\partial_i f(x)|^p \le L^{p-1} \int_0^L |\partial_i^2 f|^p d\xi_i \\
&\Rightarrow \quad \sum_{i=1}^d \int_{\Omega_L} |\partial_i f(x)|^p\, dx \le L^p \sum_{i=1}^d \int_{\Omega_L} |\partial_i^2 f|^p\, dx \\
&\Rightarrow \quad |f|_{1,p,\Omega_L} \le L\, |f|_{2,p,\Omega_L}
\end{aligned}$$

where we note that this in fact holds $\forall\ 1 \le p \le \infty$.

The result $|D^{m-1}g|_{1,p,\Omega_L} \le L\, |D^{m-1}g|_{2,p,\Omega_L}$ is now obvious, and so when we sum over the multi-index and re-translate all the derivative multi-indicies into the norm, we get (3.57) with a non-trivial constant $c_2(d, p, m)$. □

Corollary 3.2.24 *For functions $f \in C^m_{per}(\Omega_L)$, for all $m \ge 1$ with mean zero, $|f|_{m,p,\Omega_L}$ is equivalent to the functional $\|f\|_{m,p,\Omega_L}$, for $1 \le p \le \infty$.*

3.3 The Gagliardo-Nirenberg Inequality

We will mainly be considering functions with compact support in some domain $\Omega \subseteq \mathbb{R}^d$ for which we will prove an inequality of the Gagliardo-Nirenberg type for semi-norms. Two other versions are also presented, with the use of extension theorems – the last of which is *most* relevant to subsequent chapters. However, I have outlined some of the proofs in as great a generality as possible, in particular, presenting some examples of how one might prove the Gagliardo-Nirenberg inequality for a bounded domain such as $\Omega_L \equiv [0, L]^d$ (with no specified boundary conditions) without resorting to extension theorems.

Remark: Note that the operator of zero extension outside $\Omega \subseteq \mathbb{R}^d$ maps $W^{m,p}_0(\Omega)$ isometrically *into $W^{m,p}(\mathbb{R}^d)$.*

3.3.1 For Functions with Compact Support

Theorem 3.3.1 (Gagliardo-Nirenberg) *Let q, r be any real numbers satisfying $1 \le q, r \le \infty$. For any integer j, $0 \le j < m$, and for any a in the interval $j/m \le a \le 1$, set*

$$\frac{1}{p} = \frac{j}{d} + a\left(\frac{1}{r} - \frac{m}{d}\right) + \frac{1-a}{q} . \tag{3.59}$$

Assume $m - j - d/r$ is not a non-negative integer.

For $u \in C_0^m(\mathbb{R}^d)$ we have

$$|u|_{j,p,\mathbb{R}^d} \le c|u|^a_{m,r,\mathbb{R}^d}\, |u|^{1-a}_{0,q,\mathbb{R}^d} . \tag{3.60}$$

If $m - j - d/r$ is a non-negative integer, then the above inequality holds only for $a = j/m$.

Remark: We see that we can use the density results outlined in the last section, namely, that $C_0^\infty(\mathbb{R}^d) \cap W^{m,r}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ is dense in $W^{m,r}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ for $1 \le q, r < \infty$, i.e. we can assert that given $u \in W^{m,r}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ and $\epsilon > 0$, there exists $\phi \in C_0^\infty(\mathbb{R}^d) \cap W^{m,r}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ such that

$$|u - \phi|_{m,r} < \epsilon \ \text{ and } \ |u - \phi|_q < \epsilon .$$

(See for example Friedman, Section 10.) This allows us to relax the assumption in the above theorem from $u \in C_0^m(\mathbb{R}^d)$ to $u \in W^{m,r}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. It is important to note that these density results DO NOT extend to $q, r = \infty$.

Remark: The theorem above is equally valid for $u \in C_0^m(\Omega)$ for some bounded domain $\Omega \subset \mathbb{R}^d$ (see proof). Consequently, we can use that $C_0^m(\Omega) \cap W_0^{m,r}(\Omega) \cap L_0^q(\Omega)$ is dense in $W_0^{m,r}(\Omega) \cap L_0^q(\Omega)$ for $1 \le q, r < \infty$ to relax our assumption to $u \in W_0^{m,r}(\Omega) \cap L_0^q(\Omega)$. Again we exclude $q, r = \infty$.

Also Note: For bounded domain Ω, the semi-norm $|\cdot|_{m,p,\Omega}$ is equivalent to the full norm $\|\cdot\|_{m,p,\Omega}$ on $W_0^{m,p}(\Omega)$, $1 \le p < \infty$ (use the L^p form of Poincaré inequality (3.36) for functions with compact support – see previous section).

3.3.2 For Functions on Ω_L – No Specified Boundary Conditions

With the extension theorems of Section 3.2.9 and the remarks of the last section in mind, we can prove the following two alternative results,

Theorem 3.3.2 *Under all the conditions outlined in Theorem 3.3.1 (including the special treatment of the case when $m - j - d/r$ is a non-negative integer) but, where we need only assume $u \in W^{m,r}(\Omega_L) \cap L^q(\Omega_L)$ and with the restriction $1 < q, r < \infty$, we have*

$$|u|_{j,p,\Omega_L} \le C\|u\|^a_{m,r,\Omega_L}\, \|u\|^{1-a}_{0,q,\Omega_L} . \tag{3.61}$$

Proof: The result follows immediately from Theorem 3.3.1, where we include the remark that if we restrict ourselves to $1 \le q, r < \infty$ we need only assume $u \in W^{m,r}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, and we make additional use of the extension theorems in Section 3.2.9, indeed:

Assume $u \in W^{m,r}(\Omega_L) \cap L^q(\Omega_L)$. By Section 3.2.9 we know there exists an extension operator $E : W^{m,r}(\Omega_L) \cap L^q(\Omega_L) \to W^{m,r}(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$. Apply Theorem 3.3.1 to Eu, to get

$$|Eu|_{j,p,\mathbb{R}^d} \le C\|Eu\|^a_{m,r,\mathbb{R}^d}\|Eu\|^{1-a}_{0,q,\mathbb{R}^d} \tag{3.62}$$

and so by the properties of the extension operator E, we see that

$$|u|_{j,p,\Omega_L} \leq |Eu|_{j,p,\mathbb{R}^d} \leq C_2 \|u\|^a_{m,r,\Omega_L} \|u\|^{1-a}_{0,q,\Omega_L}$$

and we have proved the theorem. □

However, if we are willing to make some alternative assumptions about the type of domain we are considering, we can apply Friedman's version of the extension theorem in the following way:

Theorem 3.3.3 *Let Ω be a bounded domain with the segment property such that $\partial\Omega$ is C^m. Then, under all the conditions outlined in Theorem 3.3.1 (including the special treatment of the case when $m - j - d/r$ is a non-negative integer) but, where we need only assume $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$ with $1 \leq q, r < \infty$, we have*

$$|u|_{j,p,\Omega} \leq C \|u\|^a_{m,r,\Omega} \|u\|^{1-a}_{0,q,\Omega} \ . \tag{3.63}$$

Proof: Assume $u \in C^m(\overline{\Omega})$. By the extension theorem in Section 3.2.9 there exists an extension operator $E : C^m(\overline{\Omega}) \to C^m_0(\mathbb{R}^d)$ and so, since $Eu \in C^m_0(\mathbb{R}^d)$, we can apply Theorem 3.3.1 to Eu to get

$$|Eu|_{j,p,\mathbb{R}^d} \leq C \|Eu\|^a_{m,r,\mathbb{R}^d} \|Eu\|^{1-a}_{0,q,\mathbb{R}^d} \tag{3.64}$$

and so

$$|u|_{j,p,\Omega} \leq |Eu|_{j,p,\mathbb{R}^d} \leq C_2 \|u\|^a_{m,r,\Omega} \|u\|^{1-a}_{0,q,\Omega} \ . \tag{3.65}$$

We can now use that for the domain considered, $C^\infty(\Omega) \cap W^{m,r}(\Omega) \cap L^q(\Omega)$ is dense in $W^{m,r}(\Omega) \cap L^q(\Omega)$. □

3.3.3 For Space-periodic, Mean-zero Functions on Ω_L

Theorem 3.3.4 *Under all the conditions outlined in Theorem 3.3.1 (including the special treatment of the case when $m - j - d/r$ is a non-negative integer) but, where we assume $u \in W^{m,r}(\Omega_L) \cap L^q(\Omega_L)$, with the restriction $1 < q, r < \infty$, and that our functions u, are mean-zero and periodic on Ω_L, we have*

$$|u|_{j,p,\Omega_L} \leq c_1 |u|^a_{m,r,\Omega_L} |u|^{1-a}_{0,q,\Omega_L} \ . \tag{3.66}$$

Proof: We use Theorem 3.3.2 together with the fact (proved in Section 3.2.10) that $|u|_{m,p,\Omega_L}$ is an equivalent norm to $\|u\|_{m,p,\Omega_L}$ (when $1 \leq p < \infty$) for the space of functions considered. □

Important Remark: There are many generalizations of the Sobolev spaces considered and also corresponding extension theorems. See Chapter 7 of Adams [1], and also the book by S.M. Nikol'skiĭ, *Approximation of Functions of Several Variables and Imbedding Theorems*, Springer-Verlag, 1975 – in particular Theorem 1 on page 381 of this book.

For the cases $r = 1$ or ∞, we will add a term $\tilde{c}(\Omega_L)\|u\|_{\tilde{q}}$ (for any $\tilde{q} > 0$) with $\tilde{c}$ constant, to the right-hand side in (3.66) – see Nirenberg [70]. This could prove to be unnecessary. If we apply (3.66) to the Navier-Stokes equations for either of these cases, this will always be a term of lower order which we will ignore.

3.4 Proof of the Gagliardo-Nirenberg Theorem

We will now proceed to prove Theorem 3.3.1.

3.4.1 Case $a = 1$

We shall consider the case $a = 1$. Also, let us specialize to the case $j = 0$, $m = 1$, since we can use induction on m to get the general case.

Lemma 3.4.1 *Assume $r \neq d$. For all $u \in C^1(\Omega_L)$ we have*

$$|u|_{0,\frac{dr}{d-r},\Omega_L} \leq \frac{2d^{3/2}r}{d^{1/r}|d-r|}\,|u|_{1,r,\Omega_L} + \frac{1}{L}|u|_{0,r,\Omega_L}\ ; \tag{3.67}$$

for $u \in C_0^1(\mathbb{R}^d)$ we have

$$|u|_{0,\frac{dr}{d-r},\mathbb{R}^d} \leq \frac{2d^{3/2}r}{d^{1/r}|d-r|}\,|u|_{1,r,\mathbb{R}^d}\ . \tag{3.68}$$

Proof of lemma: We can immediately distinguish two distinct cases: $r < d$ and $r > d$. The case $r = d$ is the exceptional case and we *do* prove this case below, however, we *do not* investigate it further.

(A) Case $r < d$

Lemma 3.4.2 *For all $u \in C^1(\Omega_L)$ we have*

$$|u|_{0,\frac{dr}{d-r},\Omega_L} \leq \frac{r(d-1)}{2d^{1/r}(d-r)}|u|_{1,r,\Omega_L} + \frac{1}{L}|u|_{0,r,\Omega_L}\ ; \tag{3.69}$$

for $u \in W^{1,r}(\Omega_L)$ (but in this case (only) we need to restrict $r < \infty$),

$$||u||_{0,\frac{dr}{d-r},\Omega_L} \leq \frac{2^{1-1/r}r(d-1)}{(d-r)}\,||u||_{1,r,\Omega_L}\ ; \tag{3.70}$$

for $u \in C_0^1(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ we have

$$|u|_{0,\frac{dr}{d-r},\Omega} \leq \frac{r(d-1)}{2d^{1/r}(d-r)}|u|_{1,r,\Omega}\ . \tag{3.71}$$

Proof: Assume $u \in C^1(\Omega_L)$.

Definition 3.4.1 *By λ, we denote a real interval of length $|\lambda| > 0$.*

Definition 3.4.2 *By $\lambda_i \equiv [a_i, b_i]$ we denote a real interval of length $|\lambda_i| = b_i - a_i > 0$ along the x_i direction. In general, $\lambda_i = \lambda_i(x_j)$, $\forall j \neq i$.*

Definition 3.4.3 *Let ξ_i^1 and ξ_i^2 denote two arbitrary distinct points in the interval λ_i, i.e. $\xi_i^1, \xi_i^2 \in \lambda_i$ with $\xi_i^1 \neq \xi_i^2$.*

By the Fundamental Theorem of Calculus we see that:

$$u(x_1, \ldots, x_i, \ldots, x_d) - u(\cdots, \xi_i^1, \cdots) = \int_{\xi_i^1}^{x_i} \frac{\partial u}{\partial x_i} dx_i\ , \tag{3.72}$$

$$u(\cdots, x_i, \cdots) - u(\cdots, \xi_i^2, \cdots) = \int_{\xi_i^2}^{x_i} \frac{\partial u}{\partial x_i} dx_i \tag{3.73}$$

for $i = 1, \ldots, d$ and where $x_i \in \lambda_i$.

Without loss of generality, assume $\xi_i^2 > \xi_i^1$, and combining the above,

$$2u(x) - u(\cdots, \xi_i^1, \cdots) - u(\cdots, \xi_i^2, \cdots) = \int_{\xi_i^1}^{x_i} \frac{\partial u}{\partial x_i} dx_i - \int_{x_i}^{\xi_i^2} \frac{\partial u}{\partial x_i} dx_i \tag{3.74}$$

which implies

$$|2u(x) - u(\cdots, \xi_i^1, \cdots) - u(\cdots, \xi_i^2, \cdots)| \leq \int_{\xi_i^1}^{\xi_i^2} \left| \frac{\partial u}{\partial x_i} \right| dx_i \tag{3.75}$$

so

$$|2u(x)| \leq \int_{\lambda_i} \left| \frac{\partial u}{\partial x_i} \right| dx_i + |u(\cdots, \xi_i^1, \cdots)| + |u(\cdots, \xi_i^2, \cdots)| \; . \tag{3.76}$$

Now integrate both sides with respect to ξ_i^1 and ξ_i^2 over the range of the interval

$$\Rightarrow \quad |u(x)| \leq \frac{1}{2} \int_{\lambda_i} \left| \frac{\partial u}{\partial x_i} \right| dx_i + \frac{1}{|\lambda_i|} \int_{\lambda_i} |u| dx_i \; . \tag{3.77}$$

Remark: Note what happens when u is a non-zero constant on λ_i.

From (3.77) we can write

$$|u(x)|^{\frac{d}{d-1}} \leq \prod_{i=1}^{d} \left(\frac{1}{2} \int_{\lambda_i} \left| \frac{\partial u}{\partial x_i} \right| dx_i + \frac{1}{|\lambda_i|} \int_{\lambda_i} |u| dx_i \right)^{\frac{1}{d-1}} \tag{3.78}$$

Let us suppose $|\lambda_1| = |\lambda_2| = \ldots = |\lambda_d| = |\lambda|$ and call $\Omega_\lambda \equiv [0, |\lambda|]^d \subset \mathbb{R}^d$.

Integrate (3.78) with respect to x_1 up to x_d over Ω_λ and apply Hölder's inequality each time,

$$\int_{\Omega_\lambda} |u(x)|^{\frac{d}{d-1}} dx \leq \prod_{i=1}^{d} \left(\frac{1}{2} \int_{\Omega_\lambda} \left| \frac{\partial u}{\partial x_i} \right| dx + \frac{1}{|\lambda|} \int_{\Omega_\lambda} |u| dx \right)^{\frac{1}{d-1}} \; . \tag{3.79}$$

Now using Young's inequality we get

$$\int_{\Omega_\lambda} |u(x)|^{\frac{d}{d-1}} dx \leq \left(\frac{1}{2d} \sum_{i=1}^{d} \int_{\Omega_\lambda} \left| \frac{\partial u}{\partial x_i} \right| dx + \frac{1}{|\lambda|} \int_{\Omega_\lambda} |u| dx \right)^{\frac{d}{d-1}} \; . \tag{3.80}$$

Remark: Again, consider the case u constant on Ω_λ, we get $|u|_{0,\frac{d}{d-1},\Omega_\lambda} \leq \frac{1}{|\lambda|} |u|_{0,1,\Omega_\lambda}$.

It seems obvious to take $|\lambda| = L$ so as to minimize $1/|\lambda|$, and so we see that

$$|u|_{0,\frac{d}{d-1},\Omega_L} \leq \frac{1}{2d} |u|_{1,1,\Omega_L} + \frac{1}{L} |u|_{0,1,\Omega_L} \; . \tag{3.81}$$

Now, the usual norm for $W^{m,p}(\Omega)$ is normally taken to be (as stated previously)

$$||u||_{m,p} = \left(\sum_{0 \leq j \leq m} |u|_{j,p}^p \right)^{1/p} \tag{3.82}$$

and so with this definition for the norm (3.81) would give us

$$||u||_{0,\frac{d}{d-1},\Omega_L} \leq \max \left[\frac{1}{2d}, \frac{1}{L} \right] ||u||_{1,1,\Omega_L} \; . \tag{3.83}$$

An equivalent norm to that defined above (and perhaps more appropriate as it is dimensionally uniform) would be

$$\|u\|^{(2)}_{1,1,\Omega_L} = |u|_{1,1,\Omega_L} + \frac{1}{L}|u|_{0,1,\Omega_L} \tag{3.84}$$

in which case we would write

$$\|u\|_{0,\frac{d}{d-1},\Omega_L} \leq \|u\|^{(2)}_{1,1,\Omega_L} \,. \tag{3.85}$$

Note that if u had compact support in any domain $\Omega \subseteq \mathbb{R}^d$ then the second term on the right-hand side in (3.81) would be non-existent (go back to step before (3.74) and use the zero boundary conditions) and we could write

$$|u|_{0,\frac{d}{d-1},\Omega} \leq \frac{1}{2d}|u|_{1,1,\Omega} \,. \tag{3.86}$$

Now, for $r < d$, consider the function, $v = |u|^{\frac{r(d-1)}{d-r}}$, which is continuously differentiable if $r > 1$.

If we substitute for v in (3.81) and apply a Hölder inequality we see that we get

$$\begin{aligned}\left(\int_{\Omega_L} |u|^{\frac{dr}{d-r}}dx\right)^{1-1/d} &\leq \frac{r(d-1)d^{1-1/r}}{2d(d-r)}|u|_{1,r,\Omega_L}\left(\int_{\Omega_L} |u|^{\frac{dr}{d-r}}dx\right)^{1-1/r} \\ &\quad + \frac{1}{L}\left(\int_{\Omega_L} |u|^{\frac{r(d-1)}{d-r}}dx\right)\end{aligned} \tag{3.87}$$

and so if we apply a Hölder inequality to the last term as follows

$$\int_{\Omega_L} |u|^{\frac{r(d-1)}{d-r}}dx = \int_{\Omega_L} |u|^{\frac{d(r-1)}{d-r}}\,|u|\,dx \leq \left(\int_{\Omega_L} |u|^{\frac{dr}{d-r}}dx\right)^{1-1/r}\left(\int_{\Omega_L} |u|^r dx\right)^{1/r}\,, \tag{3.88}$$

we get

$$|u|_{0,\frac{dr}{d-r},\Omega_L} \leq \frac{r(d-1)}{2d^{1/r}(d-r)}|u|_{1,r,\Omega_L} + \frac{1}{L}|u|_{0,r,\Omega_L} \,. \tag{3.89}$$

Thus, if we define $\|u\|_{1,r,\Omega_L} = (|u|^r_{1,r,\Omega_L} + |u|_{0,r,\Omega_L})^{1/r}$ and more naturally,

$$\|u\|^{(2)}_{1,r,\Omega_L} = (|u|^r_{1,r,\Omega_L} + \frac{1}{L^r}|u|^r_{0,r,\Omega_L})^{1/r}$$

then we get,

$$\|u\|_{0,\frac{dr}{d-r},\Omega_L} \leq 2^{1-1/r}\max\left[\frac{r(d-1)}{2d^{1/r}(d-r)},\frac{1}{L}\right]\|u\|_{1,r,\Omega_L} \tag{3.90}$$

and

$$\|u\|^{(2)}_{0,\frac{dr}{d-r},\Omega_L} \leq 2^{1-1/r}\frac{r(d-1)}{(d-r)}\|u\|^{(2)}_{1,r,\Omega_L} \,. \tag{3.91}$$

Remark: At this point we can make use of the density theorems outlined in the previous section, i. e. that $C^\infty(\Omega_L) \cap W^{m,p}(\Omega_L)$ is dense in $W^{m,p}(\Omega_L)$, so that we need only assume $u \in W^{1,r}(\Omega_L)$ for the case when we assume no specific boundary conditions – the Gagliardo-Nirenberg inequality in terms of full norms.

Also, if $u \in C^1_0(\Omega)$, $\Omega \subseteq \mathbb{R}^d$

$$|u|_{0,\frac{dr}{d-r},\Omega} \leq \frac{r(d-1)}{2d^{1/r}(d-r)}|u|_{1,r,\Omega} \,.$$

Remark: Since $r < d$, and $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{1,r}(\mathbb{R}^d)$ $(1 \leq r < \infty)$, we can relax our assumption to $u \in W^{1,r}(\mathbb{R}^d)$. □

(B) Case $r > d$.

We see that it suffices to prove

Lemma 3.4.3 *For $r > d$ and $u \in C^1(\Omega_L)$*

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{1-d/r}} \leq \frac{2rd^{3/2-1/r}}{r-d}|u|_{1,r,\Omega_L} . \tag{3.92}$$

Proof:

Notation: *For $0 < t < 1$, Ω_{Lt} = cube of edge Lt with faces parallel to Ω_L and such that $\overline{\Omega}_{Lt} \subset \Omega_L$.*

Let $x, y \in \Omega_L$ with $|x-y| = \sigma L < L$, $(\sigma < 1)$. Hence, there exists a cube $\Omega_{\sigma L}$ with $x, y \in \overline{\Omega}_{\sigma L} \subset \Omega_L$.

If $z \in \Omega_{\sigma L}$, then

$$u(x) = u(z) - \int_0^1 \frac{d}{dt}u(x+t(z-x))dt \tag{3.93}$$

$$\Rightarrow |u(x)-u(z)| \leq d^{1/2}\sigma L \int_0^1 |\nabla u(x+t(z-x))|dt \tag{3.94}$$

where the $\nabla = \nabla_w$ operator is with respect to $w = x + t(z-x)$. Now, since

$$\int_{\Omega_{\sigma L}} dz = (\sigma L)^d \tag{3.95}$$

we see that

$$\begin{aligned}\left|u(x) - \frac{1}{(\sigma L)^d}\int_{\Omega_{\sigma L}} u(z)dz\right| &= \left|\frac{1}{(\sigma L)^d}\int_{\Omega_{\sigma L}} (u(x)-u(z))dz\right| \\ &\leq \frac{d^{1/2}}{(\sigma L)^{d-1}}\int_{\Omega_{\sigma L}} dz \int_0^1 |\nabla u(x+t(z-x))|dt . \end{aligned} \tag{3.96}$$

Thus with $w = x + t(z-x)$, we get

$$\begin{aligned}\left|u(x) - \frac{1}{(\sigma L)^d}\int_{\Omega_{\sigma L}} u(z)dz\right| &\leq \frac{d^{1/2}}{(\sigma L)^{d-1}}\int_{\Omega_{\sigma Lt}}\int_0^1 t^{-d}|\nabla u(w)|dt\, dw \\ &\leq \frac{d^{1/2}}{(\sigma L)^{d-1}}\int_0^1 t^{-d}\int_{\Omega_{\sigma Lt}} |\nabla u(z)|\, dz\, dt \\ &\leq \frac{d^{1/2}}{(\sigma L)^{d-1}}\left[\int_{\Omega_L} |\nabla u(z)|^r dz\right]^{1/r}\int_0^1 \left[\int_{\Omega_{\sigma Lt}} dz\right]^{1/q} t^{-d}\, dt\end{aligned}$$

by using Hölder's inequality with $\frac{1}{r}+\frac{1}{q} = 1$. Also note that, by $|\nabla u(w)|$ we mean $\sum_{i=1}^{d} |\frac{\partial}{\partial w_i}u(w)|$.

Then we see that

$$\begin{aligned}\left|u(x)-\frac{1}{(\sigma L)^d}\int_{\Omega_{\sigma L}}u(z)dz\right| &\leq \frac{d^{1/2}}{(\sigma L)^{d-1}}|\nabla u|_{0,r,\Omega_L}\int_0^1(\text{vol }\Omega_{\sigma Lt})^{1/q}t^{-d}\,dt\\ &= \frac{d^{1/2}}{(\sigma L)^{d-1}}\int_0^1(\sigma L)^{d/q}\,t^{d/q-d}dt\,|\nabla u|_{0,r,\Omega_L}\\ &= (\sigma L)^{1-d/r}\,K(d,r)\,|\nabla u|_{0,r,\Omega_L} \qquad (3.97)\end{aligned}$$

where

$$K(d,r)=d^{1/2}\int_0^1 t^{-d/r}dt=\frac{d^{1/2}r}{r-d}\,. \qquad (3.98)$$

A similiar inequality holds for y in place of x, and so

$$\begin{aligned}|u(x)-u(y)| &\leq \left|u(x)-\frac{1}{(\sigma L)^d}\int_{\Omega_{\sigma L}}u(z)dz\right|+\left|u(y)-\frac{1}{(\sigma L)^d}\int_{\Omega_{\sigma L}}u(z)dz\right|\\ &\leq 2(\sigma L)^{1-d/r}\,K(d,r)\,|\nabla u|_{0,r,\Omega_L} \qquad (3.99)\end{aligned}$$

So that in fact, since $|\nabla u|_{0,r,\Omega_L}\leq d^{1-1/r}|u|_{1,r,\Omega_L}$, we get

$$\frac{|u(x)-u(y)|}{|x-y|^{1-d/r}}\leq 2K(d,r)d^{1-1/r}|u|_{1,r,\Omega_L} \qquad (3.100)$$

but this is for $|x-y|=\sigma L$ with $\sigma<1$.

Now, let $x,y\in\Omega_L$ and $|x-y|=\sigma L$ with $1\leq\sigma\leq d^{1/2}$. If $z\in\Omega_L$ then we see that all arguments coincide with those outlined above (with L replacing σL in all instances) and so the inequality corresponding to (3.97) is

$$\left|u(x)-\frac{1}{L^d}\int_{\Omega_L}u(z)dz\right|\leq L^{1-d/r}\,K(d,r)\,|\nabla u|_{0,r,\Omega_L}\,. \qquad (3.101)$$

So, if we use that $L\leq\sigma L$ for $1\leq\sigma\leq d^{1/2}$ then we see that we can proceed exactly as before to get the same inequality . □

Remark: This argument also holds (in a slightly modified form) for $u\in C_0^1(\Omega)$, $\Omega\subseteq\mathbb{R}^d$ (with some extra assumptions about the type of domain), but, in particular, for $\Omega=\mathbb{R}^d$.

(C) Case $r=d$.

Lemma 3.4.4 *For $q\geq 1$, $\frac{p(d-1)}{d}\geq 1$, $p-q\geq\frac{d}{(d-1)}$ and $u\in C^1(\Omega_L)$ we have*

$$|u|_{0,p,\Omega_L}\leq c_d\,|u|_{1,d,\Omega_L}^{1-q/p}\,|u|_{0,q,\Omega_L}^{q/p}+L^{-(1-q/p)}|u|_{0,\frac{pd}{p-q+d},\Omega_L}\,, \qquad (3.102)$$

whereas for $u\in C_0^1(\Omega)$,

$$|u|_{0,p,\Omega_L}\leq c_d\,|u|_{1,d,\Omega_L}^{1-q/p}\,|u|_{0,q,\Omega_L}^{q/p}\,. \qquad (3.103)$$

Remark: Note the restrictions outlined, and also that $p\geq q$ implies $p\geq\frac{dp}{p-q+d}$.

Proof of lemma: Apply (3.81) to the continuously differentiable function $v=|u|^{\frac{p(d-1)}{d}}$ (provided $\frac{p(d-1)}{d}\geq 1$), and use Hölder's inequality,

$$\begin{aligned}\left(\int_{\Omega_L}|u|^p dx\right)^{1-1/d} &\leq c'_d\sum_{i=1}^{d}\left(\int_{\Omega_L}\left|\frac{\partial u}{\partial x_i}\right|^d dx\right)^{1/d}\left(\int_{\Omega_L}|u|^{p-\frac{d}{d-1}}dx\right)^{1-1/d}\\ &\quad+\frac{1}{L}\int_{\Omega_L}|u|^{\frac{p(d-1)}{d}}dx \qquad (3.104)\end{aligned}$$

Now notice that, if we write

$$\begin{aligned}\left(\int_{\Omega_L} |u|^{p-\frac{d}{d-1}} dx\right)^{1-1/d} &= \left(\int_{\Omega_L} |u|^{\frac{(p-q)(p(d-1)-d)}{(p-q)(d-1)}} dx\right)^{1-1/d} \\ &\leq \left(\int_{\Omega_L} |u|^{\frac{P_1 qd}{(d-1)(p-q)}} dx\right)^{\frac{d-1}{P_1 d}} \left(\int_{\Omega_L} |u|^{\frac{P_2 p[(p-q)(d-1)-d]}{(d-1)(p-q)}} dx\right)^{\frac{d-1}{P_2 d}}\end{aligned}$$

where we have used Hölder's inequality with $\frac{1}{P_1}+\frac{1}{P_2}=1$. We choose $P_1 = \frac{(p-q)(d-1)}{d}$.

Note: this means we require $P_1 \geq 1$, i.e.

$$p-q \geq \frac{d}{d-1}\,. \tag{3.105}$$

So we get

$$\left(\int_{\Omega_L} |u|^{p-\frac{d}{d-1}} dx\right)^{1-1/d} \leq \left(\int_{\Omega_L} |u|^q dx\right)^{\frac{1}{p-q}} \left(\int_{\Omega_L} |u|^p dx\right)^{\frac{(p-q)(d-1)-d}{(p-q)d}}\,. \tag{3.106}$$

Again, we have to use an intricate Hölder inequality to deal with the term

$$\int_{\Omega_L} |u|^{\frac{p(d-1)}{d}} dx \leq \left(\int_{\Omega_L} |u|^{\frac{pQ_1((p-q)(d-1)-d)}{d(p-q)}} dx\right)^{1/Q_1} \left(\int_{\Omega_L} |u|^{\frac{pQ_2}{p-q}} dx\right)^{1/Q_2} \tag{3.107}$$

and where $\frac{1}{Q_1}+\frac{1}{Q_2}=1$ with $\frac{p}{p-q} \leq \frac{p(d-1)}{d}$ also true since we have the restriction (3.105).

So, if we choose, $Q_1 = \frac{d(p-q)}{(p-q)(d-1)-d}$, we find

$$\int_{\Omega_L} |u|^{\frac{p(d-1)}{d}} dx \leq \left(\int_{\Omega_L} |u|^p dx\right)^{\frac{(p-q)(d-1)-d}{d(p-q)}} \left(\int_{\Omega_L} |u|^{\frac{pd}{p-q+d}} dx\right)^{\frac{p-q+d}{d(p-q)}}\,. \tag{3.108}$$

And thus we see that (after using Hölder's inequality with $\sum_i$) we get the required result. □

Remark: This is the exceptional case, and we will not investigate it further!

Hence the proof of the original Lemma 3.4.1. □

We are now in a position to prove the following lemma:

Lemma 3.4.5 *For all $u \in C_0^m(\mathbb{R}^d)$, $0 \leq j < m$, $1 \leq r \leq \infty$ and $\forall\ m-j-d/r \notin \mathbb{N}\cup\{0\}$, we have*

$$|u|_{j,p,\mathbb{R}^d} \leq K(j,m,d,r)\,|u|_{m,r,\mathbb{R}^d} \tag{3.109}$$

where

$$\frac{1}{p} = \frac{1}{r} - \frac{m-j}{d}\,. \tag{3.110}$$

Proof: By induction on m. We have already proved that $\forall\ u \in C_0^1(\mathbb{R}^d)$, $r \neq d$, that (taking the worst of two constants)

$$|u|_{0,p} \leq \frac{2d^{3/2}r}{d^{1/r}|d-r|}\,|u|_{1,r} \quad \text{with} \quad p = \frac{dr}{d-r} \tag{3.111}$$

i.e. (3.109) is true for $m = 1$.

For all $u \in C_0^{\tilde{m}+1}(\mathbb{R}^d)$, $r \neq d$, we can obviously substitute $D^{\tilde{m}}u$ into (3.111) to get (for any $\tilde{m} \in \mathbb{N}$)

$$|u|_{\tilde{m},p} \leq K'(\tilde{m},d,r)\,|u|_{\tilde{m}+1,r} \ \text{ with } \ p = \frac{dr}{d-r} \tag{3.112}$$

and with $K'(\tilde{m},d,r)$ a *non-trivial* constant (since the generalized derivative D^n and gradient ∇^n are not the same operator under L^p-norms, $\forall\ 1 \leq p \leq \infty$).

Assume (3.109) is true for some m ($\forall\ 0 \leq j < m$). From (3.112) we can write

$$|u|_{m,r} \leq K'(m,d,r')\,|u|_{m+1,r'} \ \text{ with } \ \frac{1}{r} = \frac{1}{r'} - \frac{1}{d}\,. \tag{3.113}$$

Using this in (3.109) and noting that $r = dr'/(d-r')$, we soon see that we get (3.109) for $m+1$. □

Remark: It is important to note the restriction $m-j-d/r \notin \mathbb{N}\cup\{0\}$, as this condition allows us to exclude any occurance of the exceptional case (outlined in the previous section) in our induction proof. To clarify this point, let us examine how the induction proof proceeds when p is negative (when p is positive this problem is not present):

Given $p < 0$, we would first prove $|u|_{j,p} \leq c\,|u|_{j+1,r_1}$ with $1/p = 1/r_1 - 1/d$ and of course we exclude $r_1 = d$.

Note that $r_1 > d \geq 1$. We then iterate the right-hand side, which proceeds with the usual positive norms, and as we increase to m derivatives, this allows us to reduce our resulting $r = r_{m-j}$ so that

$$\frac{1}{p} = \frac{1}{r} - \frac{m-j}{d} \equiv -\frac{1}{d}\left[m - j - \frac{d}{r}\right] \tag{3.114}$$

and of course the condition $m-j-d/r \notin \mathbb{N}\cup\{0\}$ excludes $r_1 = d$. The rest is induction!

3.4.2 Case $a = j/m$

We shall consider the case $a = j/m$. Further, we shall concentrate on the case $j = 1, m = 2$, since we can use induction on m to get the general m case.

First consider the following Corollary to Lemma 3.2.12:

Corollary 3.4.6 *On an interval λ (length $|\lambda| > 0$) we have for $u \in C^2(\lambda)$*

$$\int_\lambda |Du|^p dx \leq c^p |\lambda|^{1+p-p/r} \left(\int_\lambda |D^2u|^r dx\right)^{p/r} + c^p 8^p |\lambda|^{-(1+p-p/r)} \left(\int_\lambda |u|^q dx\right)^{p/q} \tag{3.115}$$

with $c^p \equiv 2^{p-1}$, $\frac{1}{2q} + \frac{1}{2r} = \frac{1}{p}$ and $1 \leq q < \infty$, $1 < r < \infty$.

Proof: Proceeding exactly as in the Lemma 3.2.12, we can integrate (3.30) over λ to get

$$\int_\lambda |Du|^p dx \leq c^p |\lambda| \left(\int_\lambda |D^2u| dx\right)^p + c^p 8^p |\lambda|^{1-2p} \left(\int_\lambda |u| dx\right)^p\,. \tag{3.116}$$

Now, for $r \geq 1$, $q \geq 1$,

$$\int_\lambda |D^2u| dx \leq |\lambda|^{1-1/r} \left(\int_\lambda |D^2u|^r\right)^{1/r} \tag{3.117}$$

and

$$\int_\lambda |u|dx \le |\lambda|^{1-1/q}\left(\int_\lambda |u|^q dx\right)^{1/q} . \tag{3.118}$$

If we now identify $\frac{1}{p} = \frac{1}{2q} + \frac{1}{2r}$ and substitute (3.117) and (3.118) in (3.116), then the corollary follows. □

Lemma 3.4.7 *For all $u \in C_0^2(\mathbb{R}^d)$ and $1 \le r \le \infty$, $1 \le q \le \infty$, we have*

$$|u|_{1,p,\mathbb{R}^d} \le d^{1/2q}2^{5/2}\,|u|_{2,r,\mathbb{R}^d}^{1/2}|u|_{0,q,\mathbb{R}^d}^{1/2} \tag{3.119}$$

where $\dfrac{1}{2r} + \dfrac{1}{2q} = \dfrac{1}{p}$.

Proof: We will first prove the result (for $d = 1$) that: for $1 \le q < \infty$, $1 < r < \infty$,

$$\int |Du|^p dx \le c^p 2^{3p/2+1}\left(\int |D^2u|^r dx\right)^{p/2r}\left(\int |u|^q dx\right)^{p/2q} . \tag{3.120}$$

We see that it is sufficient for us to prove that: for any $L > 0$,

$$\int_0^L |Du|^p dx \le c^p 2^{3p/2+1}\left(\int_0^\infty |D^2u|^r dx\right)^{p/2r}\left(\int_0^\infty |u|^q dx\right)^{p/2q} . \tag{3.121}$$

Let $k \in \mathbb{N}$ and consider the interval $\lambda :\ 0 \le x \le L/k$ (assume $u \not\equiv 0$ in λ).

If the first term on the right-hand side of (3.115) is greater than the second, then take $\lambda_1 = \lambda$. If not, increase λ, keeping the left point fixed, until both terms on the right-hand side of (3.115) are equal – call this interval λ_1.

In the first case we have

$$\int_{\lambda_1} |Du|^p dx \le 2c^p\left(\frac{L}{k}\right)^{1+p-p/r}\left(\int_0^L |D^2u|^r dx\right)^{p/r} , \tag{3.122}$$

and in the latter

$$\int_{\lambda_1} |Du|^p dx \le 2c^p 8^{p/2}\left(\int_{\lambda_1} |D^2u|^r dx\right)^{p/2r}\left(\int_{\lambda_1} |u|^q dx\right)^{p/2q} . \tag{3.123}$$

If $\lambda_1 \ge L$ we have finished. If not, proceed to $\lambda_2, \lambda_3, \ldots$ etc. until $[0, L]$ is covered by the λ_i's. Case (3.122) happens at most $k' \le k$ times, and so in general we would get something which looks like

$$\begin{aligned}\int_0^L |Du|^p dx \quad &\le \quad k'2c^p\left(\frac{L}{k}\right)^{1+p-p/r}\left(\int_0^L |D^2u|^r dx\right)^{p/r} \\ &\quad +2c^p 8^{p/2}\left(\int_0^\infty |D^2u|^r dx\right)^{p/2r}\left(\int_0^\infty |u|^q dx\right)^{p/2q}\end{aligned} \tag{3.124}$$

where we have used Hölder's inequality. Taking the limit $k \to \infty$ then (3.121) follows.

For $d > 1$, apply (3.120) to each derivative $D_i u$ treating all the x_j $(j \ne i)$ as parameters. Then integrate with respect to the x_j $(j \ne i)$ and use Hölder's inequality. Then simply sum over i and take the p^{th} root.

The cases $q = \infty$, $r = 1, \infty$ follow by taking these limits in the inequality (3.120). □

Lemma 3.4.8 *For all $u \in C_0^m(\mathbb{R}^d)$, and integers $0 \le j < m$ we have*

$$|u|_{j,p,\mathbb{R}^d} \le c(d,j,m,q,r)\,|u|^{j/m}_{m,r,\mathbb{R}^d}|u|^{1-j/m}_{0,q,\mathbb{R}^d} \quad \textit{with} \quad \frac{1}{p} = \frac{j}{mr} + \frac{m-j}{mq} \tag{3.125}$$

Proof: By induction on $m \ge 2$.

We know the lemma is true for $m = 2$ and we will assume it is true (i.e. (3.125) holds) for some m ($\forall\ 0 \le j \le m-1$).

Note that if we *assume* $u \in C_0^{\tilde{m}+1}(\mathbb{R}^d)$ we can substitute $D^{\tilde{m}-1}u$ into (3.119) to get ($\forall\ \tilde{m} \in \mathbb{N}$)

$$|u|_{\tilde{m},r} \le c_1(d,\tilde{m},q',r')\,|u|^{1/2}_{\tilde{m}+1,r'}|u|^{1/2}_{\tilde{m}-1,q'} \quad \text{with} \quad \frac{1}{r} = \frac{1}{2r'} + \frac{1}{2q'} \tag{3.126}$$

and with $c_1(d,\tilde{m},q',r')$ a *non-trivial* constant.

Since by assumption (3.125) holds for $j = m-1$ (for the m we have chosen) we can write

$$|u|_{m-1,q'} \le c_2(d,m,q,r)\,|u|^{\frac{m-1}{m}}_{m,r}|u|^{\frac{1}{m}}_{0,q} \quad \text{with} \quad \frac{1}{q'} = \frac{m-1}{mr} + \frac{1}{mq}\ . \tag{3.127}$$

Substitute (3.127) into (3.126) *with* $\tilde{m} \equiv m$, to get

$$|u|_{m,r} \le c_3(d,m,q,r')\,|u|^{\frac{m}{m+1}}_{m+1,r'}|u|^{\frac{1}{m+1}}_{0,q} \quad \text{with} \quad \frac{1}{r} = \frac{m}{(m+1)r'} + \frac{1}{(m+1)q}\ . \tag{3.128}$$

If we substitute this into (3.125) then we get

$$|u|_{j,p} \le c_4(d,m,j,q,r')\,|u|^{\frac{j}{m+1}}_{m+1,r'}|u|^{1-\frac{j}{m+1}}_{0,q} \quad \text{with} \quad \frac{1}{p} = \frac{j}{(m+1)r'} + \frac{m+1-j}{(m+1)q} \tag{3.129}$$

i.e. the result of the lemma for $m+1$. □

3.4.3 Combining the $a = 1$ and $a = j/m$ Cases

In this subsection, we will bring together the last two subsections in order to prove the Gagliardo-Nirenberg inequality for general a between $j/m \le a \le 1$ (for $m - j - d/r$ not a non-negative integer). First, consider the following lemma

Lemma 3.4.9 *For $-\infty < \lambda \le \mu \le \nu < \infty$ and any $\Omega \subseteq \mathbb{R}^d$ we have*

$$|u|_{\frac{1}{\mu}} \le c_{(3)}\,|u|^{\frac{\nu-\mu}{\nu-\lambda}}_{\frac{1}{\lambda}}|u|^{\frac{\mu-\lambda}{\nu-\lambda}}_{\frac{1}{\nu}} \tag{3.130}$$

when the right-hand side is meaningful and where $c_{(3)}$ is independent of u.

Remark: The proof of the result for $\lambda > 0$ is an obvious application of Hölder's inequality, and also we note that $c_{(3)} = 1$ for this case.

For $\lambda < 0$, the situation is a little more delicate and $c_{(3)}$ is a non-trivial constant (but, of course, still independent of u) – see Nirenberg [70] or Friedman [33].

The two previous sections (in particular Lemmas 3.4.5 and 3.4.8) have shown us that

$$|u|_{j,p} \le c_{(1)}|u|_{m,r} \quad \text{with} \quad \frac{1}{p} = \frac{j}{d} + \left(\frac{1}{r} - \frac{m}{d}\right) \tag{3.131}$$

and

$$|u|_{j,p} \le c_{(2)} |u|_{m,r}^{j/m} |u|_{0,q}^{1-j/m} \quad \text{with} \quad \frac{1}{p} = \frac{j}{mr} + \frac{m-j}{mq}. \tag{3.132}$$

where we must assume $m - j - d/r \notin \mathbb{N} \cup \{0\}$ in (3.131).

There are several ways to proceed via (3.130).

Clearly, we can use (3.130) in the form

$$|u|_{j,p} \le c^*_{(3)} \, |u|^{\alpha}_{j,p_1} |u|^{\beta}_{j,p_2} \quad \text{with} \quad \alpha = \frac{1/p_2 - 1/p}{1/p_2 - 1/p_1} \quad \text{and} \quad \beta = 1 - \alpha \, . \tag{3.133}$$

Note that we *cannot* restrict both of p_1 and p_2 to being positive!

Now use (3.131) with p_2 and (3.132) with p_1 in the right-hand side of (3.133) to get

$$|u|_{j,p} \le c \, |u|_{m,r}^{\frac{\alpha j}{m}+\beta} |u|_{0,q}^{\alpha(1-j/m)} \tag{3.134}$$

with

$$\frac{1}{p_2} = \frac{j}{d} + \left(\frac{1}{r} - \frac{m}{d}\right) \tag{3.135}$$

and

$$\frac{1}{p_1} = \frac{j}{mr} + \frac{m-j}{mq} \, . \tag{3.136}$$

Identify

$$a := \frac{\alpha j}{m} + \beta = \alpha(j/m - 1) + 1 \tag{3.137}$$

and recall

$$\alpha = \frac{1/p_2 - 1/p}{1/p_2 - 1/p_1} \, . \tag{3.138}$$

Note $0 \le \alpha \le 1$ if and only if $j/m \le a \le 1$, and so solving (3.135)–(3.138) we get the result of Theorem 3.3.1. □

3.5 Conclusions and Further Work

Nearly all of the theory of this chapter is relevant for the rest of this thesis. The most important points are

1. We have defined Lebesgue and Sobolev spaces and outlined the elementary inequalities of Young and Hölder. We know $W^{m,p}(\Omega) = H^{m,p}(\Omega)$ and that $H^m = W^{m,2}$ is a Hilbert space. Some important density results were also introduced.

2. Semi-norms and results concerning equivalent norms for particular domains were discussed in detail as were extension theorems.

3. A Poincaré type of inequality was then proved and we were subsequently able to show that for mean-zero, space-periodic (and sufficiently regular) functions the semi-norm is equivalent to the usual Sobolev norm.

4. We stated and proved the Gagliardo-Nirenberg inequality for functions with compact support on $\mathbb{R}^d$. We then use extension theorems to prove a similar inequality in terms of full norms for functions with no specified boundary conditions. We combine this second theorem with the results in point 3 (above) to prove a Gagliardo-Nirenberg inequality in terms of semi-norms for mean-zero, space-periodic functions (also see all the other necessary assumptions in that theorem). We shall use this version repeatedly in the following chapters.

5. The original Gagliardo-Nirenberg inequality for functions with compact support is then proved, in as full generality as possible. Multiplicative constants are calculated as far as possible. If we know $c_{(1)}$, $c_{(2)}$ and $c_{(3)}$ (as outlined in the last section above), we can explicitly give the Gagliardo-Nirenberg constant for the general case.

Further Work:

1. Can we improve the extension theorem results and calculate the multiplicative constants explicitly (in addition to finding $c_{(1)}$, $c_{(2)}$ and $c_{(3)}$ explicitly)?

2. Is there any way to generalize or improve the inequality ? See, for example, Nirenberg [71].

Chapter 4

The Navier-Stokes Equations

In this chapter we introduce the Navier-Stokes equations and specify the initial and boundary value problem associated with them that we are going study. A detailed account of the necessary specific function spaces with some useful lemmas is then provided along with some initial a-priori estimates. We then determine the existence and uniqueness of "weak" and "strong" solutions to our problem.

Through this examination we realise that in the two dimensional case strong solutions exist for all time, but in the three dimensional case, we cannot extend the time interval of existence of strong solutions for arbitrary time, without making further assumptions – these consist of assuming the solution lies in a certain $L^p(\Omega_L)$ space for all time in order to show the existence of strong solutions for all time. Further, we show that strong solutions are also regular (i.e. smooth) solutions while they exist. This means that we can prove regular solutions for all time for the two dimensional case but only for a finite interval of time (depending on the initial data) in the three dimensional case (again without making the further assumptions mentioned).

We relate the finite-time existence of strong solutions to the well-posedness of the three dimensional problem and show that when we know strong solutions exist for all time and initial data, the problem is well-posed and we can prove the existence of a global attractor.

Lastly, we introduce an infinite set of a-priori estimates (these are norm estimates for smooth solutions) which we use to consider the minimum assumptions sufficient to show the existence of an attractor consisting of regular functions. The stage is then set for the analysis of Chapter 5, where we will try to relax these minimum assumptions.

4.1 Equations and Boundary Conditions

Consider the domain $\Omega_L \equiv [0, L]^d \subset \mathbb{R}^d$. Let $x = (x_1, \ldots, x_d) \in \Omega_L$ and $t \in \mathbb{R}, t > 0$.

Suppose $u(x,t) \equiv (u_1(x,t), \ldots, u_d(x,t)) : \Omega_L \times [0,\infty) \to \mathbb{R}^d$ is the velocity field for the fluid considered at some point (x,t) and $\mathcal{P} = \mathcal{P}(x,t) : \Omega_L \times [0,\infty) \to \mathbb{R}$ is the pressure field. Further, $f = f(x) : \Omega_L \to \mathbb{R}^d$ is the *given* time-independent forcing function (force per unit volume). Further in this thesis, we will assume that f is divergence-free and $f \in C^\infty(\Omega_L)^d$.

Assuming uniform density (which without loss of generality we rescale to unity) and with $\nu > 0$ as the viscosity (really the kinematic viscosity or dissipation coefficient), then if our flow is restricted to Ω_L, the incompressible Navier-Stokes equations are

$$\begin{aligned} u_t + (u \cdot \nabla)u &= \nu \Delta u - \nabla \mathcal{P} + f \\ \text{div } u &= 0 \end{aligned} \tag{4.1}$$

If we non-dimensionalize the above equations, they would have the same form, except now ν^{-1} would represent the Reynold's number for the flow, denoted by $\mathrm{Re} = UL/\nu$, where L is the representative length and U is the representative velocity used for non-dimensionalization.

For $d = 2$ we know that there exists a well-posed boundary value problem associated with (4.1) (cf. Temam [86]). While in the $d = 3$ case this problem is still open, it is sensible (by analogy with the $d = 2$ case) to complete (4.1) by the initial and boundary conditions (for $d = 2, 3$):

$$u(x,0) = u_0(x) \quad \forall\, x \in \Omega_L \tag{4.2}$$

$$u(x,t) = \phi(x,t) \quad \forall\, x \in \partial\Omega_L,\ t > 0$$

(4.1)–(4.2) with this boundary condition are the classical formulation of the Navier-Stokes equations.

Here we will assume space-periodic boundary conditions:

$$u(x + Le_i, t) = u(x,t) \quad \forall\, x \in \mathbb{R}^d,\ \forall\, t > 0 \tag{4.3}$$

where $e_1, \ldots, e_d$ is the canonical basis for $\mathbb{R}^d$ and L is the period in the i^{th} direction. These boundary conditions provide a simpler functional setting, but at the same time they leave the many mathematical difficulties unchanged.

Remark: The issue of boundary layers is not considered here.

We shall take $\int_{\Omega_L} f\,dx = 0$, so that there is no overall translation of the fluid, i.e.

$$\int_{\Omega_L} u\,dx = 0 \tag{4.4}$$

(the fluid has mean zero – see for example, Temam [86].)

4.2 The Functional Setting

In this section we introduce some important function spaces which we will use to examine the existence, uniqueness and regularity of our problem.

4.2.1 The Function Spaces H and V

Consider the following two function spaces:

$$V = \{u \in \dot{H}^1_{per}(\Omega_L)^d, \mathrm{div} u = 0 \text{ in } \mathbb{R}^d\}\ , \tag{4.5}$$

$$H = \{u \in \dot{H}^0_{per}(\Omega_L)^d, \mathrm{div} u = 0 \text{ in } \mathbb{R}^d\}\ . \tag{4.6}$$

Remark: $X^d = \prod_{i=1}^{d} X$ endowed with the product structure.

We equip H and V with the scalar products and norms (respectively)

$$(u,v) \equiv (u,v)_H = \int_{\Omega_L} u(x)v(x)\,dx \quad \text{and} \quad \|u\|_H = \{(u,u)_H\}^{1/2}\ , \tag{4.7}$$

$$((u,v)) \equiv (u,v)_V = \sum_{i=1}^{d} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_j}\right) \quad \text{and} \quad \|u\|_V = \{((u,u))\}^{1/2}\ . \tag{4.8}$$

where $||u||_V$ is equivalent to the norm induced by $\dot{H}^1_{per}(\Omega_L)^d$.

V is a Hilbert space for this norm.

The dual of V is $V' = \{u \in \dot{H}^{-1}_{per}(\Omega_L)^d \equiv (\dot{H}^1_{per}(\Omega_L)^d)', \text{div } u = 0 \text{ in } \mathbb{R}^d\}$.

We see that $V \subseteq H \subseteq V'$ where the inclusions (injections) are continuous, and each space is dense in the following one.

Also, $\mathfrak{V} = V \cap C^\infty(\mathbb{R}^d)^d$ (test function space) is dense in V, H and V'.

We shall also identify, for some domain $\Omega \subseteq \mathbb{R}^d$, $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ and so $\mathcal{D}'(\Omega)$ = the space of distributions on Ω.

4.2.2 The Operator A

Recall the theory of the section on linear operators in Chapter 2. We see that we can apply the general theory outlined there to the spaces H and V defined above.

Identify the bi-linear form $a(\cdot,\cdot)$ on $V \times V$ with the scalar product defined on $V \times V$

$$a(u,v) = ((u,v)) \equiv (u,v)_V \quad \forall\, u,v \in V \tag{4.9}$$

and so A is the (strictly) positive, self-adjoint, unbounded, linear operator from $D(A)$ onto H defined by

$$(Au,v) = \langle Au,v\rangle = a(u,v) = (u,v)_V \quad \forall\, u,v \in V\ . \tag{4.10}$$

Furthermore, with this definition $A : D(A) \to H$ is an isomorphism, with

$$D(A) = \{u \in V, Au \in H\} \tag{4.11}$$

or more precisely,

$$D(A) = \dot{H}^2_{per}(\Omega_L)^d \cap V\ . \tag{4.12}$$

Specifically,

$$Au = -\Delta u \quad \forall\, u \in D(A)\ . \tag{4.13}$$

(Also see Temam [86] or Constantin and Foias [16] for a more explicit treatment of the characterisation of $D(A)$ – though in the latter the boundary conditions under consideration are somewhat different.)

As outlined before, $D(A)$ is equipped with the norm $||Au||_H$, which is equivalent to the norm induced by $\dot{H}^2_{per}(\Omega_L)^d$ – the Poincaré inequality for mean-zero, space-periodic functions is needed to show this.

Further, since A is a strictly positive, self-adjoint, unbounded, linear operator, we can define powers A^s, $s \in \mathbb{R}$, which are, similarly, strictly positive, self-adjoint, unbounded, linear operators in H, whose domain is $D(A^s) \subseteq H$ (the inclusion being dense in H).

We set

$$V_s = D(A^{s/2}) \quad \forall\, s \in \mathbb{R}\ . \tag{4.14}$$

V_s is a closed subspace of $\dot{H}^s_{\text{per}}(\Omega_L)^d$, in fact:

$$V_s = \{u \in \dot{H}^s_{per}(\Omega_L)^d,\ \text{div } u = 0\}\ . \tag{4.15}$$

We call $V_{-s} = (V_s)'$, $s > 0$, the dual of V_s.

And so we see that we can write

$$\begin{array}{ccccccccccc}
s>2 & & s=2 & & s=1 & & s=0 & & s=-1 & & s=-s'<-1 \\
 & & & & & & & & & & \\
D(A^{s/2}) & \subseteq & D(A) & \subseteq & D(A^{1/2}) & \subseteq & D(A^0) & \subseteq & D(A^{-1/2}) & \subseteq & D(A^{-s'/2}) \\
|| & & || & & || & & || & & || & & || \\
V_s & \subseteq & V_2 & \subseteq & V_1 = V & \subseteq & V_0 = H & \subseteq & V_{-1} & \subseteq & V_{-s'}
\end{array}$$

Further, not only is A an isomorphism from V onto V' as well as from $D(A)$ onto H, more generally, A is an isomorphism from V_{s+2} onto V_s, $\forall\ s \in \mathbb{R}$.

Also, the norm $||A^{s/2}||_H$ on V_s is equivalent to that induced by $\dot{H}^s_{per}(\Omega_L)^d$

$$c_1||u||_{s,2} \leq ||A^{s/2}||_H \leq c_2||u||_{s,2} \quad \forall\ u \in D(A^{s/2}) \tag{4.16}$$

where c_1 and c_2 depend on s and L.

Note that the injection of V onto H is compact.

Further, the injection of V_s into $V_{s-\epsilon}$ is compact for all $s \in \mathbb{R}$, $\epsilon > 0$.

Also, as in Chapter 2, we can consider $A^{-1} : H \to D(A)$ as a self-adjoint, compact operator in H and all the theory of Section 2.5 applies to our linear operator A defined by (4.10).

4.3 A Priori Estimates

Assuming u is a sufficiently regular solution of the Navier-Stokes equations, we can make the following (*energy-type*) a-priori estimates:

4.3.1 The Energy A-priori Estimate

We can rewrite the equation (4.1) (with ω = curl u) as:

$$u_t + \omega \wedge u = \nu\Delta u - \nabla(\mathcal{P} + \frac{1}{2}|u|^2) + f \tag{4.17}$$

where '$\wedge$' is the vector product. Consider the scalar product of (4.17) with u and integrating over Ω_L

$$\frac{1}{2}\partial_t||u||^2_{L^2(\Omega_L)} = -\nu||Du||^2_{L^2(\Omega_L)} + (u, f)_{\Omega_L} \tag{4.18}$$

where $||u||_{L^2(\Omega_L)}$ is equivalent to the norm on H, the square of which equals $\int_{\Omega_L} |u|^2 dx$, i.e. the total *energy* of our system.

Also, $||Du||_{L^2(\Omega_L)} \equiv ||u||_{\dot{H}^1_{per}(\Omega_L)} \equiv |u|_{1,2,\Omega_L}$ – the H^1 semi-norm which is equivalent to the norm on V.

Remark: I will persist with this new norm notation as a matter of convenience for later in this chapter and throughout the following chapters.

Note: For divergence free functions such as u on periodic boundary conditions,

$$\int_{\Omega_L} |\omega|^2\, dx = \int_{\Omega_L} |\nabla u|^2\, dx = |u|^2_{1,2,\Omega_L} \tag{4.19}$$

A Cauchy-Schwarz inequality in (4.18) gives

$$\frac{1}{2}\partial_t||u||^2_{L^2(\Omega_L)} + \nu||Du||^2_{L^2(\Omega_L)} \leq ||u||_{L^2(\Omega_L)}||f||_{L^2(\Omega_L)} \tag{4.20}$$

The Poincaré inequality for mean-zero functions tells us that

$$||u||_{L^2(\Omega_L)} \leq L||Du||_{L^2(\Omega_L)} \tag{4.21}$$

and so if we use Young's inequality we see that

$$\frac{1}{2}\partial_t||u||^2_{L^2(\Omega_L)} + \nu||Du||^2_{L^2(\Omega_L)} \leq \frac{\nu}{2L^2}||u||^2_{L^2(\Omega_L)} + \frac{L^2}{2\nu}||f||^2_{L^2(\Omega_L)} \leq \frac{\nu}{2}||Du||^2_{L^2(\Omega_L)} + \frac{L^2}{2\nu}||f||^2_{L^2(\Omega_L)} \tag{4.22}$$

$$\Rightarrow \partial_t||u||^2_{L^2(\Omega_L)} + \nu||Du||^2_{L^2(\Omega_L)} \leq \frac{L^2}{\nu}||f||^2_{L^2(\Omega_L)} \;. \tag{4.23}$$

If we now integrate with respect to time and recall that $f \in C^\infty(\Omega_L)^d$ is time-independent, then

$$||u||^2_{L^2(\Omega_L)}(t) + \nu\int_0^t ||Du||^2_{L^2(\Omega_L)}(s)\,ds \leq ||u||^2_{L^2(\Omega_L)}(0) + \frac{L^2 t}{\nu}||f||^2_{L^2(\Omega_L)} \;. \tag{4.24}$$

Thus we could deduce that if the initial energy is finite, then the energy is finite for all $t > 0$ and also $\int_0^t ||Du||^2_{L^2}(s)\,ds$ is bounded for all $t > 0$. If we use Poincaré's inequality in (4.23) and apply the Uniform Gronwall Lemma (see Temam [88], Chapter 3, Lemma 1.1), then

$$||u||^2_{L^2}(t) \leq ||u||^2_{L^2}(0)\exp[-\nu L^{-2}t] + \nu^{-2}L^4||f||^2_{L^2}(1 - \exp[-\nu L^{-2}t])$$

i.e. the energy $||u||^2_{L^2}$ is uniformly bounded.

4.3.2 A Second A-priori Estimate

Result for $d = 2$

Note that we can write (4.1) in index notation (commas denoting partial derivatives):

$$u_{i,t} + u_k u_{i,k} = \nu u_{i,kk} - \mathcal{P}_{,i} + f_i \tag{4.25}$$

where $u = (u_1, u_2)$ and also $u_{1,1} + u_{2,2} = 0$ $(d = 2)$.

If we differentiate the above with respect to the j^{th} variable, multiply by $u_{i,j}$ and integrate over Ω_L then:

$$\frac{1}{2}\partial_t||Du||^2_{L^2} + \int_{\Omega_L} u_{i,j}u_{k,j}u_{i,k}dx = -\nu||D^2u||^2_{L^2} + ((u,f))_{\Omega_L} \tag{4.26}$$

where $||D^2u||_{L^2(\Omega_L)} \equiv ||u||_{\dot{H}^2_{per}(\Omega_L)} \equiv |u|_{2,2,\Omega_L}$ i.e. the H^2 semi-norm, which is equivalent to $||Au||_H = ||\Delta u||_H$.

Crucially, in the $d = 2$ case (using the divergence property) $\int_{\Omega_L} u_{i,j}u_{k,j}u_{i,k}\,dx = 0$ (which is not true in the $d = 3$ case). Thus for $d = 2$, if we use integration by parts, a Cauchy-Schwarz inequality and then Young's inequality on the last term in (4.26), we get

$$\partial_t||Du||^2_{L^2} + \nu||D^2u||^2_{L^2} \leq \nu^{-1}||f||^2_{L^2} \;.$$

Poincaré's inequality and then integrating with respect to time shows us that $||Du||^2_{L^2(\Omega_L)}(t)$ and $\int_0^t ||D^2u||^2_{L^2(\Omega_L)}(s)\, ds$ are bounded for all $t > 0$ provided $||Du||^2_{L^2}(0)$ is finite, while the Uniform Gronwall Lemma implies

$$||Du||^2_{L^2}(t) \leq ||Du||^2_{L^2}(0) \exp[-c\nu L^{-2}t] + \frac{L^2}{c\nu^2}||f||^2_{L^2}(1 - \exp[-c\nu L^{-2}t])$$

and so $||Du||_{L^2} \equiv ||\omega||_{L^2}$ is uniformly bounded in time.

Result for $d = 3$

Again, assuming our solutions to (4.1)–(4.4) are sufficiently regular, note that by using a Hölder inequality as well as integration by parts, we can write the $d = 3$ version of (4.26) as:

$$\frac{1}{2}\partial_t||Du||^2_{L^2} + \nu||D^2u||^2_{L^2} \leq c(n)\, ||Du||_{L^2}||Du||^2_{L^4} + (-\Delta u, f)_{\Omega_L} \; . \tag{4.27}$$

The Gagliardo-Nirenberg inequality of the last chapter allows us to majorise the $||Du||_{L^4}$ term as follows

$$||Du||_{L^4} \leq c\, ||D^2u||^{3/4}_{L^2}||Du||^{1/4}_{L^2} \; . \tag{4.28}$$

Using this in (4.27), applying Young's inequality and dealing with the last term on the right-hand side appropriately, we see that we get

$$\frac{d}{dt}||Du||^2_{L^2} + \nu||D^2u||^2_{L^2} \leq 2c_1\nu^{-3}||Du||^6_{L^2} + \frac{2}{\nu}||f||^2_{L^2} \tag{4.29}$$

where c_1 is independent of ν and L.

Now, if we define

$$\tilde{F}_1(t) = ||Du||^2_{L^2} + \left(\frac{L}{\nu}\right)^2 ||f||^2_{L^2} \; , \tag{4.30}$$

then (4.29) becomes

$$\frac{d\tilde{F}_1}{dt} \leq c_2(\nu, L)\, \tilde{F}_1^3 \tag{4.31}$$

and integration soon reveals that

$$\tilde{F}_1(t) \leq \frac{\tilde{F}_1(0)}{(1 - 2c_2\tilde{F}_1^2(0)\, t)^{1/2}} \tag{4.32}$$

which holds as long as the denominator is non-zero: in fact we can deduce that provided $0 \leq t \leq T_1(||Du||_{L^2}(0))$, where

$$T_1(||Du||_{L^2}(0)) = \frac{3}{8}\left[c_2\left(||Du||^2_{L^2}(0) + \left(\frac{L}{\nu}\right)^2 ||f||^2_{L^2}\right)^2\right]^{-1} \tag{4.33}$$

then

$$\tilde{F}_1(t) \leq 2\, \tilde{F}_1(0) \tag{4.34}$$

i.e.

$$||Du||^2_{L^2}(t) + \left(\frac{L}{\nu}\right)^2 ||f||^2_{L^2} \leq 2\left(||Du||^2_{L^2}(0) + \left(\frac{L}{\nu}\right)^2 ||f||^2_{L^2}\right) \tag{4.35}$$

and so obviously $||Du||_{L^2}(t)$ is bounded for the interval of time indicated.

4.4 Weak and Strong Solutions

4.4.1 Weak Form of Navier-Stokes Equations (due to J. Leray)

Let $T > 0$ be given.

First, let us introduce the tri-linear continuous form b (defined on $H^1_{per}(\Omega_L)^d \times H^1_{per}(\Omega_L)^d \times H^1_{per}(\Omega_L)^d$ and in particular on $V \times V \times V$):

$$b(u, v, w) = \sum_{i,j=1}^{d} \int_{\Omega_L} u_i(D_i v_j) w_j \, dx \tag{4.36}$$

whenever the integrals make sense.

We see that, if $u \in V$

$$b(u, v, v) = 0 \quad \forall\, v \in H^1_{per}(\Omega_L)^d \ . \tag{4.37}$$

Further, let us define the bi-linear operator B from $V \times V$ into V' by

$$\langle B(u, v), w \rangle = b(u, v, w) \quad \forall\, u, v, w \in V \tag{4.38}$$

and set

$$B(u) = B(u, u) \ \in V' \quad \forall\, u \in V \ . \tag{4.39}$$

Remark: See Temam [87] and [86] for the various continuity properties of the tri-linear form b.

Let us suppose that u and $\mathcal{P}$ are classical solutions of (4.1)–(4.4), and in particular,

$$u \in C^2(\Omega_L \times [0, T])^d \quad \text{and} \quad \mathcal{P} \in C^1(\Omega_L \times [0, T]) \ . \tag{4.40}$$

We can immediately see (via the a-priori estimates above) that $u \in L^2(0, T; V)$, and further, if we multiply (4.1) by a test function $v \in \mathfrak{V}$ and integrate over Ω_L (using periodic boundary conditions) then we get

$$\frac{d}{dt}(u, v) + \nu((u, v)) + b(u, u, v) = \langle f, v \rangle \tag{4.41}$$

By continuity (4.41) holds for all $v \in V$.

This suggests the following weak formulation:

The Weak Problem (WP): *For $f \in L^2(0, T; V')$ and $u_0 \in H$ given, find u satisfying*

$$u \in L^2(0, T; V) \tag{4.42}$$

and

$$\frac{d}{dt}(u, v) + \nu((u, v)) + b(u, u, v) = \langle f, v \rangle \quad \forall\, v \in V \ , \tag{4.43}$$

$$u(0) = u_0 \ . \tag{4.44}$$

Remark: If $u \in L^2(0, T; V)$ only, then the initial condition (4.44) does not make sense. We will prove below, that if $u \in L^2(0, T; V)$ and also satisfies (4.43) then u is a.e. equal to a continuous function from $[0, T]$ into V', so that (4.44) is meaningful.

Consider the following two lemmas provided in Temam [87]:

Lemma 4.4.1 *Suppose X is a Banach space with dual X' and $u, g \in L^1(a, b; X)$. Then the following three conditions are equivalent:*

1. *u is a.e. equal to a primitive function of g,*
$$u(t) = \xi + \int_a^t g(s)\, ds \ , \quad \xi \in X, \quad a.\ e.\ \ t \in [a, b] \ . \tag{4.45}$$

2. *For each test function $\phi \in \mathcal{D}((a, b))$,*
$$\int_a^b u(t)\phi'(t)\, dt = - \int_a^b g(t)\phi(t)\, dt \ , \qquad \phi' = \frac{d\phi}{dt} \ . \tag{4.46}$$

3. *For each $\eta \in X'$,*
$$\frac{d}{dt} \langle u, \eta \rangle = \langle g, \eta \rangle \tag{4.47}$$
in the scalar distributional sense, on (a, b).

If 1-3 are satisfied, we know that u is a.e. equal to a continuous function from $[a, b]$ into X, and g is the (X-valued) distributional (weak) derivative of u.

Lemma 4.4.2 *Assume $d \leq 4$ and that $u \in L^2(0, T; V)$. Then the function Bu defined by*
$$\langle Bu(t), v \rangle = b(u(t), u(t), v) \qquad \forall\ v \in V \ \ a.e.\ in\ \ t \in [0, T] \tag{4.48}$$
belongs to $L^1(0, T; V')$.

If u satifies (4.42)–(4.43), then (4.10) and the lemma above imply
$$\frac{d}{dt} \langle u, v \rangle = \langle f - \nu Au - Bu, v \rangle \qquad \forall\ v \in V \ . \tag{4.49}$$
Since A is linear, continuous (in fact an isomorphism) from V into V', and $u \in L^2(0, T; V)$, then $Au \in L^2(0, T; V')$ and so using Lemma 4.4.2 we see that $f - \nu Au - Bu \in L^1(0, T; V')$.

Lemma 4.4.1 with $X = V'$ tells us
$$\frac{du}{dt} \in L^1(0, T; V') \tag{4.50}$$
and
$$\frac{du}{dt} = f - \nu Au - Bu \tag{4.51}$$
(the latter satisfied in the distibutional sense in V'). Further, Lemma 4.4.1 implies that u is a.e. equal to a continuous function from $[0, T]$ into V' and (4.44) is therefore meaningful.

Clearly, the weak problem is equivalent to:

(WP) Alternative: *Given $f \in L^2(0, T; V')$ and $u_0 \in H$, find u satisfying*
$$u \in L^2(0, T; V) \ , \qquad \frac{du}{dt} \in L^1(0, T; V') \tag{4.52}$$
and
$$\frac{du}{dt} + \nu Au + B(u) = f \qquad on \ \ (0, T) \ , \tag{4.53}$$

$$u(0) = u_0 \ . \tag{4.54}$$

Since the solutions we are considering may not be very regular functions, (4.53) is therefore satisfied in a distributional sense in V'.

Remark: Since we will assume f is independent of t, this means that the dynamical system associated with (4.53) is autonomous.

4.4.2 Strong Solutions

We now consider a class of more regular solutions:

The Strong Problem (SP): *For $f \in L^2(0,T;H)$ and $u_0 \in V$ given, find u satisfying*

$$u \in L^2(0,T;D(A)) \cap L^\infty(0,T;V) \tag{4.55}$$

as well as (4.42) - (4.44).

Remark: If u is a strong solution, then by interpolation (see Temam [86], Section 2.4),

$$Bu \in L^4(0,T;H) \ . \tag{4.56}$$

We know $f \in L^2(0,T;H)$ and $Au \in L^2(0,T;H)$ so

$$\frac{du}{dt} = f - \nu Au - Bu \in L^2(0,T;H) \ . \tag{4.57}$$

The condition $u \in L^2(0,T;D(A))$ along with that last condition (4.57), allows us to deduce (see Temam [87] Chapter 3, Section 1.4; or Temam [86], Section 2.4) that u is a.e. equal to a continuous function from $[0,T]$ into V,

$$u \in C([0,T];V) \ . \tag{4.58}$$

4.5 Existence and Uniqueness Theorems

I quote the two following theorems from Temam [86]. They collect results from Leray [57, 56, 58], Hopf [42], Ladyzhenskaya [52], Lions [59], Lions and Prodi [61] and also Serrin [79].

Let $T > 0$ be given.

Theorem 4.5.1 (Weak Solutions) *For $f \in L^2(0,T;V')$ and $u_0 \in H$ given, there exists a weak solution u to the Navier-Stokes equations ($d = 2$ and 3) satisfying*

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H) \ . \tag{4.59}$$

For d=2: *u is unique and*

$$u \in C([0,T];H) \ , \tag{4.60}$$

$$\frac{du}{dt} \in L^2(0,T;V') \tag{4.61}$$

and we can assert: $u(t) \to u_0$ in H as $t \to 0$.

For d=3: *u is weakly continuous from* $[0,T]$ *into* H,

$$u \in C([0,T]; H_w) , \tag{4.62}$$

$$\frac{du}{dt} \in L^{4/3}(0,T;V') . \tag{4.63}$$

Remark: By u is weakly continuous from $[0,T]$ into H (i.e. (4.62)), we mean: $\forall\ v \in H$, $t \mapsto (u(t), v)$ is a continuous scalar function.

Theorem 4.5.2 (Strong Solutions) (i) d=2: *For any* $T > 0$, $f \in L^\infty(0,T;H)$ *and* $u_0 \in V$ *given, there exists a unique strong solution to the Navier-Stokes equations satisfying*

$$u \in L^2(0,T;D(A)) \cap C([0,T];V) , \tag{4.64}$$

$$\frac{du}{dt} \in L^2(0,T;H) . \tag{4.65}$$

(ii) d=3: *For* $f \in L^\infty(0,T;H)$ *and* $u_0 \in V$ *given, there exists a unique strong solution to the Navier-Stokes equations on* $[0,T^*]$, *with* $T^* = T^*(u_0) = \min\{T, T_1(||Du_0||_{L^2})\}$ *(and* T_1 *given by (4.33)) which satisfies*

$$u \in L^2(0,T^*;D(A)) \cap C([0,T^*];V) , \tag{4.66}$$

$$\frac{du}{dt} \in L^2(0,T^*;H) . \tag{4.67}$$

The Proofs of Theorems 4.5.1 and 4.5.2: We consider a so-called Galerkin method. We look for an approximate solution u_m of the weak and strong problems

$$u_m = \sum_{j=1}^{m} g_{j,m}(t) w_j \quad \forall\ m \in \mathbb{N} , \tag{4.68}$$

$$u_m : [0,T] \longrightarrow W_m = \text{space spanned by } w_1, \ldots, w_m \tag{4.69}$$

where the w_j, $j \in \mathbb{N}$ are the eigenfunctions of the operator A.

Note that the w_j, $j \in \mathbb{N}$ are a basis for H.

If P_m is the orthogonal projector in H onto W_m, then u_m satisfies

$$\frac{d}{dt}(u_m, v) + \nu((u_m, v)) + b(u_m, u_m, v) = (f, v) \quad \forall\ v \in W_m , \tag{4.70}$$

$$u_m(0) = P_m u_0 . \tag{4.71}$$

Existence and uniqueness of the solution u_m for the system (4.70), (4.71) is well known (they represent a system of ODEs) defined on an interval $(0, T_m)$, $T_m > 0$.

We use a-priori estimates for the u_m (analogous to those outlined above) to show that the u_m lie in the appropriate function spaces and we extract a subsequence $u_{m'}$ of u_m so that $u_{m'} \longrightarrow u$ (the unique limit) in the right spaces as $m' \longrightarrow \infty$ (and to show $T_m = T$). Passage to the limit in (4.70) shows that u is a solution of the weak problem.

For the details, I refer the reader to Temam [87] and [86]. □

Remarks:

1. We see that for $d = 2, 3$ a weak solution for the Navier-Stokes equations exists on $[0, T]$, $\forall\, T > 0$.
   (a) $d = 2$: Weak solution is also unique,
   (b) $d = 3$: It is not known if the weak solution is unique (or what further assumption could make it unique).
2. As far as strong solutions are concerned:
   (a) $d = 2$: There exists a unique strong solution to the Navier-Stokes equations on $[0, T], \forall\, T > 0$.
   (b) $d = 3$: There exists a unique strong solution to the Navier-Stokes equations on $[0, T^*]$, T^* as above – we cannot extend the interval of existence to an arbitrary time T. This is an existence result local in time.
3. Some of the a-priori estimates for u_m (referred to in the proof) imply that (analogous to the a-priori energy estimate derived above for smooth functions)
$$||u||^2_{L^2}(t) + 2\nu \int_0^t ||Du||^2_{L^2}(s)\, ds \leq ||u||^2_{L^2}(0) + 2 \int_0^t \langle f(s), u(s) \rangle\, ds\ . \tag{4.72}$$
   This is an energy inequality satisfied by the weak solution u for $d = 2, 3$. For $d = 2$ weak solutions in fact satisfy (4.72) as an equality – as do strong solutions for $d = 2, 3$.
4. Obviously, if u is a weak solution for the Navier-Stokes equations, for $d = 2$, and if u satisfies the hypotheses of Theorem 4.5.2 above, then uniqueness $\Rightarrow u$ is a strong solution.
5. J. Sather and J. Serrin (see [80]) proved: *For $d = 3$, assuming $f \in L^2(0, T; H)$ and $u_0 \in H$ given, then if there exists a solution of (4.42), (4.43) satisfying (4.55), (4.59) as well as (4.72) as an equality (i.e. a strong solution), then there does not exist any other solution u of (4.42), (4.43) satisfying (4.59), (4.72) (i.e. a weak solution)* – i.e. as long as the strong solution exists ($d = 3$), it is unique in the class of weak solutions. Temam [87] points out that the uniqueness of a weak solution is related to the regularity properties we know of the weak solution.
6. We can investigate further regularity properties of strong solutions if we make further assumptions on the given data (this may *not* involve further restrictions on the initial data). According to Temam this is "due to a regularizing effect" of the Navier-Stokes equations for strong solutions – we shall clarify these statements further on.
7. It is possible to "lift" a weak solution to a strong solution by assuming a further regularity property for the weak solution. An example is given below.

4.5.1 An Example of How to Lift a Weak Solution to a Strong One.

In $d = 3$, recall from Section 4.3 that we can make the following a-priori estimate,
$$\frac{1}{2}\partial_t ||Du||^2_{L^2} + \nu ||D^2 u||^2_{L^2} = - \int_{\Omega_L} u_{i,j} u_{k,j} u_{i,k}\, dx + (u, f)_{1,\Omega_L} \tag{4.73}$$
and using a Cauchy-Schwarz inequality, then a Young's inequality and finally Poincaré's inequality, we get
$$\frac{1}{2}\partial_t ||Du||^2_{L^2} + \frac{\nu}{2} ||D^2 u||^2_{L^2} \leq ||Du||^3_{L^3} + \frac{L^2}{2\nu} ||Df||^2_{L^2}\ . \tag{4.74}$$

Now consider the following Gagliardo-Nirenberg inequality

$$|u|_{1,3} \leq c\, |u|_{2,2}^{a} |u|_{0,q}^{1-a} \tag{4.75}$$

where $a = 6/(q+6)$ and hence the restriction $1 < q \leq 6$.

Using this in (4.74), applying a Young's inequality (which incurs the restriction $3a < 2 \Rightarrow q > 3$) and then integrating with respect to t we get

$$\begin{aligned} &\|Du\|_{L^2}^2(t) + c_1\nu \int_0^t \|D^2u\|_{L^2}^2(s)\, ds \\ \leq\ & \|Du\|_{L^2}^2(0) + c_2\nu^{-\frac{3a}{2-3a}} \int_0^t \|u\|_{L^q}^{\frac{2(1-a)}{2-3a}}(s)\, ds + \frac{tL^2}{2\nu}\|Df\|_{L^2}^2 \ . \end{aligned} \tag{4.76}$$

From this inequality we can conclude that if $\|u\|_{L^{3+\epsilon}}$ is uniformly bounded then $u \in L^\infty([0,t]; V)$, for all $t > 0$. In fact, $\|Du\|_{L^2}(t)$ is uniformly bounded in time (apply (4.75), Poincaré's inequality and the Uniform Gronwall Lemma to (4.74)).

Recall that Theorem 4.5.2 shows that for u a weak solution of the Navier-Stokes equations in three dimensions, u is also a strong solution on $[0, T^*(\|Du_0\|_{L^2})]$.

Let $T > 0$ be given and assume $\|u\|_{L^{3+\epsilon}}$ is bounded on $[0,T]$.

Now, suppose T' is the largest possible value of T^* and assume that $T' < T$.

Then

$$\limsup_{t \nearrow T'} \|Du(t)\|_{L^2} = \infty \ . \tag{4.77}$$

However, the a-priori estimate (4.76) above implies that if $\|Du(t)\|_{L^2}$ becomes unbounded then $\|u\|_{L^{3+\epsilon}}$ is also unbounded – a contradiction! (Or in other words, (4.76) $\Rightarrow$ $\|Du\|_{L^2}(t)$ is bounded for $t \nearrow T'$, contradicting the assumption $T' < T$.)

In fact, with this assumption, we know $u \in C([0,T], V)$, $\forall\, T > 0$, via Theorem 4.5.2.

We also remark that if we *naturally assume* that $\|Du\|_{L^2}(t)$ is uniformly bounded, then u is a strong solution of the Navier-Stokes equations for an arbitrary interval of time ($d = 3$).

Thus, to summarize (a result which is analogous to that of Serrin [79]): if u is a weak solution of the three dimensional Navier-Stokes equations and if we assume u is uniformly bounded in $L^{3+\epsilon}$ then u is a strong solution for the Navier-Stokes equations with $u \in C([0,T]; V)$, $\forall\, T > 0$, and further $\|u\|_V(t)$ is uniformly bounded.

In fact, the result that Serrin [79] proves is the following:

Let u be a solution of the d-dimensional Navier-Stokes equations (assume f is at least in $L^1((0,T); L^1(\Omega))$ and is a conservative external force) in some open region $\Omega \times (0,T)$ of space-time, with $u \in L^\infty((0,T); L^2(\Omega))$, $\omega \in L^2((0,T); L^2(\Omega))$. Suppose further that

$$u \in L^{s'}((0,T); L^s(\Omega)) \tag{4.78}$$

where

$$\frac{d}{s} + \frac{2}{s'} < 1 \tag{4.79}$$

Then u is of class C^∞ in the space variables, and each derivative is bounded in compact subregions of $\Omega \times (0,T)$.

If we further assume that $u_t \in L^p((0,T); L^2(\Omega))$, where $p \geq 1$, then the space derivatives of u are absolutely continuous functions of time, and there exists a strongly differentiable function $\mathcal{P} = \mathcal{P}(x,t)$ such that (4.1) is satisfied a.e. in $\Omega \times (0,T)$.

Remarks:

1. This result is more general. However, Serrin did not directly use the results of Theorems 4.5.1 and 4.5.2, which we were able to use in our derivation above.

2. The proof is via a complicated series of estimates for convolution integrals to show $\omega \in L^\infty((0,T); L^\infty(\Omega))$ and then successively showing spatial Hölder continuity (with arbitrary exponent $\theta < 1$) of ω, ω_x, ω_{xx}, ... etc.

Temam [86] provides a version which states that the assumption $u \in L^4(0,T;V)$ is sufficient to "lift" u to the status of a strong solution for all finite intervals, while Ladyzhenskaya [52], shows (a result which we essentially reproduce in Chapter 6) that if we assume the initial data is sufficiently small (i.e. within a particular ball $B(0,R^*)$) or if we assume the viscosity is sufficiently large, then we can show the existence of strong solutions for all time.

Ladyzhenskaya [52] also provides some other results: the existence (for $d = 3$) of so-called "generalized solutions" (defined as a solution which (essentially) lies in $L^\infty(0,T;L^4(\Omega))$) on some finite interval of time $[0,T^*]$. However, this result has been superceded by those mentioned in the last few sections.

4.5.2 Some Further Remarks:

1. For $d = 2$ everything is essentially known: We know a unique strong solution exists for an arbitrary interval of time $[0,T]$. The two following points analogously apply to the $d = 2$ case, except that we can replace T^* by T for arbitrary $T > 0$ given.

2. Temam [86] gives a-priori estimates ($d = 3$) in terms of $|u|_r \equiv |u|_{r,2,\Omega_L}$ (cf. Temam [86] Lemma 4.1) (under the assumption u is smooth) and with analogous estimates for u_m proceeds (via a Galerkin method) to prove the result ($d = 3$):

   Lemma 4.5.3 *If* $u_0 \in V$ *and* $f \in L^\infty(0,T;V_{r-1})$, $r \geq 1$ *then* $u \in C((0,T^*];V_r)$.

   This lemma is related to the results in the Sections 4.7 and 4.8 below. However, it also exemplifies the next point:

3. If we examine carefully the lemma outlined above, we see that it implicitly tackles (and answers) the question of regularity at $t = 0$: If we assume the initial data $u_0 \in W_1$ (i. e. further regularity properties) for some space $W_1 \subseteq H$, can we show that the solution $u(t)$ enters into the space $W_2 \subseteq W_1$ in an arbitrary short time? This question is naturally related to the *compatability conditions* of the data at $t = 0$: these are the necessary and sufficient conditions on the data for the solution u to be smooth up to time $t = 0$.

   This has been proved in the result above ($W_1 = V$ and $W_2 = V_r$, $\forall\, r \geq 1$).

   In other words, even if we assume initial data which is not regular ($u_0 \in V$ and f appropriately regular (as above)) within an arbitrary small interval of time the solution becomes regular satisfying the conclusions of Lemma 4.5.3.

   Thus we do not need to assume C^∞ initial data in order for our solutions to be smooth on $(0,T^*]$ ($d = 3$) and in fact need only make the assumptions outlined in Lemma 4.5.3. (Of course, if we assume $u_0 \in V_r$ then $u \in C([0,T^*],V_r)$.)

   See Temam [85] where $t = 0$ regularity for $d = 2,3$ is more explicitly investigated and proved for the Navier-Stokes equations.

4. We can also derive a-priori estimates for weak solutions which take a different form to the a-priori estimates used to prove Lemma 4.5.3 above – see Foias, Guillope and Temam [30] and also Temam [86], Theorem 4.2.

4.5.3 Summary

1. For $d = 2$, we can show the existence of a semi-group of operators for the Navier-Stokes equations:
$$S(t) : u_0 \mapsto u(t) \tag{4.80}$$
which are continuous from H into itself (in fact from H into $D(A)$).

2. For $d = 3$, the boundary value problem is not well-posed and the situation is a little more complex. We can make an extra assumption in order to show the existence of such a semi-group of operators, for example, that u is uniformly bounded in $L^{3+\epsilon}(\Omega_L)$ holds for the flow. Then having established the existence of the semi-group of operators, we must investigate any further assumptions we might have to make in order to show the existence of a C^∞ attractor. We proceed to discuss this in detail below.

For a good proportion of the material above, I have relied on the classical texts by Temam, [86] and [87].

4.6 Non-well-posed Problems

We have seen that for the two dimensional Navier-Stokes equations (4.1) (supplemented with the initial and boundary conditions indicated) are well-posed and there exist strong solutions globally in time, which means that we can immediately proceed to the two sections which follow this one, where we show the existence of a C^∞ global attractor.

However in three spatial dimensions, we have found that (4.1)–(4.4) is not necessarily a well-posed initial boundary value problem and we only know that there exists a unique strong solution on some finite interval of time (provided we do not restrict our data further).

We have indicated how we can "lift" a weak solution to a strong one for a time interval of arbitrary length by making some extra assumptions on the flow e.g. $u \in L^{3+\epsilon}(\Omega_L)$ uniformly or more obviously, $u \in V$ uniformly. We can also show the existence of a strong solution for all time if we assume the initial data to be sufficiently small (in certain norms) or that the viscosity is very large (this is Ladyzhenskaya's result [52]). In either case, we are then able to establish the existence of a well defined continuous semi-group of operators $S(t) : H \to H$. (We in fact have $u \in C([0,T]; V)$.)

Subsequently, we are then able to proceed to discuss further regularity properties of the solution as well as the regularity of the attractor.

Let us be more careful with these assertions.

When the initial value problem (IVP) with which we are presented is ill-posed, i.e. a semi-group of operators cannot be defined everywhere, we can still proceed to define invariant sets and attractors – *recall* the theory of semi-groups for non-well-posed problems in Chapter 2.

In the following exposition we present the relationship between whether an initial value problem is well-posed, and the corresponding existence of an absorbing set and global attractor; more precisely, we will show that for some specific IVPs, *if a particular initial value problem is well-posed for all initial data, then there exists an absorbing set and a global attractor.*

The specific equations mentioned are those which fall into the category of IVPs which can be written in the form

$$\begin{aligned} \tfrac{du}{dt} + Au + B(u) + Ru &= f \,, \\ u(0) &= u_0 \end{aligned} \tag{4.81}$$

in H (for f and u_0 in H).

As before, we have assumed:

1. We are given two Hilbert spaces $V \subseteq H$ (the injection being compact, dense and continuous) and so $V \subseteq H \subseteq V'$.
2. We consider the coercive, bi-linear continuous form $a(u, v)$. So, the associated linear operator $A \in \mathcal{L}(V, V')$ is an isomorphism from V onto V', and
$$D(A) = \{u \in V, \quad Au \in H\} \subseteq V \tag{4.82}$$
the injection being dense and continuous.
3. $R : V \longrightarrow V'$ is a continuous linear operator which maps $D(A)$ onto H, and which also satisfies a certain set of conditions. Similarly, $B : V \times V \longrightarrow V'$ is a bi-linear continuous operator which satisfies a certain set of conditions (B also maps $D(A) \times D(A)$ onto H). I refer the interested reader to Chapter 7 of Temam [88] since the exact nature of these conditions is not important for a discussion of the ideas presented here. However, for the $d = 3$ Navier-Stokes equations (4.1)–(4.4) we know that $R = 0$ and B certainly satisfies the conditions mentioned.

Note that the IVP (4.81) includes the Navier-Stokes equations (4.1)–(4.4).

We can prove the following theorem for the IVP (4.81):

Theorem 4.6.1 *Under the assumptions mentioned for the operators R and B, for $f \in H$ and $u_0 \in H$ given, there exists a solution u of (4.81) satisfying*

$$u \in L^2(0, T; V) \cap L^\infty(0, T; H) \qquad \forall\, T > 0 \,. \tag{4.83}$$

Such a solution is not guaranteed to be unique.

If we further assume that $u_0 \in V$, then $\exists\, T^* = T^*(\|u_0\|_V, f) > 0$ such that there exists a unique strong solution u of (4.81) on $[0, T^*]$ satisfying

$$u \in L^2(0, T^*; D(A)) \cap C([0, T^*]; V) \tag{4.84}$$

For the proof, see Temam [88], Chapter 7.

We see from the theory of Section 2.7 of Chapter 2, that there exists an operator $S(t)$ which is the mapping

$$S(t) : u_0 \in V \mapsto u(t) \in V \tag{4.85}$$

whenever the mapping is defined.

Remark: Some a-priori estimates derived in the last two sections (and applied in the proof of the theorem above) show that the absorbing ball of V, $B(0, M) \subseteq \mathcal{D}(S(T^*(M)))$.

With reference to the theory of Section 2.7 of Chapter 2 and the above theorem, we will say that the IVP (4.81) is well-posed in V on $[0, T]$ for some $u_0 \in V$, if $u_0 \in \mathcal{D}(S(T); V)$. In that case, there exists a unique solution u of (4.81) such that

$$u \in C([0, T]; V) \cap L^2(0, T; D(A)) \,. \tag{4.86}$$

Theorem 4.6.2 *Assume the hypotheses of Theorem 4.6.1 are satisfied. If the IVP (4.81) is well-posed in* V, $\forall\, u_0 \in V$ *and* $\forall\, T > 0$, *then the corresponding semi-group* $\{S(t), t \geq 0\}$ *possesses an absorbing set in* V *and a global attractor.*

Remarks:

1. In other words, as Temam states: "if we know the solution u of [4.81] remains bounded in V, $\forall\, u_0$ and all finite intervals $[0, T]$, then we know that $\|u(t)\|_V$ remains bounded $\forall\, t \geq 0$ and uniformly for u_0 in a bounded set in V."

2. Thus, if we can show the existence of a unique strong solution for all time, for all initial data, then Theorem 4.6.2 tells us that our system is well-posed and possesses a global attractor. For the three dimensional Navier-Stokes equations, this means that we must assume u is uniformly bounded in V or, as we already know, it is sufficient to assume u is uniformly bounded in $L^{3+\epsilon}$.

3. We could proceed to investigate the dimension of functional invariant sets for non-well-posed problems (cf. Temam [88], Chapter 7).

4. This theory along with that concerning the existence and uniqueness of weak and strong solutions exposes the first two fundamental questions we should *now* ask ourselves concerning the Navier-Stokes equations (4.1)–(4.4):

   (a) Do unique strong solutions exist for all time ($T > 0$) or if they only exist on some finite interval $[0, T^*]$, what minimum assumptions must we make in order to prove that they exist $\forall\, T > 0$? We can then use lemmas such as 4.5.3 to show these strong solutions are regular (smooth) on all finite intervals of time.

   (b) The theory of this section then tells us that, having assumed the sufficient condition to show (well-posedness) the existence and uniqueness of a strong solution (and so we further know $u \in C([0, T]; V)$), we can then say that the corresponding semi-group $\{S(t), t \geq 0\}$ has an absorbing set in V and a global attractor. We then ask ourselves ($d = 3$ case): What is the minimal assumption we need to make to show that the maximal attractor is bounded in H^n, for some or all $n \in \mathbb{N}$?

5. We now investigate some a-priori estimates which will provide us with partial answers to both questions. We will go on to improve these estimates and what we can deduce from them in Chapter 5.

4.7 The Ladder Theorem

In this section we construct a set of a-priori estimates for solutions of the Navier-Stokes equations which are based on those in Bartuccelli, Doering and Gibbon [4].

Note: We must assume u is a sufficiently regular function for all time in the three dimensional case.

These estimates take the form of differential inequalities for various semi-norms of u.

Definition 4.7.1 *For* $d = 2, 3$ *we define the functionals*

$$H_N = \sum_{i=1}^{d} \sum_{|n|=N} \int |D^n u_i|^2 dx \equiv |u|^2_{N,2,\Omega_L} \tag{4.87}$$

which are equivalent to the square of the usual norms defined on $\dot{H}^N_{per}(\Omega_L)^d$ or V_N (see above and Chapter 3).

Note (reiteration): For mean-zero, space periodic functions (as here) this semi-norm H_N is equivalent to the full Sobolev norm on $H^N(\Omega_L) \equiv W^{N,2}(\Omega_L)$ – see Chapter 3.

Lemma 4.7.1 *On periodic boundary conditions, for u sufficiently regular and $\forall\ r, s, N \in \mathbb{N}$ with $s \leq N$ we have*

$$H_N \leq H_{N+r}^{\frac{s}{r+s}} H_{N-s}^{\frac{r}{r+s}} . \tag{4.88}$$

Proof: Step 1:

Firstly, let $\tilde{H}_M = \sum_{|m|=M} \int |D^m \phi_i|^2\, dx$. Now we show that (with ϕ_i periodic)

$$\tilde{H}_M \leq \tilde{H}_{M+1}^{1/2} \tilde{H}_{M-1}^{1/2} . \tag{4.89}$$

$$\begin{aligned} \tilde{H}_M &= -\sum_{i,M} \int (D^{m+1}\phi_i)\,(D^{m-1}\phi_i)\, dx \\ &\leq \left[\sum \int (D^{m+1}\phi_i)^2\, dx\right]^{1/2} \left[\sum \int (D^{m-1}\phi_i)^2\, dx\right]^{1/2} \end{aligned} \tag{4.90}$$

using the Cauchy-Schwarz inequality.

Step 2: Secondly, we show that $\forall\, M \in \mathbb{N}$,

$$\tilde{H}_M \leq \tilde{H}_{M+1}^{\frac{M}{(M+1)}} \tilde{H}_0^{\frac{1}{(M+1)}} . \tag{4.91}$$

To achieve this, we know from (4.89) that (4.91) holds for $M = 1$. Assume (4.91) holds for M. Then

$$\tilde{H}_{M+1} \leq \tilde{H}_{M+2}^{1/2} \tilde{H}_M^{1/2} \leq \tilde{H}_{M+2}^{1/2} \tilde{H}_{M+1}^{\frac{M}{2(M+1)}} \tilde{H}_0^{\frac{1}{2(M+1)}} \tag{4.92}$$

so

$$\tilde{H}_{M+1} \leq \tilde{H}_{M+2}^{\frac{(M+1)}{(M+2)}} \tilde{H}_0^{\frac{1}{(M+2)}} . \tag{4.93}$$

Hence (4.91) is true $\forall\, M \in \mathbb{N}$ by induction.

Step 3: Thirdly, we show that $\forall\, M, r \in \mathbb{N}$

$$\tilde{H}_M \leq \tilde{H}_{M+r}^{\frac{M}{(M+r)}} \tilde{H}_0^{\frac{r}{(M+r)}} . \tag{4.94}$$

We know from (4.91) that (4.94) holds for $r = 1$. Assume (4.94) holds for r. Then

$$\tilde{H}_M \leq \tilde{H}_{M+r}^{\frac{M}{(M+r)}} \tilde{H}_0^{\frac{r}{(M+r)}} \leq \tilde{H}_{M+r+1}^{\frac{M}{(M+r+1)}} \tilde{H}_0^{\frac{(r+1)}{(M+r+1)}} \tag{4.95}$$

where we have used (4.91). Hence (4.94) is true $\forall\, M, r \in \mathbb{N}$, by induction.

Step 4: Now suppose that originally $\phi_i = D^{n-m} u_i$, then

$$\tilde{H}_M = H_N \quad \text{and} \quad \tilde{H}_0 = H_{N-M} \tag{4.96}$$

and so with $M = s$ we have

$$H_N \leq H_{N+r}^{\frac{s}{s+r}} H_{N-s}^{\frac{r}{s+r}} .$$

□

Now recall that our periodic forcing function $f \in C^\infty(\Omega_L)^d$ is divergence-free and independent of time, and so f has a cut-off in its wave number spectrum at $k_{max} = 2\pi/\lambda_f$, where we regard λ_f as the smallest scale in the forcing.

For our system, we can introduce a natural time-scale as follows: $\tau_0 = L^2/\nu$.

Thus we can introduce velocity and vorticity fields associated with the forcing function:

$$u_f = \tau_0 f \quad \text{and} \quad \omega_f = \text{curl} u_f \ . \tag{4.97}$$

So, we can define the following set of dimensionally uniform semi-norms:

Definition 4.7.2

$$F_N := H_N + |u_f|^2_{N,2,\Omega_L} \ . \tag{4.98}$$

For these semi-norms we can prove the following a-priori estimates:

Theorem 4.7.2 (Ladder Theorem) *For all $t > 0$, $N \geq 1$, $1 \leq s \leq N$ and $d = 2, 3$, we have*

$$\frac{1}{2}\dot{F}_N \leq -\nu \frac{F_N^{1+1/s}}{F_{N-s}^{1/s}} + (c^{(1)}_{N,s}|u|_{1,\infty,\Omega_L} + \nu\lambda_0^{-2})F_N \tag{4.99}$$

and

$$\frac{1}{2}\dot{F}_N \leq -\frac{\nu}{2}\frac{F_N^{1+1/s}}{F_{N-s}^{1/s}} + (c^{(2)}_{N,s}\frac{|u|^2_{0,\infty,\Omega_L}}{\nu} + \nu\lambda_0^{-2})F_N \tag{4.100}$$

where $\lambda_0^{-2} = L^{-2} + \lambda_f^{-2}$ and $c^{(j)}_{N,s}$ $(j = 1, 2)$ are constants which depend only on N and s.

Proof: In [4] the following a-priori estimates were established:

$$\frac{1}{2}\dot{H}_N \leq -\nu H_{N+1} + c^{(1)} H_N |u|_{1,\infty} + H_N^{1/2}|f|_{N,2,\Omega_L} \ , \tag{4.101}$$

$$\frac{1}{2}\dot{H}_N \leq -\frac{\nu}{2} H_{N+1} + c^{(2)}\frac{H_N|u|^2_{0,\infty}}{\nu} + H_N^{1/2}|f|_{N,2,\Omega_L} \ . \tag{4.102}$$

Now recall our definition for F_N, and also the assumed properties of f. I will indicate the proof of (4.99) as the proof of (4.100) follows analogously (from (4.102)).

Add and subtract $\nu|u_f|_{N+1,2}$ to (4.101) to get

$$\frac{1}{2}\dot{F}_N \leq -\nu F_{N+1} + c^{(1)} H_N |u|_{1,\infty} + \nu|u_f|^2_{N+1,2} + H_N^{1/2}|f|_{N,2,\Omega_L} \ . \tag{4.103}$$

Now consider the last term in (4.103). Young's inequality $\Rightarrow$

$$H_N^{1/2}|f|_{N,2} \leq \tau_0^{-1} H_N + \tau_0|f|^2_{N,2} = \tau_0^{-1} F_N = L^{-2}\nu F_N \ . \tag{4.104}$$

The third term on the right-hand side of (4.103) can be majorised as follows:

$$\nu|u_f|^2_{N+1,2} \leq \nu\frac{|u_f|^2_{N+1,2}F_N}{|u_f|^2_{N,2}} \leq \nu\lambda_f^{-2}F_N \tag{4.105}$$

where λ_f is the cut-off scale described above.

Thus, if we include both these results in (4.103) and use (4.88), we get (4.99). □

4.8 Consequence of the Ladder Structure.

Note: Henceforth, we will assume (without loss of generality) $u_0 \in C^\infty(\Omega_L)$ – recall our discussion on the question of regularity at $t = 0$.

We are interested in the long-time behaviour of solutions to the Navier-Stokes equations.

Consider the following strict definitions for the 'time asymptotic upper bound' and 'time average' of the function $f(u(t))$:

$$\overline{f} := \sup_{u_0 \in C^\infty} \limsup_{t\to\infty} f(S(t)u_0) \tag{4.106}$$

and

$$\langle f\rangle := \limsup_{t\to\infty} \sup_{u_0\in C^\infty} \frac{1}{t}\int_0^t f(S(\tau)u_0)\,d\tau \,. \tag{4.107}$$

Recall the results of Section 4.3.

1. $d = 2, 3$: The energy H_0 is uniformly bounded in time, provided that it is bounded initially. Furthermore, we see that there exists an absorbing ball in $\dot{L}^2_{per}(\Omega_L)^d$ whose radius is given by
$$\overline{H}_0 \le \frac{L^4}{\nu^2}\|f\|^2_{L^2} \tag{4.108}$$
and we also have the time average result
$$\langle H_1\rangle \le \frac{L^2}{\nu}\|f\|^2_{L^2} \,. \tag{4.109}$$

2. $d = 2$: The 'enstrophy' $H_1 \equiv |u|_{1,2,\Omega_L}$ is uniformly bounded in time, provided it is initially bounded. Also, there exists an absorbing ball in $\dot{H}^1_{per}(\Omega_L)^d$ whose radius is given by
$$\overline{H}_1 \le \frac{L^4}{\nu^2}|f|^2_{1,2} \tag{4.110}$$
or even
$$\overline{H}_1 \le \frac{L^2}{\nu^2}\|f\|^2_{L^2} \tag{4.111}$$
and we also have
$$\langle \|D^2u\|^2_{L^2}\rangle \le \frac{\|f\|^2_{L^2}}{\nu^2} \,. \tag{4.112}$$

The latter of the following two lemmas will prove useful in what follows.

Lemma 4.8.1 *For $y \ge 0$, $\alpha > \beta > 1$, and $\epsilon > 0$*

$$y^\beta \le \left(\frac{\beta-1}{\alpha-1}\right)\epsilon^{\frac{\alpha-1}{\beta-1}}\,y^\alpha + \left(\frac{\alpha-\beta}{\alpha-1}\right)\epsilon^{-\frac{\alpha-1}{\alpha-\beta}}\,y \,. \tag{4.113}$$

Proof: Via Young's inequality (under the hypotheses of the lemma),

$$y^\beta = y^{\mu+\nu} \le \frac{1}{p}\left(y\epsilon^{1/\mu}\right)^{\mu p} + \frac{1}{q}\left(\frac{y}{\epsilon^{1/\nu}}\right)^{\nu q} \tag{4.114}$$

with $\mu + \nu = \beta$, $1/p + 1/q = 1$ and $1 < p, q < \infty$. Now choose $\mu p = \alpha$ and $\nu q = 1$, so we find that $\mu = \frac{\alpha(\beta-1)}{\alpha-1}$ and $\nu = \frac{\alpha-\beta}{\alpha-1}$. By making the appropriate substitutions, the result follows. □

Lemma 4.8.2 *For any solution $y(t) > 0$, $t > 0$ of a differential inequality*

$$x\dot{y} \leq -y^{\alpha} + x\,z\,y^{\beta}\ , \qquad t \in \mathbb{R}^{+} \tag{4.115}$$

where $\alpha > \beta > 1$ and $x = x(t) > 0$, $z = z(t) > 0$; we can make the following estimate

$$y(t) \leq y(0) \exp\left[c \int_0^t g(\tau)d\tau\right] \tag{4.116}$$

where

$$g(t) \equiv x(t)^{\frac{\beta-1}{\alpha-\beta}}\, z(t)^{\frac{\alpha-1}{\alpha-\beta}} \qquad \text{and} \qquad c = \frac{\alpha-\beta}{\alpha-1}\ . \tag{4.117}$$

Proof: Apply Lemma 4.8.1 to the right-hand side of (4.115) and choose

$$\epsilon(t) = [x(t)z(t)]^{-\frac{\beta-1}{\alpha-1}} \tag{4.118}$$

so that we get

$$\dot{y}(s) \leq c\,g(s)y(s) \quad \Rightarrow \quad \frac{d}{ds}\left(y(s)\exp\left[-c\int_{t_0}^{s} g(\tau)d\tau\right]\right) \leq 0\ . \tag{4.119}$$

Integrate between s and $t_0 + r$

$$\Rightarrow y(t_0 + r) \leq y(s) \exp\left[c\int_{t_0}^{t_0+r} g(\tau)d\tau\right] \tag{4.120}$$

which gives (4.116). However note that we can make the following alternative estimate (by integrating with respect to s between t_0 and $t_0 + r$):

$$y(t_0 + r) \leq \frac{1}{r}\left(\exp\left[c\int_{t_0}^{t_0+r} g(\tau)d\tau\right]\right)\int_{t_0}^{t_0+r} y(s)ds \tag{4.121}$$

(by analogy with the Uniform Gronwall Lemma). □

Remark: Oliver [72] (Appendix C, Lemmas 8 and 9) provides a much more general argument for the existence of absorbing balls for differential inequalities and we will use these two lemmas for large time absorbing ball radii.

Now let us consider the Ladder Theorem in the previous section.

For either of the ladders (4.99) or (4.100) we see that we can close the hierarchy of the structure of (4.103) via the following Gagliardo-Nirenberg inequalities:

$$|u|_{1,\infty}^2 \leq c\, F_{N+1}^{a} F_0^{(1-a)}, \quad a = (d+2)/[2\,(N+1)]\ ; \tag{4.122}$$

$$|u|_{0,\infty}^2 \leq c\, F_N^{b} F_0^{(1-b)}, \quad b = d/2N \tag{4.123}$$

where in each case we incur the restiction $2N > d$ (comes from requiring $a < 1$ and also $b < 1$).

Let us concentrate on the $|u|_{1,\infty}$ ladder (this in fact gives us sharper results). Closing the hierarchy as described, we get:

$$\frac{1}{2}\dot{F}_N \leq -\nu F_{N+1} + cF_{N+1}^{a/2} F_N F_0^{\frac{1-a}{2}} + \nu\lambda_0^{-2} F_N\ . \tag{4.124}$$

Young's inequality and the inequality in Lemma 4.7.1 give

$$\frac{1}{2}\dot{F}_N \leq -\frac{(2-a)\nu}{a}\frac{F_N^2}{F_{N-1}} + c\frac{(2-a)}{a}\left[\frac{F_N^2 F_0^{1-a}}{\nu^a}\right]^{\frac{1}{2-a}} + \nu\lambda_0^{-2}F_N \tag{4.125}$$

where for convenience we have taken $s = 1$. From this inequality we find (via Lemma 4.8.2) that we can bound the $F_N(t)$ pointwise in time by $F_{N-1}(t)$ and $F_0(t)$, $\forall\ t > 0$ (*but* subject to the restriction $a < 1$, i.e. $N > d/2$) and further we see that we can find absorbing balls in all the $\dot{H}^n_{per}(\Omega_L)^d$, $\forall\ n \in \mathbb{N}$ (subject to the same restriction) with radii

$$\overline{F}_N \leq c\frac{\overline{F}_{N-1}^{\frac{2-a}{2(1-a)}}\ \overline{F}_0^{1/2}}{\nu^{1-a}} \tag{4.126}$$

Remarks:

1. For a strict treatment on how to find absorbing balls from differential inequalities of the type (4.125) I refer the reader to Oliver [72] where a very adequate account is provided.
2. Formally we can ignore the last term in (4.125) as this is of lower order.

Important Remarks:

1. The restriction $N > d/2$ is crucial. It means that we must have $N \geq 2$ in (4.126). So $\overline{F}_1$ is our "bottom rung", i.e. starting point of the ladder.
2. Thus if there exists an absorbing ball in $\dot{H}^1_{per}(\Omega_L)^d$, then there exist absorbing sets in all the $\dot{H}^n_{per}(\Omega_L)^d$, $\forall\ n \in \mathbb{N}$.
3. From the estimates included above we deduce that for $d = 2$ there exist absorbing balls in all the $\dot{H}^n_{per}(\Omega_L)^d$ and so via the theory outlined in Chapter 2, *for the $d = 2$ Navier-Stokes equations, there exists a compact, connected global attractor, which is included in C^∞ and bounded in $\dot{H}^n_{per}(\Omega_L)^d$, $\forall\ n \in \mathbb{N}$.*
4. For the $d = 3$ case, we know that if we assume $u \in V$ uniformly then via the theory of Section 4.6 there exists a global attractor. The estimates above re-iterate this assertion and also show that (with this assumption) *the global attractor is compact, connected, included in C^∞ and bounded in $\dot{H}^n_{per}(\Omega_L)^d$, $\forall\ n \in \mathbb{N}$.*

This is not the weakest assumption (in $d = 3$), as indicated earlier, and we investigate this in the next chapter.

4.9 Conclusions

In this chapter we have,

1. Rigorously outlined the Navier-Stokes problem we are going to investigate.
2. Defined the function spaces H and V, and also the specific form of the linear operator A from $D(A)$ onto H (defined via the inner product on V). We also defined the spaces V_s, $s \in \mathbb{R}$.

3. Provided a-priori estimates which indicated that the energy $||u||_H$ for the system ($d = 2$ and 3) is bounded for all time, and that the solution to the $d = 2$ Navier-Stokes problem was bounded in V for all time $t > 0$, whereas this was only true for a finite time in the $d = 3$ case.

4. Introduced the concepts of "weak" and "strong" solutions, which we then considered in detail. We were able to show the existence
   (a) of unique, strong solutions for all time in the $d = 2$ case,
   (b) of unique (while they exist), strong solutions in the three dimensional case for a pre-determined finite-time interval, the length of which depended on the initial data.

5. Examined how in three dimensions, after making certain assumptions (such as the solution u to our Navier-Stokes problem is assumed bounded in a certain function space), we could show the existence of strong solutions for all time. We also provided a theorem which shows how strong solutions are in fact regular solutions (while they exist).

6. Related the non-well-posedness of the three dimensional Navier-Stokes problem to our inability to show that unique strong solutions exist for all time.

7. Outlined a series of a-priori estimates, which we called the 'Ladder Theorem' and we showed how we can prove the existence of regular strong solutions for all time and the existence of a C^{∞} attractor, provided we assume the solution of our Navier-Stokes problem is uniformly bounded in V (i.e. we assume uniformly bounded strong solutions exist for all time).

It is important to note that there exists a vast amount of functional-analytic theory surrounding these (and other equations) and I refer the interested reader to, in particular, the books by Temam [86, 87, 88], Constantin and Foias [16], Ladyzhenskaya [52] and also Serrin [80, 79].

Chapter 5

The Lattice and Regularity Assumptions

We have seen how to "lift" weak solutions to strong ones via a-priori estimates like (4.76) and we have also witnessed the usefulness of the set of a-priori estimates provided by the Ladder Theorem. The Ladder Theorem, however, did not naturally expose the $L^{3+\epsilon}$ result of Section 4.5.1 but instead, required u to be uniformly bounded in V as a minimum assumption to prove the existence of strong solutions. We can then assert via Lemma 4.5.3 the regularity of strong solutions for all finite intervals of time, and further we use the Ladder Theorem to show the existence of absorbing sets in $\dot{H}^n_{per}(\Omega_L)^d$, for all $n \in \mathbb{N}$.

Recall that we want to address the following questions:

1. What are the minimum assumptions sufficient to show the existence and uniqueness of strong solutions on $[0, T]$, $\forall\, T > 0$?

2. What are the minimum assumptions sufficient to show the existence of a global attractor of C^∞ functions?

To this end, we then naturally ask ourselves:

3. Can we improve the Ladder Theorem in any way?

4. Can we reduce the minimum assumption for regularity of the solution u to the Navier-Stokes equations from u uniformly bounded in $L^{3+\epsilon}(\Omega_L)$ (Serrin's result [80]) to u uniformly bounded in some other space $W \supseteq L^{3+\epsilon}(\Omega_L)$? (Our ultimate goal would be $W = \dot{L}^2_{per}(\Omega_L)^d$, which we already know the solution u lies in, and we would therefore obtain regular solutions *without any assumptions.*)

By introducing an additional degree of freedom into the class of functionals considered, we will generalize the Ladder Theorem to a Lattice Theorem. Unfortunately this reproduces Serrin's result, however, it does shed some new light on the problem and in particular, we are able to provide some *alternative* minimum assumptions for regularity.

5.1 Notation and Brief Assessment of the Problem

The complete functional setting for the Navier-Stokes equations provided in the last chapter continues to apply here as well as in the next chapter.

Definition 5.1.1 *For $N \in \mathbb{N} \cup \{0\}$, $m \geq 1$ we define the functionals*

$$H_{N,m} = \sum_{i=1}^{d} \sum_{|n|=N} \int_{\Omega_L} |D^n u_i|^{2m} dx \equiv \|D^N u\|_{2m}^{2m} \equiv |u|_{N,2m,\Omega_L}^{2m} \tag{5.1}$$

which are equivalent to the full Sobolev norms on the spaces $\dot{W}_{per}^{N,2m}(\Omega_L)^d$ i.e. the Sobolev space of periodic functions with zero mean (recall the theory of Chapter 3).

When one calculates the ladder a-priori estimates, the pressure $\mathcal{P}$ is naturally removed via a divergence theorem result with space-periodic boundary conditions. As we will see when we begin to calculate the Lattice a-priori estimates, this is no longer the case. However we are able to deal with the pressure quite naturally from the Navier-Stokes equations and this in turn, releases the idea of transferring the minimum assumptions for regularity onto the pressure. This constitutes the latter half of this chapter. I also refer the reader to Bartuccelli et al. [6].

5.2 The Lattice Theorem

Theorem 5.2.1 (Lattice Theorem) *Assuming u is a smooth solution of the Navier-Stokes equations (4.1)-(4.4), then for $d = 3$, $N \geq 1$ and $1 \leq m \leq 2$, we have*

$$\frac{1}{2m}\dot{H}_{N,m} \leq -\nu\, c_1 \frac{H_{N,m}^{1+1/m}}{H_{N-1,m}^{1/m}} + c_{N,m}\, \nu^{-3/5}\, H_{N,m}\, \|Du\|_4^{8/5} \tag{5.2}$$

or

$$\frac{1}{2m}\dot{H}_{N,m} \leq -\nu\, c_1 \frac{H_{N,m}^{1+1/m}}{H_{N-1,m}^{1/m}} + c_{N,m}\, \nu^{\frac{-p}{(1-p)}}\, H_{N,m}^{1+\frac{1}{2mq}}\, \|u\|_4^{(N-1)/q} \tag{5.3}$$

where $p = \dfrac{3[m(N-1)+2]}{8mN}$ and $q = N(1-p)$.

5.3 Proof of Lattice Theorem

From the incompressible Navier-Stokes equations and our definition for $H_{N,m}$,

$$\begin{aligned} \frac{1}{2m}\dot{H}_{N,m} &= \sum_{i=1}^{d} \sum_{|n|=N} \int_{\Omega_L} (D^n u_i)^{2m-1} D^n \left[-(u\cdot\nabla)u_i + \nu\Delta u_i - \mathcal{P}_{,i} + f_i\right] dx \\ &= T_{NL} + T_L + T_P + T_F\,. \end{aligned} \tag{5.4}$$

5.3.1 The Laplacian Term T_L

Integration by parts gives,

$$\begin{aligned} T_L &= \nu \sum_i \sum_{|n|=N} \int_{\Omega_L} (D^n u_i)^{2m-1} D^n \Delta u_i \, dx \\ &= -\nu \frac{(2m-1)}{m^2} \sum_{i,k} \sum_{|n|=N} \int_{\Omega} \left[D_k \left((D^n u_i)^m \right) \right]^2 dx \ . \end{aligned} \tag{5.5}$$

Consequently, if we define

$$B_{i,n}^{(m)} = (D^n u_i)^m \tag{5.6}$$

so

$$\|B_{i,n}^{(m)}\|_2^2 = H_{N,m} \tag{5.7}$$

then we find that

$$T_L = -\nu \frac{(2m-1)}{m^2} \|DB_{i,n}^{(m)}\|_2^2 \ . \tag{5.8}$$

Now, note that if we perform integration by parts on $H_{N,m}$ and then use the Cauchy-Schwarz inequality,

$$H_{N,m}^2 \leq (2m-1)^2 \|DB_{i,n}^{(m)}\|_2^2 \sum_i \sum_{|n|=N} \int_{\Omega_L} |D^n u_i|^{2(m-1)} |D^{n-1} u_i|^2 \, dx \ . \tag{5.9}$$

Using Hölder's inequality, it turns out that

$$H_{N,m}^2 \leq (2m-1)^2 \|DB_{i,n}^{(m)}\|_2^2 \, H_{N,m}^{1-1/m} \, H_{N-1,m}^{1/m} \ . \tag{5.10}$$

Consequently,

$$- \|DB_{i,n}^{(m)}\|_2^2 \leq - \frac{1}{(2m-1)^2} \frac{H_{N,m}^{1+1/m}}{H_{N-1,m}^{1/m}} \ . \tag{5.11}$$

which gives us the expression for the Laplacian term in the Lattice Theorem.

5.3.2 The Pressure Term T_P

$$|T_P| = \left| \sum_i \sum_{|n|=N} \int_{\Omega_L} (D^n u_i)^{2m-1} D^n (\mathcal{P}_{,i}) \right| dx \leq H_{N,2m-1}^{1/2} \, T_S^{1/2} \tag{5.12}$$

where

$$\begin{aligned} T_S &= \sum_i \sum_{|n|=N} \int_{\Omega_L} (D^n \mathcal{P}_{,i})^2 \, dx \\ &= \sum_{|n|=N} \int_{\Omega_L} (D^n \nabla \mathcal{P})^2 \, dx \\ &= \sum_{|(n-1)|=N-1} \int_{\Omega_L} (D^{(n-1)} \Delta \mathcal{P})^2 \, dx \ . \end{aligned} \tag{5.13}$$

Now we prove,

Lemma 5.3.1

1. $\Delta\mathcal{P} = -\sum_{i,j} u_{i,j}u_{j,i}$,

2. $||\Delta\mathcal{P}||_r \leq c\, H_{1,r}^{1/r}$, $\forall\ r \geq 1$.

Proof:

1. Taking the divergence of the Navier-Stokes equations gives

$$\Delta\mathcal{P} = -\nabla\cdot(u\cdot\nabla u) = -\sum_{i,j}\partial_i(u_j u_{i,j}) = -\sum_{i,j} u_{i,j}u_{j,i}\ . \tag{5.14}$$

2. Now simply take L^r-norm of both sides and apply the Cauchy-Schwarz inequality,

$$||\Delta\mathcal{P}||_r^r \leq \sum_{i,j}\int_{\Omega_L} |u_{i,j}|^r |u_{j,i}|^r\, dx \leq H_{1,r} \tag{5.15}$$

hence the result.

□

Remark: Note that we have assumed a divergence-free forcing function – we do *not* need to make this assumption, the Lattice Theorem would still be correct (we simply get lower order terms in $H_{N,m}$).

With the first result of the lemma

$$T_S = \sum_{|n-1|=N-1}\int_{\Omega_L}\left|\sum_{i,j} D^{n-1}(u_{i,j}u_{j,i})\right|^2 dx\ . \tag{5.16}$$

Using the Schwarz inequality and a Leibniz expansion, we get

$$T_S \leq \sum_{i,j}\sum_{|(n-1)|=N-1}\int_{\Omega_L}\left|\sum_{\ell} C_\ell^{n-1} D^\ell(u_{i,j})D^{n-1-\ell}(u_{j,i})\right|^2 dx \tag{5.17}$$

and we will define

$$A_{i,j}^{(\ell)} = \sum_{|(n-1)|=N-1}\int_{\Omega_L} |D^{\ell+1_j}u_i|^2 |D^{(n-1)-\ell+1_i}u_j|^2\, dx \tag{5.18}$$

where $i,j = 1,\ldots,d$; $n = (n_1, n_2, \ldots, n_d)$, $\ell = (\ell_1, \ell_2, \ldots, \ell_d)$ are multi-indicies. $(n-1)$ is also a multi-index (given in this form for notational purposes only) such that $|(n-1)| = (n-1)_1 + \cdots + (n-1)_d = N-1$. From the Leibniz operation we must have $\ell_i \leq (n-1)_i$, $\forall\ i$.

Consequently,

$$T_S \leq \sum_{i,j}\sum_{\ell} C_\ell^N\, A_{i,j}^{(\ell)}\ . \tag{5.19}$$

A Hölder inequality gives

$$A_{i,j}^{(\ell)} \leq ||D^{\ell+1_j}u_i||_p^2 ||D^{(n-1)-\ell+1_i}u_j||_q^2 \tag{5.20}$$

where $\dfrac{1}{p}+\dfrac{1}{q}=\dfrac{1}{2}$.

There are two paths we can now follow:

(i) The $\|Du\|_4$ Lattice

Consider the following set of Gagliardo-Nirenberg inequalities,

$$\|D^{\ell+1_j}u_i\|_p \le c\,\|D^N u_i\|_r^a\,\|Du_i\|_s^{1-a} \tag{5.21}$$

$$\|D^{(n-1)-\ell+1_i}u_j\|_q \le c\,\|D^N u_j\|_r^b\,\|Du_j\|_s^{1-b} \tag{5.22}$$

where $\dfrac{1}{p}+\dfrac{1}{q}=\dfrac{1}{2}$, and we also require

$$\frac{1}{p}=\frac{L}{d}+a\left(\frac{1}{r}-\frac{N-1}{d}\right)+\frac{1-a}{s} \qquad 0\le \tfrac{L}{N-1}\le a<1\,, \tag{5.23}$$

$$\frac{1}{q}=\frac{N-L-1}{d}+b\left(\frac{1}{r}-\frac{N-1}{d}\right)+\frac{1-b}{s} \qquad 0\le \tfrac{N-L-1}{N-1}\le b<1\,. \tag{5.24}$$

Choose

$$\frac{1}{ap}=\frac{1}{r}+\frac{1-a}{as} \qquad \Rightarrow a=\tfrac{L}{N-1}\,, \qquad \forall\, d\,; \tag{5.25}$$

$$\frac{1}{bq}=\frac{1}{r}+\frac{1-b}{bs} \qquad \Rightarrow b=\tfrac{N-L-1}{N-1}\,, \qquad \forall\, d\,. \tag{5.26}$$

Hence

$$0\le a<1 \quad\Leftrightarrow\quad 0\le L<N-1 \tag{5.27}$$

$$0\le b<1 \quad\Leftrightarrow\quad 0\le N-L-1<N-1 \tag{5.28}$$

where equality also holds above. Since $a+b=1$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ we see that we must have

$$\frac{1}{r}+\frac{1}{s}=\frac{1}{2}\,. \tag{5.29}$$

Therefore

$$A_{i,j}^{(\ell)} \le c\,\|D^N u\|_r^2\|Du\|_s^2 \tag{5.30}$$

and we get

$$T_P \le c\,H_{N,2m-1}^{1/2}\,H_{N,r/2}^{1/r}\,\|Du\|_s\,. \tag{5.31}$$

Since $\dfrac{1}{r}+\dfrac{1}{s}=\dfrac{1}{2}$, a convenient natural choice here is $s=r=4$, which gives

$$T_P \le c\,H_{N,2m-1}^{1/2}\,H_{N,2}^{1/4}\,\|Du\|_4\,. \tag{5.32}$$

Now, note that we can write

$$H_{N,2m-1}^{1/2} \equiv \|B_{i,n}^{(m)}\|_{\frac{2}{m}(2m-1)}^{(2m-1)/m} \quad\text{and}\quad H_{N,r/2}^{1/r} \equiv \|B_{i,n}^{(m)}\|_{r/m}^{1/m} \tag{5.33}$$

and we can perform the following two Gagliardo-Nirenberg inequalities:

$$\|B_{i,n}^{(m)}\|_{\frac{2}{m}(2m-1)} \;\le\; c\,\|DB_{i,n}^{(m)}\|_2^{a_1}\,\|B_{i,n}^{(m)}\|_2^{1-a_1} \tag{5.34}$$

$$\|B_{i,n}^{(m)}\|_{r/m} \;\le\; c\,\|DB_{i,n}^{(m)}\|_2^{a_2}\,\|B_{i,n}^{(m)}\|_2^{1-a_2} \tag{5.35}$$

where $a_1 = 3(m-1)/2(2m-1)$ and $a_2 = 3(r-2m)/2r$ and where we must restrict ourselves to $1 \le m \le 2$ when $r = 4$. Combining these two inequalities in our expression above for T_P, we find

$$T_P \le c\left[\|DB_{i,n}^{(m)}\|_2^2\right]^{3/8} H_{N,m}^{5/8}\,\|Du\|_4 \quad \text{where } 1 \le m \le 2\,. \tag{5.36}$$

If we now use a Young's Inequality (multiply and divide by $\nu^{3/8}$) and combine the $\|DB_{i,n}^{(m)}\|_2$ term in (5.36) with the Laplacian term, we obtain the $\|Du\|_4$ lattice.

(ii) The $\|u\|_4$ Lattice

Instead of the Gagliardo-Nirenberg inequalities employed above, we consider the following set

$$\|D^{\ell+1_j}u_i\|_p \;\le\; c\,\|D^N u_i\|_r^a\,\|u_i\|_s^{1-a} \tag{5.37}$$
$$\|D^{(n-1)-\ell+1_i}u_j\|_q \;\le\; c\,\|D^N u_j\|_r^b\,\|u_j\|_s^{1-b} \tag{5.38}$$

where $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$; $r,s\ge 1$, and we require

$$\frac{1}{p} = \frac{L+1}{d} + a\left(\frac{1}{r}-\frac{N}{d}\right) + \frac{1-a}{s}\,, \quad 0 \le \tfrac{L+1}{N} \le a < 1 \tag{5.39}$$
$$\frac{1}{q} = \frac{N-L}{d} + b\left(\frac{1}{r}-\frac{N}{d}\right) + \frac{1-b}{s}\,, \quad 0 \le \tfrac{N-L}{N} \le b < 1\,. \tag{5.40}$$

If we choose

$$\frac{1}{ap} = \frac{1}{r} + \frac{1-a}{as} \quad\Rightarrow\quad a = \frac{L+1}{N}\,, \quad \forall\, d \tag{5.41}$$
$$\frac{1}{bq} = \frac{1}{r} + \frac{1-b}{bs} \quad\Rightarrow\quad b = \frac{N-L}{N}\,, \quad \forall\, d \tag{5.42}$$

then,

$$0 \le a < 1 \;\Leftrightarrow\; 0 \le L+1 < N \tag{5.43}$$
$$0 \le b < 1 \;\Leftrightarrow\; 0 \le N-L < N \tag{5.44}$$

where equality also holds here. Since we require $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, this means we must have

$$r = \frac{2s\,(N+1)}{sN - 2\,(N-1)}\,. \tag{5.45}$$

Hence, since $a+b=(N+1)/N$ we get

$$T_P \le d\,c\,H_{N,2m-1}^{1/2}\,H_{N,r/2}^{(N+1)/rN}\,\|u\|_s^{(N-1)/N}\,. \tag{5.46}$$

We can now see that when $N=1$, then we must have $r=4$, independent of s, and further, for general N, if we choose $s=4$ then r is again exactly equal to 4 (independent of N).

Remark: Another choice could be $r = 2\,(2m-1)$ or $r = 2m$ and to leave the parameter s free. The results which we outline in the next main section below turn out to be the same with either of these choices.

With the choice $r = s = 4$

$$T_P \le d\,c\,H_{N,2m-1}^{1/2}\,H_{N,2}^{(N+1)/4N}\,\|u\|_4^{(N-1)/N}\,. \tag{5.47}$$

Since from (5.33) we know that $H_{N,r/2} \equiv \|B_{i,n}^{(m)}\|_{r/m}^{r/m}$, we can use the inequalities (5.34) and (5.35) to get

$$T_P \leq d\,c \left[\|DB_{i,n}^{(m)}\|_2^2\right]^{\tilde{p}} H_{N,m}^{1-\tilde{p}+\frac{1}{2mN}} \|u\|_4^{(N-1)/N} \tag{5.48}$$

where $\tilde{p} = 3\,[m(N-1)+2]/8mN$. Consequently, an application of Young's inequality gives us the $\|u\|_4$ lattice.

5.3.3 The Non-linear and Forcing Terms T_{NL} and T_F

Although it is possible to bound the non-linear term above as in [4], it is also possible to show that it has an upper bound proportional to the pressure term.

$$T_{NL} = -\sum_{i,j}\sum_{|n|=N}\int_{\Omega_L} |D^n u_i|^{2m-1} |D^n(u_j u_{i,j})|\,dx \qquad N \geq 1\,. \tag{5.49}$$

Consequently, we can use a Leibniz expansion

$$T_{NL} = -\sum_{i,j}\sum_{|n|=N}\int_{\Omega_L} |D^n u_i|^{2m-1} |\sum_{\ell\neq 0} C_\ell^n\, D^\ell u_j\, D^{n-\ell} u_{i,j}|\,dx$$

where integration by parts reveals that the $\ell = 0$ term is zero. Cauchy's inequality implies

$$T_{NL} \leq H_{N,2m-1}^{1/2} \left[\sum_{i,j}\sum_{|n|=N}\sum_{\ell\neq 0} C_\ell^n \int_{\Omega_L} |D^\ell u_j|^2\, |D^{n-\ell} u_{i,j}|^2\,dx\right]^{1/2} \tag{5.50}$$

It is now easy to see that we can deal with the non-linear term by following a procedure very similiar to that previously used for the pressure term. Note that $\ell \neq 0$ allows us to find the appropriate upper bound for the non-linear term.

The forcing term can be bounded with a single application of Hölder's inequality, as follows:

$$\begin{aligned} T_F &\leq \sum_i \sum_{|n|=N}\int_{\Omega_L} |D^n u_i|^{2m-1}\, |D^n f_i|\,dx \\ &\leq H_{N,m}^{1-\frac{1}{2m}} \|D^N f\|_{2m}\,. \end{aligned} \tag{5.51}$$

Thus we have now proved both parts of the Lattice Theorem. □

Remark: Any forcing terms are of lower order and we therefore omit them from the statement of the theorem.

5.4 The $L^{3+\epsilon}$ Result From the Lattice

Firstly, set $N = 1$ in the second version (5.3) of the lattice. We see that the differential inequality takes the form

$$\frac{1}{2m}\dot{H}_{1,m} \leq -\nu\, c\, \frac{H_{1,m}^{1+1/m}}{H_{0,m}^{1/m}} + c_{1,m}\, \nu^{\frac{-p}{(1-p)}}\, H_{1,m}^{1+\frac{1}{2mq}} \tag{5.52}$$

where $p = 3/4m$ and $q = 1 - p$.

Recall Lemma 4.8.2 of Chapter 4. We immediately see that $H_{1,m}(t)$ will remain bounded for all $t \geq 0$ provided $H_{0,m}(t)$ also does so for all $t \geq 0$, with the restriction $2q > 1 \Rightarrow m > 3/2$.

This result is analogous to that of the a-priori estimate (4.76), and in fact, we could use this alternative in the subsequent arguments of Chapter 4 to "lift" weak solutions to strong ones.

Further, we see that if we assume $H_{0,3/2+\epsilon}$ ($\epsilon > 0$) is uniformly bounded, then we can find an absorbing set in $H_{1,3/2+\epsilon}$ by analogy with the arguments of Section 4.8.

For more general $N \geq 1$, we can close the hierarchy of the lattice (5.3) via the following Gagliardo-Nirenberg inequality

$$||u||_4 \leq c\, H_{N,m}^{a_2/2m} H_{0,m}^{(1-a_2)/2m} \tag{5.53}$$

where $a_2 = 3(2-m)/4Nm$, and we have the restriction $1 \leq m \leq 2$.

Then we find upper bounds (point-wise in time) for the $H_{N,m}(t)$ in terms of $H_{N-1,m}(t)$ and $H_{0,m}(t)$ via Lemma 4.8.2 (with $m > 3/2$) and in fact, an absorbing ball argument gives

$$\overline{H}_{N,m} \leq c\left[\nu^{-8N^2m^2}\overline{H}_{N-1,m}^{N\alpha}\overline{H}_{0,m}^{(\alpha-Nm)(N-1)}\right]^{\frac{1}{\beta}} \tag{5.54}$$

where $\alpha = [5Nm + 3m - 6]$, $\beta = [\alpha(N-1) - 6N + 4Nm]$ and $N \geq 1$. Thus we have recovered Serrin's result.

Alternatively, considering (5.2), $||Du||_4$ can be controlled by:

$$||Du||_4 \leq c\, H_{N,m}^{a_1/2m} H_{0,m}^{(1-a_1)/2m} \tag{5.55}$$

where $a_1 = (m+6)/4Nm$ and $\frac{1}{N} \leq a_1 < 1$, i.e. for $N = 2$ we must restrict ourselves to $1 \leq m \leq 2$. Also

$$H_{N-1,m} \leq c\, H_{N,m}^{1-\frac{1}{N}} H_{0,m}^{1/N}\,. \tag{5.56}$$

If we substitute (5.55) and (5.56) into the lattice (5.2) and look for upper bounds point-wise in time (via Lemma 4.8.2) as well as an absorbing ball, we find that

$$\overline{H}_{N,m} \leq c\left[\nu^{-4m^2N}\overline{H}_{0,m}^{[2m(N+1)-3]}\right]^{\frac{1}{[2m-3]}} \quad \text{for } N \geq 2,\ m > 3/2 \tag{5.57}$$

Thus $H_{0,3/2+\epsilon}$ is the bottom point of this lattice, and again, we have recovered Serrin's result.

5.5 Boundedness of $||\mathcal{P}||_s$ and Regularity

The first theorem of this section is an a-priori estimate for the functionals $H_{0,m}$ and we use this estimate to show that we can prove the existence of strong solutions for all times provided we assume that $||\mathcal{P}||_s$ is uniformly bounded in time, with s given. The second theorem given in this section uses a result of the Lattice Theorem to relax some of the assumptions we must make in order to show a C^∞ attractor.

Theorem 5.5.1 *For smooth solutions u of the three dimensional Navier-Stokes equations (4.1)–(4.4), provided $m \geq 2$ and with $\frac{m-1}{2m+1} < \delta \leq m-1$ we get*

$$\frac{1}{2m}\dot{H}_{0,m} \leq -\nu\, c_{1,m}\frac{H_{0,m}^{(2m+1)/3}}{H_{0,m-1}^{2m/3}} + c_{2,m}\, H_{0,m}^{\beta_m/\gamma_m}\left(||\mathcal{P}||_{2(1+\delta)}\right)^{2m/\gamma_m} \tag{5.58}$$

where

$$\beta_m = (1-a)(m-1) \quad and \quad \gamma_m = m - a(m-1) \tag{5.59}$$

with

$$a = \frac{3}{2}\left[1 - \frac{m\delta}{(m-1)(\delta+1)}\right] . \tag{5.60}$$

Proof:

Beginning with $H_{0,m} = \int_{\Omega_L} |u|^{2m}\, dx$ and differentiating with respect to time gives

$$\frac{1}{2m}\dot{H}_{0,m} \leq -\nu\,(2m-1)\int_{\Omega_L} |Du|^2|u|^{2(m-1)}\, dx \;+\; \left|\int_{\Omega_L} (\nabla\mathcal{P})\, u^{2m-1}\, dx\right| . \tag{5.61}$$

Now we take the last term, integrate by parts and use a Hölder inequality

$$\begin{aligned} T_{\mathcal{P}} &= \left|\int_{\Omega_L} (\nabla\mathcal{P})\, u^{2m-1}\, dx\right| \\ &\leq (2m-1)\int_{\Omega_L} |\mathcal{P}|\,|Du|\,|u|^{2(m-1)}\, dx \\ &\leq (2m-1)\,\|\mathcal{P}\|_{2(1+\delta)}\,\|u\|_{2m\eta}^{m-1}\left[\int_{\Omega_L} |Du|^2\,|u|^{2(m-1)}\, dx\right]^{1/2} \end{aligned} \tag{5.62}$$

where $\frac{\delta}{1+\delta} = \frac{m-1}{m\eta}$. Now

$$\|u\|_{2m\eta} = \|u^m\|_{2\eta}^{1/m} \leq c\left(\int_{\Omega_L} |Du|^2\,|u|^{2(m-1)}\, dx\right)^{a/2m} H_{0,m}^{(1-a)/2m} \tag{5.63}$$

where we have used a Gagliardo-Nirenberg inequality with $a = 3\,[\eta-1]/2\eta$. Since $0 \leq a < 1$ we find that $1 \leq \eta < 3$ which implies that δ must lie in the range

$$\frac{m-1}{2m+1} < \delta \leq m-1 . \tag{5.64}$$

The pressure term becomes

$$T_{\mathcal{P}} \leq c\,(2m-1)\|\mathcal{P}\|_{2(1+\delta)}\left(\int_{\Omega_L} |Du|^2\,|u|^{2(m-1)}\, dx\right)^{[m+a\,(m-1)]/2m} H_{0,m}^{(1-a)(m-1)/2m} . \tag{5.65}$$

We use a Young's inequality in (5.65) and combine the $\int_{\Omega_L} |Du|^2|u|^{2(m-1)}\, dx$ term with the Laplacian term and then use interpolation:

$$-\, m^2\int_{\Omega_L} |Du|^2\,|u|^{2(m-1)}\, dx = -\int_{\Omega_L} |D(u^m)|^2\, dx \leq -\,c\frac{H_{0,m}^{(2m+1)/3}}{H_{0,m-1}^{2m/3}} . \tag{5.66}$$

We have now proved the theorem. □

Important Remark: Consider the consequences of the theorem: If we take $m = 2$ then we can show that $H_{0,2}(t)$ is bounded point-wise in time $\forall\, t \geq 0$ provided that $\|\mathcal{P}\|_{2(1+\delta)}$ is bounded for all $t \geq 0$ (via Lemma 4.8.2) and further an absorbing ball argument gives

$$\overline{H}_{0,2} \leq c\,\nu^{-\frac{3(7\delta+1)}{4(5\delta+2)}}\left(\overline{H}_{0,1}\right)^{\frac{7\delta+1}{5\delta+2}}\left(\overline{\|\mathcal{P}\|}_{2(1+\delta)}\right)^{\frac{6(1+\delta)}{5\delta+2}} \tag{5.67}$$

where δ lies in the range $1/5 < \delta \leq 1$. Thus $\|\mathcal{P}\|_{2(1+\delta)}$ uniformly bounded implies $H_{0,2} = \|u\|_4$ is uniformly bounded.

Corollary 5.5.2 *A sufficient condition to show the existence of strong (and so regular) solutions as well as the existence of a global attractor included in C^∞ and bounded in H^n, $\forall\ n \in \mathbb{N}$ is $\overline{||\mathcal{P}||}_{2(1+\delta)}$ uniformly bounded, $1/5 < \delta \leq 1$.*

Proof: This result follows from the remark above combined with the Lattice Theorem (5.3) (with $m = 2$), using all the techniques previously outlined to lift weak solutions to strong ones. □

Theorem 5.5.3 *The ball denoted by $\mathcal{B} = B_{\dot{H}^{0,4}_{p\acute{e}r}(\Omega_L)^d}(0, R)$ is an absorbing set in $\dot{H}^{0,4}_{p\acute{e}r}(\Omega_L)^d$ for the semi-group $S(t)$ (defined via the assumptions of the corollary above) where*

$$R = \overline{H}_{0,2} \leq c\left[\nu^{-\gamma/4}\overline{H}_{0,1}^{(7\delta+1)(s+6)}\overline{||\mathcal{P}||}_s^{6s(\delta+4)}\right]^{1/D} \tag{5.68}$$

with $s > 15/8$ and

$$\begin{aligned} D &= 5\delta s - 60\delta + 47s - 78 & (5.69)\\ \gamma &= 21\delta s + 1026\delta - 573s + 594 & (5.70)\end{aligned}$$

Furthermore all the absorbing sets in $\dot{H}^{n,4}_{p\acute{e}r}(\Omega_L)^d$, $\forall\ n \in \mathbb{N}$ are included in $\mathcal{B}$.

Proof:

1. Use the first result of Lemma 5.3.1 in a Gagliardo-Nirenberg inequality to obtain

$$||\mathcal{P}||_{2(1+\delta)} \leq c\,||\Delta\mathcal{P}||_2^{\frac{b}{2}}||\mathcal{P}||_s^{1-b} \leq c\,H_{1,2}^{b/2}||\mathcal{P}||_s^{1-b} \tag{5.71}$$

   where $s \leq 2(1+\delta)$ is to be determined. The exponent $b(s, \delta)$ is given by

$$b(s,\delta) = 3\frac{2(1+\delta) - s}{(1+\delta)(s+6)}\ . \tag{5.72}$$

2. To perform the next step, we need to control $H_{1,2}$ by $H_{0,2}$. This is conveniently furnished from the Lattice Theorem (5.54) by choosing $N = 1$ and $m = 2$ to give

$$\overline{H}_{1,2} \leq c\,\nu^{-16}\left(\overline{H}_{0,2}\right)^5\ . \tag{5.73}$$

   Note that the constant is dimensionless.

3. Using the results from 1 and 2 in (5.67) we easily find that $\overline{H}_{0,2}$ is controlled by $\overline{H}_{0,1}$ and $\overline{||\mathcal{P}||}_s$ provided

$$1 > 15b(s,\delta)\left(\frac{1+\delta}{5\delta+2}\right) \tag{5.74}$$

   which yields

$$s > \frac{78 + 60\delta}{47 + 5\delta}\ . \tag{5.75}$$

Since δ lies in the range $1/5 < \delta \leq 1$ we find that any choice of s which satisfies $s > 15/8$ will do. □

Remark: This theorem further restricts the size of the absorbing set in $\dot{H}^{0,4}_{p\acute{e}r}(\Omega_L)^d$ to within the radius $R = R(\overline{||\mathcal{P}||}_{15/8+\epsilon})$, $\forall\ \epsilon > 0$ and so we have further relaxed the classes of functions which we need assume lie in the attractor in order to show the existence of a global attractor included in C^∞ and bounded in H^n, $\forall\ n \in \mathbb{N}$ (described in the corollary above).

5.6 Conclusions and Further Work

Thus we can reproduce Serrin's $L^{3+\epsilon}$ result and also provide some concrete alternatives in the form of assumptions made on the pressure field. The last theorem indicates that it should be possible to improve Corollary 5.5.2 so that $||\mathcal{P}||_{15/8+\epsilon}$ assumed uniformly bounded should be sufficient to show the existence of strong solutions for all time – this might constitute some interesting further work as would any improvement on Serrin's result.

Chapter 6

Length Scales

We have examined carefully the mathematical questions of existence, uniqueness and regularity (smoothness) of the solution to the Navier-Stokes equations. We have seen that the existence of a regular solution in three dimensions for all time is indeterminable without some extra assumptions and further that there is a marked difference between two and three dimensional flows – which is of course reflected by a qualitative difference in physical behaviour, particularly, the vorticity is (at least) conserved for the two dimensional Navier-Stokes equations.

In this chapter, we introduce the notion of turbulence (irregular variations of the velocity field, spatially and temporally) as a physical behaviour exhibited by 'real' flows and observed in a wide variety of physical situations. The inherent nonlinearity (due to the transport term) of the Navier-Stokes equations, means that we expect temporal chaos (by analogy with finite dimensional nonlinear evolution systems, we suppose that our deterministic system will exhibit a sensitivity to initial conditions), but we also realize that for our infinite dimensional system we can also have spatial chaos.

We try to initiate an understanding of this behaviour by providing a rigorous basis for a set of minimum length scales which will hopefully 'resolve' turbulent flows.

6.1 Review of the Existing Theories of Turbulence

Let us examine the concept of turbulence and the theories derived so far used to descibe/explain it.

We are all aware of the intricate physical behaviour observed in the wake of a ship or aircraft, or the beautiful patterns occuring when two distinct fluids are mixed or the swirling eddies occuring along the edges of rapidly flowing rivers. These are features of 'high Reynolds number' flows – recall that $Re = UL/\nu$ which is inversely proportional to the viscosity (dissipation coefficient).

We attribute the common held notion of turbulence to Lewis Richardson:

"Turbulent flows consist of a hierarchical structure of entities which we call 'eddies' which have various 'sizes'. We generally accept that the forcing in the flow 'drives' the largest eddies which become unstable and branch into smaller eddies which in turn become unstable and produce even smaller eddies and so forth. It is generally considered that this process continues until the scales involved (e.g. eddy diameter) are such that molecular viscosity ν will rapidly damp further cascading and the energy is dissipated as heat."

6.1.1 Eddy Viscosity

At a simplistic level we realize that there are two scales, the molecular and the macroscopic (or hydrodynamical) levels.

Random molecular motions are characterized by the mean free path. Hydrodynamic scales are those macroscopic characteristic scales of our fluid model we wish to investigate (and on which the Navier-Stokes equations as a model for fluid flow is based – see the *Continuum Hypothesis*, for example, in Batchelor [8]). A result shown by Chapman, Enskog and others is that activity at the molecular scale tends to 'diffuse' the motions at the hydrodynamic scale (e.g. the smoothing of velocity gradients at hydrodynamic scales).

Transport coefficients such as the kinematic molecular viscosity are such that:

$$\nu = \nu_{mol} \sim \text{velocity of thermal molecular motion} \times \text{mean free path}$$

So we make the following analogy (due to Prandtl): Large-scale eddies are diffused into small-scale eddies – the diffusion coefficient (or 'eddy viscosity' – which smooths out gradients in the mean velocity) is such that

$$\nu_{eddy} \sim u_{rms} \cdot \ell \tag{6.1}$$

where u_{rms} is the root mean square of the fluctuating velocity and ℓ is the mixing length. Hence,

$$\frac{\nu_{eddy}}{\nu_{mol}} \sim \frac{u_{rms}\,\ell}{\nu_{mol}} = R = \quad \text{the (local) Reynolds number .}$$

In other words, the transport of momentum, heat and particles (and therefore kinetic energy dissipation) is enhanced (by a factor of R) in turbulent flows – "velocity gradients are smoothed out more rapidly in turbulent as opposed to laminar flows" – an observable effect.

Let ε_{loc} = local rate of (viscous) dissipation of energy.

$$\varepsilon_{loc} = \nu_{mol}|\nabla u|^2$$

on the scale L of eddying motion. On this scale, dissipation occurs via

$$\nu_{eddy} \sim u_{rms} \cdot L \tag{6.2}$$

and $|\nabla u| \sim u_{rms}/L$

$$\Rightarrow \varepsilon_{loc} \sim u_{rms} L \left(\frac{u_{rms}}{L}\right)^2 \sim \frac{u_{rms}^3}{L} = \frac{u_{rms}^2}{L/u_{rms}} \ .$$

In other words: eddying motions with kinetic energy u_{rms}^2 are dissipated in a time L/u_{rms}.

6.1.2 Scaling (Kolmogorov and Obukhov, 1941)

Equations (4.1) obviously represent the conservation laws of momentum and mass – the energy is estimated by (4.18).

Kolmogorov noticed the following scale invariance property of these equations when ν is set to zero:

Suppose we scale distance by λ, velocity by λ^h (with h arbitrary and real) and so time scales like λ^{1-h}. Then, we see that (4.1) (with $\nu = 0$) are invariant with respect to this scaling.

The following theory of scaling in turbulence relies on three basic assumptions:

1. The scale invariance property derived above (with $\nu = 0$) holds in a statistical sense (not necessarily true for detailed structures).

2. A finite flux of energy ε flows from large scales to small scales (where turbulence is dissipated) in the limit as $R \to \infty$.

3. The energy flux through the scale ℓ is assumed to depend only on ℓ and the local velocity u_ℓ of eddies of size ℓ.

   $$\text{Dimensional analysis implies } \varepsilon_\ell \sim \frac{u_\ell^3}{\ell} \text{ and } \varepsilon_\ell \text{ scales like } \lambda^{3h-1} \ .$$

   Scale invariance means that $h = 1/3$.

There are several consequences:

1. $u_\ell \sim \varepsilon^{1/3}\ell^{1/3}$, $\varepsilon \equiv \varepsilon_{loc}$ = rate of dissipation of energy.

   Then $\nabla u_\ell \sim \varepsilon^{1/3}\ell^{-2/3}$ is dominated by small scales and becomes infinite unless cascading is cut off at a minimum scale.

2. $(\delta u_\ell)^p \sim \varepsilon^{p/3}\ell^{p/3}$ where $(\delta u_\ell)^p$ is the 'structure function of order p' (= the average of the p^{th} power of velocity increments measured over distances ℓ) and $\zeta_p = p/3$ is the *scaling exponent.*

3. The energy spectrum (with k denoting the wave-number) satisfies

   $$E(k) = C\varepsilon^{2/3}k^{-5/3}$$

   where C is known as the Kolmogorov-Obukhov Constant (this theory does not predict a value for it).

4. The eddy viscosity at scale ℓ is

   $$\nu_\ell \sim \ell u_\ell \sim \varepsilon^{1/3}\ell^{4/3} \ .$$

The scaling invariance property does not hold at large scales (i.e. scales characterised by the diameter of the domain considered) as well as small scales, where the molecular viscosity ν_{mol} cannot be ignored.

Thus the above theory only applies so long as $\nu_\ell \gg \nu_{mol}$ i. e. $\ell \gg \lambda_{Ko}$, where

$$\lambda_{Ko} \sim \left(\frac{\nu_{mol}^3}{\varepsilon}\right)^{1/4}$$

is the *Kolmogorov dissipation scale.* (For $\ell \ll \lambda_{Ko}$ molecular viscosity is important and dominant).

Thus the scaling argument is only valid in the *Kolmogorov Inertial Range*

$$\lambda_{Ko} \ll \ell \ll L \ .$$

The extent of the inertial range is

$$\frac{L}{\lambda_{Ko}} \sim \frac{\varepsilon^{1/4}L}{\nu_{mol}^{3/4}} \sim \left(\frac{u_{rms}L}{\nu_{mol}}\right)^{3/4} = R^{3/4} \ .$$

See Frisch and Orszag [34] as well Landau and Lifshitz [53] for further details.

And so we have an estimate for the number of degrees of freedom (for $d = 3$) as

$$\mathcal{N}_{d=3} \sim \left(\frac{L}{\lambda_{Ko}}\right)^3 \sim R^{9/4} \tag{6.3}$$

which is a heuristic estimate for the number of modes we need to describe the fluid flow.

6.1.3 The Limitations of Kolmogorov's Theory

We now understand the concepts of Kolmogorov scaling and the corresponding theory of turbulence consequently derived. It has been one of the most influencial theories on fluid turbulence since it was introduced. However, there are some 'extenuating' points that must be made:

1. The theory has been successful as far as simple experimental verification is concerned (see for example the results of Grant, Stewart and Moillet [39] for very high Reynolds numbers flows). However, there is some further experimental evidence that scaling invariance is weakly broken for exact solutions of the Navier-Stokes equations – though the underlying mechanism for this is not clearly understood.

   Frisch and Orszag [34] provide an example: We have seen that the p^{th} order structure function scales with exponent $\zeta_p = p/3$, but there is some experimental evidence (Monin and Yaglom [68, 69] and Anselmet, Gagne, Hopfinger and Antonia [2]) which reveals that there are some flows for which ζ_p can be much smaller than $p/3$ for $p \geq 4$. Frisch and Orszag suggest that this implies increasingly non-Gaussian statistical behaviour at small scales or "inertial range intermittency".

2. The assumptions which are necessary for Kolmogorov's scaling argument are very severe – recall that after deducing the scale invariance property for the Navier-Stokes equations with zero viscosity, we made three assumptions which were needed for Kolmogorov's scaling argument.

   How can we provide a firmer, rigorous theory for Navier-Stokes minimum length scales which can also account for inertial range intermittency?

Some of the physical/geometric theories put forth so far are:

Statistical Theories (see for example, Monin and Yaglom [68, 69]) This theory involves finding equations for average quantities in the flow (such as the mean velocity etc.). However, a finite, closed set of equations involving such quantities cannot be found and so they are unable to predict Kolmogorov scaling (any closure argument involves some unacceptable approximations).

Renormalization Group Theory (due to Kenneth Wilson – and recently extended by Yakhot and Orszag [90]) Frisch and Orszag [34] point out that such theories strive to reproduce the Kolmogorov-Obukhov law in the inertial range and (likely) ignore some of the subtleties of turbulence.

Multi-fractal Models If we simultaneously examine on many scales the wave transform of some turbulent signals, we realize that a complicated structure appears to exist at nearly all scales in the inertial range. Mandelbrot [64] realized that if the branching process of the cascade continued to ever smaller scales, then as we proceed to the limit $R \to \infty$ the "fine scales of turbulence form a fractal set".

This property leads to the so-called multi-fractal models which seem to be consistent at the two levels: Firstly, with the fact that scale invariance assumptions still give a very good 'first-order' model of turbulence energetics, while on the other hand, it accounts for the existence of inertial range intermittency. Such models also appear to agree with recent experiments such as those of Parisi and Frisch [37]. However, as Frisch and Orszag point out, such models do not reveal the mechanism of symmetry breaking.

Vortex Dynamics It is known in three dimensions that infinitesimal vorticity segments are being continuously squeezed and stretched, and of course the strength of local vorticity varies accordingly – vorticity is amplified by the stretching of such vorticity segments. Vortex lines are deformed, transformed and folded in the flow which as we would expect leads to highly intricate, intermittent fine-scale vortical structures.

As we know, vorticity is generated by either by viscous action at rigid boundaries, where a no-slip condition must always hold, or by bouyancy effects in the interior of the flow. So, we expect (and observe) large scale spatially intermittent structures in flows.

It is speculated that singularities in the vorticity occur in finite time for the $d = 3$ Euler equations (the inviscid limit of the Navier-Stokes equations) – we discuss this later in this chapter in some detail. These vortex singularities may account for the symmetry breaking (breaking of scale invariance) observed in the Navier-Stokes equations.

However, another mechanism for symmetry breaking could be provided by long-lasting coherent structures on all scales (as typically observed in many turbulent flows) – which only occur when the nonlinearity of the Navier-Stokes equations is depleted such as when the vorticity field aligns with the velocity field. Such persistent coherent structures have been observed, for example, in rapidly rotating atmospheric storms.

And so as we draw this unusually long introduction to an end, the essential questions we need to address are:

1. Can we provide a model which accounts for fine-scale intermittent behaviour (inertial range intermittency) for fully general flows (which statistical turbulence descriptions cannot do)?
2. Can we make such a description mathematically rigorous?
3. Part of this could be the computation of a minimum length scale (possibly smaller than the Kolmogorov dissipation length) for the attractor. The implications for computer modelling in the case of such a scale existing are obvious!

Advances have been made in the investigation of minimum length scales:

1. Such a minimum scale is best studied through an analysis of the wave number spectrum of the velocity field.
2. We can rigorously provide a set of scales (defined through a wave-number argument), for which we can readily make estimates via the Ladder Theorem of Chapter 4 (which of course is a derived directly from the Navier-Stokes equations), and which seem to account for fine-scale intermittency (breaking of scale invariance).

3. We compare these scales against the Kolmogorov dissipation length as well as against other recent attempts to derive such a scale – for instance the scale derived by Henshaw, Kreiss and Reyna [41], or the length scale derived via the attractor dimension (Temam [88]), or that derived from the number of determining modes by Foias et al. [31] (1983) and Foias et al. [17] (1985).

We will mainly be concentrating on the three dimensional case (though we also consider the two dimensional case for which we can provide more comparisons) and, of course, we need to assume existence, uniqueness and regularity for all time throughout.

6.2 Wavenumbers and Length Scales

We have already seen an example of a minimum length scale (λ_{Ko}) for the flow, and we have argued as to why this length scale may be unsuitable in some respects.

Can we provide a definition for a length scale which is more sensitive to intermittent fluctuations? Let's investigate this question in detail.

It is clear that we can associate a characteristic length scale with each (Fourier) mode of the system and consequently we would expect the minimum length scale associated with an evolutionary PDE to be the characteristic length corresponding to the 'highest' relevant mode for the system.

We can exploit this idea more systematically: Oliver [72] points out that the problem of defining a minimum length scale can be reduced to the problem of defining relevant modes, i.e. the (Fourier) modes necessary to describe the system in some appropriate way. This can be seen as follows:

Let, say, $\{\omega_1, \ldots, \omega_N\}$ be the 'relevant' modes for our system and let ℓ_j be the length scale associated with the j^{th} mode. Then the minimum length scale we are looking for would be

$$\ell = \min_{j=1,\ldots,N} \ell_j \ . \tag{6.4}$$

But how do we determine the 'relevant' modes for our system?

Oliver [72] argues that, when it exists, a natural definition for the number of relevant modes is via the inertial manifold for the system: an inertial manifold allows us to represent the long-term behaviour (after transients have died out) of the system on a finite-dimensional linear subspace of the phase space, i.e. we can (for the long-term dynamics) reduce the PDE to a set of ODE's on a certain set of specified modes.

This argument is compelling – the author concurs with the conclusions of Oliver [72]. However, in the absense of a proof of an inertial manifold (as yet?) for the two dimensional Navier-Stokes and even regularity on a time interval of arbitrary length in the three dimensional case, we must search for an alternative natural definition for the number of relevant modes and from which we can define a minimum length scale.

6.3 A Natural Length Scale?

Consequently, consider the following argument, in which we define the number of relevant modes to be the minimum number of 'low' modes which ensure that the 'high' modes have less or equal influence on the L^∞ norm.

Let us suppose we are given a function $g(x) \in V_s$ (where we still have $A = -\Delta$) and that $\kappa > 0$. Further, let us suppose $d = 3$ (an analogous (though slightly more involved) argument applies to the $d = 2$ case). Consider the Fourier expansion of g:

$$g(x) = \sum_{\substack{k \in \mathbb{Z}^d \\ k \neq 0}} a_k e^{ik\cdot x} = \left(\sum_{[k]\leq\kappa} + \sum_{[k]>\kappa} \right) a_k e^{ik\cdot x} \tag{6.5}$$

where $k = (k_1, k_2, k_3)$ and $[k] = \max\{|k_1|, |k_2|, |k_3|\}$ and the wavenumber $\kappa > 1$ is as yet arbitrary and divides the 'low' modes $[k] \leq \kappa$ and 'high' modes $[k] > \kappa$. Applying the Cauchy-Schwarz inequality,

$$|g(x)| \leq \left(\sum_{[k]\leq\kappa} \right)^{1/2} \left(\sum_{[k]\leq\kappa} |a_k|^2 \right)^{1/2} + \left(\sum_{[k]>\kappa} k^{-2s} \right)^{1/2} \left(\sum_{[k]>\kappa} |k^s a_k|^2 \right)^{1/2} \tag{6.6}$$

for some $s > 3/2$. We can estimate (bound above)

$$\sum_{[k]\leq\kappa} \quad \text{and} \quad \sum_{[k]>\kappa} k^{-2s} \tag{6.7}$$

by simple integrals (see for example [31]) and ignoring κ's of lower order, so

$$||g||_\infty \leq c(\kappa^{3/2}||g||_2 + \kappa^{3/2-s}||D^s g||_2) \; . \tag{6.8}$$

If we now impose that we want the 'high' modes and the 'low' modes to have equal influence, we determine κ:

$$\kappa^s = \frac{||D^s g||_2}{||g||_2} \; . \tag{6.9}$$

We can immediately see that this choice of κ in (6.8) furnishes a well known interpolation inequality – see for example Foias et al. [31] where this technique is used to estimate the multiplicative constant for Agmon's inequality.

Now take $g = D^r u$ for u a solution of our dynamical system, set $s+r = N$, with $r, s, N \in \mathbb{N}\cup\{0\}$ and so:

Definition 6.3.1 *A natural definition for the number of 'relevant' modes κ^d is*

$$\kappa_{N,r}^{2(N-r)} = \frac{F_N}{F_r} \tag{6.10}$$

where $r \leq N$, with $r, N \in \mathbb{N} \cup \{0\}$ and the F_N's are defined in (4.98).

Remarks:

1. The F_r terms are naturally bounded below by $||D^N u_f||_2^2$ and so the $\kappa_{N,r}(t)$ are bounded above in terms of the F_N.

   As Oliver [72] points out, without such a lower bound, our definition would be unsuitable, unless we were able to exclude small amplitude transients, which would be the case if we were able to engage the inertial manifold idea.

2. The definition is not unique (it depends on N and r) – but it is possible that the $\kappa_{N,r}$'s are qualitively independent of N and r – we may be able to bound them above by something independent of N and r.

An alternative justification for this definition: – *Moments of the power spectrum:* Consider the energy in the flow,

$$E(t) = \int_\Omega |u(x,t)|^2 dx = \int |\hat{u}(k,t)|^2 dk \equiv \int E(k,t)dk \tag{6.11}$$

via Parseval's theorem. $E(k,t)$ is the instantaneous energy spectrum and defines the distribution of energy among wavenumbers and also allows us to consider the normalized 'probability distribution'

$$P(k,t) = \frac{E(k,t)}{E(t)} \tag{6.12}$$

We can write time-dependent moments of this distribution as

$$\langle k^{2N}\rangle_{s.a.} = \int |k|^{2N}|\hat{u}(k,t)|^2 dk / \int |\hat{u}(k,t)|^2 dk = H_N/H_0 \tag{6.13}$$

where the $s.a.$ on $\langle k^{2N}\rangle_{s.a.}$ stands for 'spatial average'. We can associate a time-dependent wavelength scale λ_N with this quantity

$$\lambda_N^{-1} \sim [\langle k^{2N}\rangle_{s.a.}]^{1/2N} \tag{6.14}$$

Not only can the forcing be included in the definitions to make the H_N's into F_N's, but the ratio of moments

$$\frac{\langle k^{2N}\rangle_{s.a.}}{\langle k^{2r}\rangle_{s.a.}} = \frac{F_N}{F_r} \tag{6.15}$$

also defines a length scale

$$(\text{length})^{-1} \sim \left(\frac{F_N}{F_r}\right)^{\frac{1}{2(N-r)}} \tag{6.16}$$

which of course, is the time-dependent length scale associated with the $\kappa_{N,r}$ wavenumbers outlined above in (6.10).

6.4 Some properties of the $\kappa_{N,r}$

In this section we make a few remarks regarding the $\langle\kappa_{N,r}\rangle$ defined in (6.10). The $\kappa_{N,r}$ have the following properties:

$$\langle\kappa_{1,0}^2\rangle \leq \lambda_0^{-2} \tag{6.17}$$

for $d = 2, 3$.

Remarks:

1. This is obviously analogous to the result (4.109).
2. In the $d = 2$ it is also true that $\langle\kappa_{2,1}^2\rangle \leq \lambda_0^{-2}$, which is the generalization of (4.112).

The $\kappa_{N,r}$ are also ordered in an obvious way,

$$\kappa_{N_1,r}(t) \leq \kappa_{N_2,r}(t) \qquad r \leq N_1 \leq N_2 \tag{6.18}$$

$$\kappa_{N,r_1}(t) \leq \kappa_{N,r_2}(t) \qquad r_1 \leq r_2 \leq N\,. \tag{6.19}$$

Theorem 6.4.1 *For the $d = 3$ Navier-Stokes equations with $r \leq N-1$*

$$(N-r)\dot{\kappa}_{N,r} \leq 2\,(c_{N,r}\|Du\|_\infty + \nu\lambda_0^{-2})\kappa_{N,r}\,. \tag{6.20}$$

Proof: Since $\kappa_{N,r}^{2(N-r)} = F_N/F_r$,

$$2\,(N-r)\kappa_{N,r}^{2(N-r)-1}\dot{\kappa}_{N,r} = \frac{\dot{F}_N}{F_r} - \frac{F_N\dot{F}_r}{F_r^2}\ . \tag{6.21}$$

Recall the F_N ladder of Chapter 4.

$$\frac{1}{2}\dot{F}_N \le -\nu F_{N+1} + (c\,\|Du\|_\infty + \nu\lambda_0^{-2})F_N\ . \tag{6.22}$$

If we go back a few steps in the proof of (6.22), then we can easily prove the reverse version of (6.22), i.e.

$$\frac{1}{2}\dot{F}_r \ge -\nu F_{r+1} - (c\|Du\|_\infty + \nu\lambda_0^{-2})F_r\ . \tag{6.23}$$

Combining (6.21)–(6.23) and using Lemma 4.7.1 we see that we get

$$(N-r)\dot{\kappa}_{N,r} \le \nu\left(\kappa_{r+1,r}^2 - \kappa_{N,r}^2\right)\kappa_{N,r} + 2\,(c\|Du\|_\infty + \nu\lambda_0^{-2})\kappa_{N,r}\ . \tag{6.24}$$

Now use that $\kappa_{r+1,r} \le \kappa_{N,r}$ provided $r+1 \le N$. □

We now investigate an a-priori estimate which reveals how upper bounds on the $\kappa_{N,r}$ can become singular in finite-time (we take the case $f = 0$ – i.e. zero forcing). Using the interpolation inequality for $N \ge 3$

$$\|Du\|_\infty \le c\kappa_{N,0}^{5/2}\|u\|_2 \tag{6.25}$$

and using this in Theorem 6.4.1, we integrate with respect to time (ignoring the $\nu\lambda_0^{-2}$ term which does not change the nature of the result – it is a term of lower order), we obtain

$$[\kappa_{(N,0)}(0)]^{-5/2} - [\kappa_{(N,0)}(t)]^{-5/2} \le c\int_0^t \|u\|_2(s)ds \tag{6.26}$$

which, from the Navier-Stokes equations, is itself bounded by

$$G(t) \equiv \int_0^t \|u\|_2(s)ds \le L^{-2}\nu^{-1}\|u\|_2(0)[1-\exp(-cL^2\nu t)] \tag{6.27}$$

(this comes from using a Poincaré inequality in (4.23) and then integrating with respect to time twice). In combination with (6.26), this gives

$$\kappa_{N,0}(t) \le c\left[\kappa_{N,0}(0)^{-5/2} - G(t)\right]^{-2/5}\ . \tag{6.28}$$

The solution has no singularities if

$$[\kappa_{N,0}(0)]^{-5/2} > L^{-2}\nu^{-1}\|u\|_2(0)\ . \tag{6.29}$$

This can be converted into

$$\kappa_{N,0}(0) < L^{-1}\left[c_N Re\right]^{-2/5} \tag{6.30}$$

where the Reynolds number Re, defined in terms of the initial conditions is

$$Re = \frac{L^{-1/2}\|u\|_2(0)}{\nu}\ . \tag{6.31}$$

This is *analogous* to the well known result of Ladyzhenskaya [52] described in Chapter 4.

Now consider the following lemma which is a modification (to periodic conditions on $[0,L]^3$) of the result

$$\|Du\|_\infty \le c\,[1 + \|\omega\|_\infty(1 + \log^+ H_3) + \|\omega\|_2] \tag{6.32}$$

proved by Beale, Kato and Majda [9] (The '+' sign on the logarithm is defined such that $\log^+ a = \log a$ for $a \ge 1$ and $\log^+ a = 0$ otherwise).

Lemma 6.4.2 *For smooth functions u on periodic boundary conditions with zero mean, we can prove*

$$\|Du\|_\infty \le c_{N,r}\left[\{1+\log^+(L\kappa_{N,r})\}\|\omega\|_\infty + L^{-3/2}F_1^{1/2}\right] \tag{6.33}$$

for $N \ge 3$ and $1 \le r < N$ and where $\omega = curl\ u$ is the vorticity.

Proof:

Since $\nabla \cdot u = 0$ and $\omega = \text{curl}u$, the Biot-Savart law implies

$$u(x) = -\frac{1}{4\pi}\int_{\Omega_L}\frac{(x-y)}{|x-y|^3}\times\omega(y)dy \equiv \int_{\Omega_L}K(x-y)\times\omega(y)dy \tag{6.34}$$

Let us introduce (as per Beale, Kato and Majda [9]) the cut-off function

$$\zeta_\rho(x) = \begin{cases} 1, & |x|<\rho \\ 0, & |x|>2\rho \end{cases}$$

and which also satisfies $|D\zeta_\rho(x)| \le c/\rho$, where ρ is a suitably small radius to be chosen later on.

Introducing $\zeta_\rho(x-y) + [1-\zeta_\rho(x-y)]$ under the integral we see that

$$\begin{aligned} Du(x) &= -\frac{1}{4\pi}\int_{\Omega_L} D\{\zeta_\rho(x-y)K(x-y)\}\omega(y)dy \\ &\quad -\frac{1}{4\pi}\int_{\Omega_L} D\{[1-\zeta_\rho(x-y)]K(x-y)\}\omega(y)dy \end{aligned} \tag{6.35}$$

and so using integration by parts (recall periodic boundary conditions assumed) we see that we get

$$Du(x) = Du^{(1)}(x) + Du^{(2)}(x) \tag{6.36}$$

where

$$Du^{(1)}(x) = -\frac{1}{4\pi}\int_{\Omega_L}\zeta_\rho(x-y)K(x-y)D\omega(y)dy \tag{6.37}$$

and

$$Du^{(2)}(x) = -\frac{1}{4\pi}\int_{\Omega_L} D\{[1-\zeta_\rho(x-y)]K(x-y)\}\omega(y)dy\ . \tag{6.38}$$

From the definition of $K(x-y)$ we have $|K(x-y)| \le c\,|x-y|^{-2}$.

Then, regarding K as a function of y, for x fixed, let $B_R = \{y : |x-y| < R\}$, then:

$$\|K\|_{L^p(B_R)} = \left(\int_{B_R}|K|^p dy\right)^{1/p} \le c\left(\int_0^R r^{2-2p}\,dr\right)^{1/p} \le c\,R^{\frac{3-2p}{p}} \tag{6.39}$$

with $r = |x-y|$. Thus

$$K \in L^p(B_{2\rho}) \quad \text{provided } p < 3/2\ .$$

Applying Hölder's inequality to (6.37) gives

$$|Du^{(1)}(x)| \le c\,\|K\|_{p,B_{2\rho}}\|D\omega\|_{q,\Omega_L} \tag{6.40}$$

where $1/p + 1/q = 1$.

$$\Rightarrow |Du^{(1)}(x)| \le c\,\rho^{1-3/q}\,\|D\omega\|_q \quad \text{with } 3 \le q \le \infty\ . \tag{6.41}$$

The remaining term is

$$Du^{(2)}(x) = \int_{\Omega_L \backslash B_{2\rho}} D\{[1-\zeta_\rho(x-y)]K(x-y)\}\omega(y)\,dy$$

Thus,

$$\begin{aligned} |Du^{(2)}(x)| &\le c\int_{\Omega_L \backslash B_{2\rho}} |DK(x-y)|\,|\omega(y)|\,dy \\ &\quad + c\int_{\Omega_L \backslash B_{2\rho}} |K(x-y)|\,|D\zeta_\rho(x-y)|\,|\omega(y)|\,dy \\ &\le C\,\|\omega\|_\infty \left\{\int_\rho^{cL} r^{-1}\,dr + \int_\rho^{2\rho} \rho^{-1}\,dr\right\} \\ &\le C'\,\|\omega\|_\infty \left\{1+\log^+(L\rho^{-1})\right\}\ . \end{aligned} \tag{6.42}$$

We now combine (6.41) and (6.42) and apply the Gagliardo-Nirenberg inequality :

$$\|D\omega\|_q \le c\,\|D^{N-1}\omega\|_2^a \|\omega\|_2^{1-a} \le c\,\kappa_{N,1}^{a(N-1)} F_1^{1/2} \tag{6.43}$$

where, with $N \ge 3$,

$$a = \frac{3}{N-1}\left(\frac{5}{6}-\frac{1}{q}\right) \tag{6.44}$$

to get

$$\|Du\|_\infty \le c\,\{\rho^{1-3/q}\kappa_{N,1}^{\frac{5}{2}-\frac{3}{q}} F_1^{1/2} + [1+\log^+(L\rho^{-1})]\|\omega\|_\infty\}\ . \tag{6.45}$$

Now choose

$$\rho^{-1} = \kappa_{N,1}^{\frac{5q-6}{2(q-3)}} L^{\frac{3q}{2(q-3)}} \tag{6.46}$$

which gives the result of the lemma. □

Remark: We see that the $\kappa_{N,r}$ $(N \ge 3)$ are thus the natural 'mediators' between $\|Du\|_\infty$ and $\|\omega\|_\infty$.

We can equate consequences of the above theorem and lemma with the results of Beale, Kato and Majda [9] for the Euler equations with zero forcing ($f = 0$).

Let us first examine, the results of that paper.

Beale, Kato and Majda [9] tackle the $d = 3$ Euler equations with zero forcing (so this would be equivalent to setting $\nu = 0$ and $f \equiv 0$ in the setting for the Navier-Stokes equations (4.1)–(4.4)).

A local existence theorem is known (analogous to the theorem for strong solutions of the Navier-Stokes equations) for the Euler equations as follows:

Theorem 6.4.3 *Assume $u_0 \in H^s(\Omega_L)$, $s \ge 3$ is given with $|u_0|_3 \le K_0$, for some $K_0 > 0$. Then there exists $T_0(K_0) > 0$ so that (4.1)–(4.4) with $\nu = 0$, $f \equiv 0$ have a solution in the class*

$$u \in C([0,T];H^s) \cap C^1([0,T];H^{s-1}) \tag{6.47}$$

at least for $T = T_0(K_0)$.

Beale, Kato and Majda go on to prove:

Theorem 6.4.4 *Suppose u is a solution of (4.1)–(4.4) with $\nu = 0$, $f \equiv 0$. Suppose further, there exists T^* which is such that u cannot be continued in the class (6.47) to $T = T^*$; and that T^* is the first such time. Then*

$$\int_0^{T^*} \|\omega(t)\|_\infty \, dt = \infty \tag{6.48}$$

and more particularly

$$\limsup_{t \nearrow T^*} \|\omega(t)\|_\infty = \infty \,. \tag{6.49}$$

There is an obvious corollary to this theorem,

Corollary 6.4.5 *Suppose there are two constants C_0 and T^*, so that on any interval of existence $[0, T]$ of the solution in class (6.47) with $T < T^*$ to the Euler equations, the vorticity satisfies the a-priori estimate*

$$\int_0^{T} \|\omega(t)\|_\infty \, dt \le C_0 \,. \tag{6.50}$$

Then the solution can be continued in the class (6.47) to the interval $[0, T^*]$.

Now recall the results of Theorem 6.4.1 and Lemma 6.4.2.

An integration of the result of Theorem 6.4.1 and Lemma 6.4.2 gives

$$\kappa_{N,r}(t) \le c_{N,r} \, L^{-1} \exp\left[\exp\left(\int_0^t \|\omega\|_\infty + g(t) \, d\tau\right)\right] \tag{6.51}$$

where

$$g(t) = \nu\lambda_0^{-2} + c\, L^{-3/2} F_1^{1/2} \tag{6.52}$$

The result for the Euler equations is recovered by putting $f = 0$ and $\nu = 0$. Here we have a solution of the differential inequality for both the Euler and the Navier Stokes equations in terms of the $\kappa_{N,r}$, instead of the H_N norms alone [9].

6.5 Important Results for the $\kappa_{N,r}$

In this section we prove some specific results for the $\kappa_{N,r}$ which we will discuss in detail in the next section.

We have strict definitions for the 'time average' and 'time-asymptotic upper bound' for a function $f(u(t))$:

$$\langle f \rangle := \limsup_{t\to\infty} \sup_{u_0 \in \mathcal{A}} \frac{1}{t} \int_0^t f(S(\tau)u_0) \, d\tau$$

and

$$\overline{f} := \sup_{u_0 \in \mathcal{A}} \limsup_{t\to\infty} f(S(t)u_0) \,.$$

Also, a useful inequality for the rest of this chapter is Jensen's inequality:

$$\langle g(x) \rangle \le g(\langle x \rangle) \quad \text{for } g \text{ concave}, \ x \ge 0 \,. \tag{6.53}$$

Theorem 6.5.1 *For $d = 2,3$, $N \geq 3$ and $r+1 \leq N$*

$$\langle \kappa_{N,r}^2 \rangle \leq c_{N,r}\, \nu^{-1} \langle \|Du\|_\infty \rangle + \lambda_0^{-2} \tag{6.54}$$

and

$$\overline{\kappa_{N,r}^2} \leq c_{N,r}\, \nu^{-1}\, \overline{\|Du\|}_\infty + \lambda_0^{-2}\ . \tag{6.55}$$

Furthermore for $N = 1,2$, we have

1. *For $d = 2$:*

$$\langle \kappa_{2,1}^2 \rangle \leq c\, \lambda_0^{-2} \quad \text{and} \quad \overline{\kappa_{2,1}^2} \leq c\, \lambda_0^{-2}\ . \tag{6.56}$$

2. *For $d = 3$:*

$$\langle \kappa_{2,1}^2 \rangle \leq c_{2,1}\, \nu^{-1} \langle \|\omega\|_\infty \rangle + \lambda_0^{-2} \tag{6.57}$$

and

$$\overline{\kappa_{2,1}^2} \leq c_{2,1}\, \nu^{-1}\, \overline{\|\omega\|}_\infty + \lambda_0^{-2} \tag{6.58}$$

as well as (for $d = 2,3$)

$$\langle \kappa_{1,0}^2 \rangle \leq \lambda_0^{-2} \tag{6.59}$$

Proof: For $N \geq 3$, this result is obvious from the Ladder Theorem of Chapter 4 – divide through by the F_N and take the time average of both sides (or in the 'limsup' case use the absorbing ball argument of Chapter 4) – this gives a very natural derivation of an upper bound for our relevant modes from the Ladder Theorem!

For $N = 2$, we can use the vorticity formulation of the Navier-Stokes equations (consider the 'curl' of 4.1 – where note that $H_1 \equiv \|\omega\|_2^2$) to prove the result above.

For $N = 1$, we have simply proved a restatement of Leray's inequality – see (4.109). □

Remark: For $N \geq 3$, the upper bounds for $\langle \kappa_{N,r}^2 \rangle$ and $\overline{\kappa_{N,r}^2}$ are uniform in (N, r), except in the constant, so we can consider such estimates for the wavenumbers to be qualitatively similiar for all N, r!

Theorem 6.5.2 *For $d = 3$ and u a smooth solution of the Navier-Stokes equations (4.1)-(4.4), we have*

$$\langle \|Du\|_\infty \rangle \;\leq\; c_1\, \nu^{-3} \left[\langle \|Du\|_2^4 \rangle + \|Du_f\|_2^4 \right] + c_2\, \nu\, \lambda_0^{-2} + \underbrace{L^{-3/2} \langle \|Du\|_2^2 \rangle^{1/2}}_{\text{comparison term}} \tag{6.60}$$

and

$$\overline{\|Du\|}_\infty \;\leq\; c_1\, \nu^{-3} \left[\overline{\|Du\|}_2^4 + \|Du_f\|_2^4 \right] + c_2 \nu \lambda_0^{-2} + \underbrace{L^{-3/2}\, \overline{\|Du\|}_2}_{\text{comparison term}} \tag{6.61}$$

Corollary 6.5.3 *For $d = 3$,*

$$\langle \|\omega\|_\infty \rangle \;\leq\; c_1\, \nu^{-3} \left[\langle \|\omega\|_2^4 \rangle + \|\omega_f\|_2^4 \right] + c_2\, \nu \lambda_0^{-2} + L^{-3/2} \langle \|\omega\|_2^2 \rangle^{1/2} \tag{6.62}$$

and

$$\overline{\|\omega\|}_\infty \;\leq\; c_1\, \nu^{-3} \left[\overline{\|\omega\|}_2^4 + \|\omega_f\|_2^4 \right] + c_2\, \nu \lambda_0^{-2} + L^{-3/2}\, \overline{\|\omega\|}_2 \tag{6.63}$$

where the last term on the right-hand side is a comparison term.

Proof of theorem:

Step 1: Proof of (6.60) and (6.62)

Recall that for space-periodic, mean-zero functions we can apply the Gagliardo-Nirenberg inequality as follows

$$\begin{aligned} \|Du\|_\infty &\leq c\,\|D^N u\|_2^a \|Du\|_2^{1-a} \\ &\leq c\,F_N^{a/2}\,F_1^{(1-a)/2} \\ &\leq c\,\kappa_{N,1}^{3/2} F_1^{1/2} \end{aligned} \tag{6.64}$$

for $N \geq 3$. Let us now add a 'comparison' term $L^{-3/2}\|Du\|_2$ to the right hand side of (6.64), so that we have

$$\|Du\|_\infty \leq \underbrace{c\,\kappa_{N,1}^{3/2} F_1^{1/2}}_{\textbf{(I)}} + \underbrace{L^{-3/2}\|Du\|_2}_{\textbf{(II)}} . \tag{6.65}$$

The reason for doing this will become much more apparent in the next section. For the moment though, assume that **(II)** is simply a term which we might want to compare **(I)** against. Note that **(II)** is of much lower order than **(I)** and if we were to ignore **(I)** we would be essentially writing $\|Du\|_\infty \sim L^{-3/2}\|Du\|_2$.

Remark: Note that (of course) (6.65) is dimensionally uniform.

We can take the time average of both sides of (6.65) to get

$$\langle\|Du\|_\infty\rangle \leq c\,\langle\kappa_{N,1}^2\rangle^{3/4}\,\langle F_1^2\rangle^{1/4} + L^{-3/2}\langle\|Du\|_2\rangle . \tag{6.66}$$

Now use Theorem 6.5.1 with $r = 1$ to get

$$\langle\|Du\|_\infty\rangle \leq \left[c\,\nu^{-1}\,\langle\|Du\|_\infty\rangle + \lambda_0^{-2}\right]^{3/4}\,\langle F_1^2\rangle^{1/4} + L^{-3/2}\langle\|Du\|_2\rangle . \tag{6.67}$$

Using Young's and Jensen's inequalities on the right-hand side of (6.67), we finally obtain

$$\langle\|Du\|_\infty\rangle \;\leq\; c_1\,\nu^{-3}\,\langle\|Du\|_2^4\rangle + c_2\,\nu^{-3}\,\|Du_f\|_2^4 + c_3\,\nu\,\lambda_0^{-2} + L^{-3/2}\langle\|Du\|_2^2\rangle^{1/2} \tag{6.68}$$

which is (6.60). The corollary to this, (6.62), follows since $\|\omega\|_\infty \leq \|Du\|_\infty$ and $\|\omega\|_2 = \|Du\|_2$.

Step 2: Proof of (6.61) and (6.63)

Again, using a Gagliardo-Nirenberg inequality

$$\begin{aligned} \|Du\|_\infty &\leq c\,H_{N+1}^{b/2}\,H_1^{(1-b)/2} \\ &\leq c\,F_{N+1}^{b/2}\,F_1^{(1-b)/2} \end{aligned} \tag{6.69}$$

where $b = \frac{3}{2N}$, and again, we will add a $L^{-3/2}\|Du\|_2$ comparison term to the right-hand side of (6.69) to get

$$\|Du\|_\infty \leq c\,F_{N+1}^{b/2}\,F_1^{(1-b)/2} + L^{-3/2}\|Du\|_2 . \tag{6.70}$$

Next, solve the Ladder Theorem of Chapter 4 to get

$$\overline{F}_{N+1} \leq \left[c\,\nu^{-1}\,\overline{\|Du\|}_\infty + \lambda_0^{-2}\right]^N \overline{F}_1 . \tag{6.71}$$

Using this in (6.70),

$$\overline{\|Du\|}_\infty \leq c\,\left[c\,\nu^{-1}\,\overline{\|Du\|}_\infty + \lambda_0^{-2}\right]^{3/4}\,\overline{F}_1^{1/2} + L^{-3/2}\overline{\|Du\|_2} . \tag{6.72}$$

Young's inequality then gives,

$$\overline{\|Du\|_\infty} \le c_1\,\nu^{-3}\,\overline{\|Du\|_2^4} + c_2\,\nu^{-3}\,\|Du_f\|_2^4 + c_3\,\nu\,\lambda_0^{-2} + L^{-3/2}\overline{\|Du\|_2}\,. \tag{6.73}$$

The corollary to (6.63) is true for the same reasons given in Step 1. □

Now let us combine the results of Theorem 6.5.1 and Theorem 6.5.2:

Corollary 6.5.4 (Main Result) *For u a smooth solution of (4.1)-(4.4), $d = 3$, $N \ge 3$ and $r+1 \le N$, we have*

$$\langle \kappa_{N,r}^2\rangle \le c\left(\underbrace{\nu^{-1}L^{-3/2}\langle \|Du\|_2^2\rangle^{1/2} + \lambda_0^{-2}}_{\text{bounded terms}}\right) + c\,\nu^{-4}\left(\underbrace{\langle \|Du\|_2^4\rangle}_{\text{no control}} + \|Du_f\|_2^4\right) \tag{6.74}$$

and

$$\overline{\kappa_{N,r}^2} \le c\left(\nu^{-1}L^{-3/2}\overline{\|Du\|_2} + \lambda_0^{-2}\right) + c\,\nu^{-4}\left(\overline{\|Du\|_2^4} + \|Du_f\|_2^4\right)\,. \tag{6.75}$$

Remark: $\langle \|Du\|_2^2\rangle$ is bounded a-priori, whereas $\langle \|Du\|_2^4\rangle$ would become singular in finite-time if we were not assuming regularity.

Further, we can make a d-dimensional restatement of the inequality (6.75):

Theorem 6.5.5 *For $d = 2, 3$ dimensions and for $N > \frac{4}{4-d}$,*

$$\overline{\kappa_{N,0}^2} \le c_N\,L^{-2}\,Q^{\frac{2(N-1)}{(4-d)N-4}} \tag{6.76}$$

where $Q = \nu^{-2}L^{4-d}\overline{F_1}$.

Proof: Consider the result (6.24) in the proof of Theorem 6.4.1. Take $r = 0$ in that result and we get

$$N\dot{\kappa}_{N,0} \le -\nu\,\kappa_{N,0}^3 + 2\left(c_N\|Du\|_\infty + \nu\lambda_0^{-2}\right)\kappa_{N,0} + \nu\kappa_{N,0}(F_1/F_0) \tag{6.77}$$

Now use that $F_0 \ge \tau^2\|f\|_2^2$ in the last term in (6.77) as well as the inequality,

$$\|Du\|_\infty \le c\,L^{\frac{d}{2(N-1)}}\,\kappa_{N,0}^{\frac{Nd}{2(N-1)}}\,F_1^{1/2} \tag{6.78}$$

(which is a combination of Gagliardo-Nirenberg and Poincaré inequalities), and then find the absorbing ball (ignoring terms of lower order) for $\kappa_{N,0}$ to get the result. □

Remark: Since the $\kappa_{N,0}$ are ordered such that $\kappa_{N_1,0} \le \kappa_{N_2,0}$ for $N_1 \le N_2$, we can take the minimum of the right-hand side of the estimate in Theorem 6.5.5, namely the limit as $N \to \infty$ (for $d = 3$) which yields Q^2. This is similar to the upper bound on the length scale derived in equation (6.75). The Q^2 estimate, in effect, when all the factors of L are accounted for, gives $\nu^{-4}\overline{F_1^2}$.

Lastly, we might ask ourselves if we can prove such results for higher derivatives:

Theorem 6.5.6 *For $p \ge 0$, $\omega_f = \mathrm{curl}\ u_f$,*

$$\left\langle \frac{\|\mathrm{curl}^p\,\omega\|_\infty}{\|\mathrm{curl}^p\,\omega\|_2 + \|\mathrm{curl}^p\,\omega_f\|_2}\right\rangle \le c_1\,\nu^{-3}\left[\langle\|\omega\|_2^4\rangle + \|\omega_f\|_2^4\right]^{3/4} + c_2\lambda_0^{-3/2} \tag{6.79}$$

where the constants c_1 and c_2 depend on p.

Proof: Consider the Gagliardo-Nirenberg inequality

$$\|(D^r u)\|_\infty \leq c\, \|D^{N-r}(D^r u)\|_2^a\, \|(D^r u)\|_2^{1-a} \tag{6.80}$$

where $a = 3/2(N-r)$ and we require $N \geq r+2$. This gives us

$$\begin{aligned} \|Du\|_\infty &\leq c\left(\frac{F_N}{F_r}\right)^{\frac{3}{4(N-r)}} F_r^{1/2} \\ &= c\,(\kappa_{N,r}^2)^{3/4}\, F_r^{1/2} \end{aligned} \tag{6.81}$$

so

$$\frac{\|D^r u\|_\infty}{\|D^r u\|_2 + \|D^r u_f\|_2} \leq c\,(\kappa_{N,r}^2)^{3/4} \tag{6.82}$$

and so we can use Theorem 6.5.1 to get

$$\begin{aligned} \left\langle \frac{\|D^r u\|_\infty}{\|D^r u\|_2 + \|D^r u_f\|_2} \right\rangle &\leq c\,\langle \kappa_{N,r}^2 \rangle^{3/4} \\ &\leq c\,[c_1 \nu^{-1} \langle \|Du\|_\infty \rangle + \lambda_0^{-2}]^{3/4} \\ &\leq c_1\, \nu^{-3}\, [\langle \|Du\|_2^4 \rangle + \langle \|Du_f\|_2^4 \rangle]^{3/4} + c_2\, \lambda_0^{-3/2} \end{aligned} \tag{6.83}$$

where we have used Theorem 6.5.2 and ignored the comparison term. With $p = r-1$ the result follows. □

6.6 Minimum Length Scales and Intermittency

So of what relevance are all the results proved for $\langle \kappa_{N,r}^2 \rangle$ and $\overline{\kappa_{N,r}^2}$ (where the $\kappa_{N,r}$ are the wave-numbers we defined in (6.10)) in the previous section, and in particular Theorems 6.5.4 and 6.5.5?

In 6.5.4 and 6.5.5 we gave ($d = 3$) a-priori upper bounds for $\langle \kappa_{N,r} \rangle$ and $\overline{\kappa_{N,r}^2}$.

We will associate two different length scales with our wave-numbers $\kappa_{N,r}$ as follows:

Definition 6.6.1

$$\ell_{N,r}^{-2} \equiv \langle \kappa_{N,r}^2 \rangle\,, \tag{6.84}$$

$$\mathcal{L}_{N,r}^{-2} \equiv \overline{\kappa_{N,r}^2}\,. \tag{6.85}$$

Now let us examine the significance of the results of the previous section with respect to the number of relevant modes and minimum length scales for the d-dimensional Navier-Stokes equations (4.1)–(4.4).

Let us enumerate all the bounded scales we have come across so far:

1. $\lambda_0^{-2} = L^{-2} + \lambda_f^{-2}$; the box length L and the smallest scale in the forcing λ_f.
2. $\lambda_{Ko} = (\nu^3/\varepsilon)^{1/4}$; the Kolmogorov length, where ε is the energy dissipation.
3. $\mu_{Ko} = (\nu^3/\varepsilon_f)^{1/4}$; the equivalent of the Kolmogorov length for the forcing.

Recall the results of (6.54) and (6.55) of Theorem 6.5.1. Note that in both cases the right-hand side upper bounds are independent of N and r, except in the constants, and so we will regard this set of length scales as qualitatively similiar. Thus we can write

$$\ell \sim \sqrt{\frac{\nu}{\langle \|Du\|_\infty \rangle}} \tag{6.86}$$

or

$$\mathcal{L} \sim \sqrt{\frac{\nu}{\overline{\|Du\|}_\infty}} \tag{6.87}$$

Remark: Recall that we are assuming global regularity holds for the $d = 3$ Navier-Stokes equations (otherwise the terms $\langle \|Du\|_\infty \rangle$ and $\overline{\|Du\|}_\infty$ may become unbounded).

These two results essentially reproduce (by different methods – though the approach is not too dissimiliar) the ($d = 3$) conclusions of Henshaw, Kriess and Reyna [41].

Henshaw, Kriess and Reyna [41] go on to make the following assertion for the scale (6.87) they derived:

If we identify the energy dissipation rate as $\varepsilon = 2\nu H_1$ and we assume that

$$\overline{\|Du\|}_\infty \sim 2\|Du\|_2 = \nu^{-1/2}\varepsilon^{1/2} \tag{6.88}$$

then we see that

$$\mathcal{L} = \lambda_{Ko} \tag{6.89}$$

i.e. the classical Kolmogorov length scale (for turbulence).

Remark: Actually, this result occurs in the pre-print of their paper [41]. In the actual [41] paper they modify the assumption to assuming ε is of order 1, and that $\overline{\|Du\|}_\infty \sim= \nu^{-1/2}$ – this is of course essentially the same and amounts to an identical result.

Thus, to summarize, we can derive a length scale $\mathcal{L}$ which corresponds to that of Henshaw, Kriess and Reyna [41], and further, if we identify the rate of energy dissipation as $\varepsilon = 2\nu H_1$ and further assume that $\overline{\|Du\|}_\infty \sim 2\|Du\|_2 = \nu^{-1/2}\varepsilon^{1/2}$, i. e. the flow is laminar – there is no strong intermittent turbulent behaviour, then the natural length scale is determined as the classical Kolmogorov length scale (as we might expect).

Equally, we could have assumed $\langle \|Du\|_\infty \rangle \sim \nu^{-1/2}\varepsilon^{1/2}$ to obtain a similiar conclusion $\ell \sim \lambda_{Ko}$.

These conclusions are expressed as

With $\varepsilon = 2\nu H_1$ as the energy dissipation rate, then if we assume:

1. $\langle \|Du\|_\infty \rangle \sim \nu^{-1/2}\varepsilon^{1/2} \quad \Rightarrow \quad \ell_{N,r}^{-2} \le c_{N,r}\lambda_{Ko}^{-2} + \lambda_0^{-2}$.
2. $\overline{\|Du\|}_\infty \sim \nu^{-1/2}\varepsilon^{1/2} \quad \Rightarrow \quad \mathcal{L}_{N,r}^{-2} \le c_{N,r}\lambda_{Ko}^{-2} + \lambda_0^{-2}$.

However, we have insinuated that these scales are in some way unsatisfactory. We now demonstrate why.

These estimates show us that if we assume the flow we are considering to have maximum fluctuations (in the vorticity) bound by the root mean square average of the vorticity of the flow the resultant scale is λ_{Ko}.

1. This is an extra assumption and is unacceptably restrictive.

2. We must still account for the existence of inertial range intermittency.

We can improve our estimates to provide a more realistic length scale.

We immediately see that we can derive two length scales from the results of Theorem 6.5.4:

1. If we identify $\varepsilon = \varepsilon_{ta} = 2\nu L^{-3}\langle H_1\rangle$ (and so there is a corresponding scale $\lambda_{Ko}^{(ta)} = (\nu^3/\varepsilon_{ta})^{1/4}$ – where 'ta' stands for 'time average') then we see from (6.74) that:

$$\ell_{N,r}^{-2} \equiv \langle \kappa_{N,r}^2 \rangle \leq c \left(\underbrace{(\lambda_{Ko}^{(ta)})^{-2} + \lambda_0^{-2}}_{\text{bounded terms}} \right) + c\,\nu^{-4} \left(\underbrace{\langle \|Du\|_2^4 \rangle}_{\text{no control}} + \|Du_f\|_2^4 \right) . \tag{6.90}$$

2. We have an alternative: If we identify $\varepsilon = \varepsilon_{ls} = 2\nu L^{-3}\overline{H_1}$ (and so correspondingly $\lambda_{Ko}^{(ls)} = (\nu^3/\varepsilon_{ls})^{1/4}$ – where 'ls' stands for 'limsup' or 'time-asymptotic upper bound') then we see from (6.75) that:

$$\mathcal{L}_{N,r}^{-2} \equiv \overline{\kappa_{N,r}^2} \leq c \left((\lambda_{Ko}^{(ls)})^{-2} + \lambda_0^{-2} \right) + c\,\nu^{-4} \left(\overline{\|Du\|_2^4} + \|Du_f\|_2^4 \right) . \tag{6.91}$$

What are the merits of both the length scales (6.90) and (6.91)?

These length scales are relevant on three levels:

1. Firstly, we have the Kolmogorov length λ_{Ko} (alternatively defined via $\lambda_{Ko}^{(ta)}$ and $\lambda_{Ko}^{(ls)}$). We have shown that this scale is prevalent when either the time average or limsup in time (as well as sup over initial conditions) of $\|Du\|_\infty$ goes like $2\|Du\|_2 = \nu^{-1/2}\varepsilon^{1/2}$ with $\varepsilon = 2\nu H_1$, i.e. it is the minimum length scale which will only resolve the flow during long time intervals when the flow remains quiescent (the maximum norm of the vorticity is bounded by the root mean square of the vorticity).

   This means that there must be long time intervals between large intermittent events. So, λ_{Ko} is the natural scale only during those time intervals between rare events when the vorticity in the fluid remains near its root mean square spatial average.

2. If we examine the length scale derived in (6.90) closely, we see that there is an additive term to the Kolmogorov scale, over which we have no explicit control. This term ($\nu^{-4}\langle\|Du\|_2^4\rangle$) could possibly be very large (in comparison with the Kolmogorov term) and so drive scales down to ones much smaller than Kolmogorov – it is the part which would become singular if regularity fails. Also note that this term will become very large for high Reynold's numbers – a natural behaviour we would expect for a faithful length scale.

   The length scale (6.90) is thus a much shorter scale associated with the possible, unpredictable intermittent bursts. There exists a very strong analogy here with the $d = 2, 3$ Complex Ginzburg-Landau equation (see [3, 72]) where the solution shows much stronger (possible) turbulent behaviour towards the inviscid limit – the Non-linear Schrödinger equation, which we know exhibits finite-time singularities. So strong intermittent turbulent behaviour is likely to occur if the Euler equations exhibit finite-time singularities (an as yet unproved proposition – see the reviews by Majda [62, 63]).

3. The length scale derived in (6.91) is possibly even shorter. The time average operations we performed in (6.90) inevitably miss out some detailed information in the flow and so (assuming regularity of course) we see that the most sensitive length scale of the attractor is that defined in (6.91) – defined via 'limsup' estimates.

There are some further remarks we should make about the scale (6.90):

In conventional turbulence theory one normally considers the time averaged energy spectrum $\langle E(k)\rangle$ (which is the quantity which is supposed to decay algebraically in an inertial range). The instantaneous distribution of energy is

$$P(k) = \frac{E(k,.)}{\int dk' E(k',.)} \tag{6.92}$$

and we see that we can define some relevant length scales via (the average distribution of energy)

$$\langle P\rangle(k) = \frac{\langle E(k,.)\rangle}{\int dk' \langle E(k',.)\rangle} \tag{6.93}$$

or via

$$\langle P(k,.)\rangle = \left\langle \frac{E(k,.)}{\int dk' E(k',.)} \right\rangle . \tag{6.94}$$

A definition through (6.93) is more conventional i.e. as a ratio of time averages (ratio of $\langle F_N\rangle$ over $\langle F_r\rangle$), however, we note that the length scale we have defined ($\ell_{N,r}$) corresponds to a time average of a ratio. Moments computed by taking the time average of the ratio will obviously tend to be more sensitive to rare, deep fluctuations in the vorticity than moments considered as a ratio of time averages.

6.6.1 Results in the $d = 2$ Case

Let us introduce the non-dimensional Grashof number:

$$\mathcal{G} := \frac{L^2 \|f\|_2}{\nu^2} \tag{6.95}$$

and in $d = 2$ we also have the estimate (recall results of Chapter 4):

$$\langle F_2\rangle \le \nu^2 \lambda_0^{-4} \mathcal{G}^2 . \tag{6.96}$$

Theorem 6.5.1 is also true in $d = 2$, $N \ge 3$. If we combine the $d = 2$ results of that theorem ($N \ge 3$) with the following ($d = 2$ only) modified version of the Brezis and Gallouet [11] logarithmic estimate (see for example Doering and Gibbon [25] for an alternative proof),

$$\|Du\|_\infty \le c\, F_2^{1/2} \left[1 + \frac{1}{2} \log L^2 \frac{F_3}{F_2}\right]^{1/2} \tag{6.97}$$

and (noting that $\kappa_{3,2}^2 \le \kappa_{N,r}^2$ for $N \ge 3$, $r \ge 2$) applying the Cauchy-Schwarz and then Jensen's inequalities:

$$\langle \|Du\|_\infty\rangle \le c\, \langle F_2\rangle^{1/2} \left[1 + \frac{1}{2} \log\left(L^2 \langle \kappa_{N,r}^2\rangle\right)\right]^{1/2} \tag{6.98}$$

we finally get (using that $\log(1 + \log Q) \le \log Q$ for $Q \ge 1$):

$$\langle \kappa_{N,r}^2\rangle \le c_{N,r}\, \lambda_0^{-2}\, \mathcal{G} \left[1 + \log\left(\frac{\lambda_0^{-2}}{L^{-2}} \mathcal{G}\right)\right]^{1/2} \tag{6.99}$$

Further from Theorem 6.5.5 we get for $d = 2$, $N \geq 3$,

$$\overline{\kappa_{N,0}^2} \leq c\,\lambda_0^{-2}\,\mathcal{G}^2 \tag{6.100}$$

and so we see that in $d = 2$ we can estimate

$$\left(\frac{\ell_{N,r}}{\lambda_0}\right)^{-2} \leq c\,\mathcal{G}\left[1 + \log\left(\frac{\lambda_0^{-2}}{L^{-2}}\mathcal{G}\right)\right]^{1/2} \tag{6.101}$$

and

$$\left(\frac{\mathcal{L}_{N,0}}{\lambda_0}\right)^{-2} \leq c\,\mathcal{G}^2\,. \tag{6.102}$$

6.7 Other Length Scale Estimates

We have already seen how the length scales we have derived compare to the classical Kolmogorov scale and at the same time that of Henshaw, Kreiss and Reyna [41]. However, there obviously exist (and have done so for some time, particularly for $d = 2$) some much more sophisticated examples of minimum length scales, and we investigate those here.

6.7.1 The Attractor Dimension

Recall that in the classical theory of turbulence a heuristical estimate for the number of degrees of freedom for the flow is given by

$$\mathcal{N} \sim \left(\frac{L}{\ell}\right)^d \tag{6.103}$$

where L is a characteristic scale associated with the physical dimensions of the domain of the fluid, and ℓ is a minimum length scale (for example, any one of the minimum scales mentioned above) for the fluid flow.

We normally identify ℓ as the diffusion length below which molecular viscosity heavily damps the motion (i.e. the scale below which we suppose no more interesting dynamics is occurring – see the arguments at the beginning of this chapter).

From the scaling arguments outlined at the beginning of this chapter, we would normally expect $\ell = \lambda_{Ko} = (\nu^3/\varepsilon)^{1/4}$ where ε is the rate of dissipation of energy.

We also know that in two dimensions, it is convenient to introduce the Kraichnan length $\lambda_{Kr} = (\nu^3/\eta)^{1/6}$ where η is the rate of enstrophy dissipation.

Temam [88] interprets the dimension of the attractor as the number of degrees of freedom of the flow, and (in Chapter 6 of [88]) provides some estimates for the dimension of the attractor which we outline below.

Note: Consider an m-dimensional infinitesimal volume element (say $V(u, m)$) in the phase space H about an arbitrary solution u on the attractor. If we can show $V(u, m)$ decays to zero volume as $t \longrightarrow \infty$, then the attractor cannot contain any m-dimensional subsets and the smallest m with this property majorises the (Hausdorff) dimension of the attractor.

With this in mind, the proof proceeds by linearizing the general evolution equation (2.1), and then considering the evolution of $V(u, m)$ under the linearized system. For the details of the general calculation (for the Hausdorff ($d_H(\mathcal{A})$) and fractal ($d_M(\mathcal{A})$) dimensions of the attractor $\mathcal{A}$), I refer the interested reader to Temam [88] and Oliver [72].

I summarize the results of Temam [88], Chapter 6 ($d = 2$).

With the rate of dissipation of energy defined as

$$\varepsilon = 2\nu L^{-2}\langle H_1\rangle \tag{6.104}$$

a simple a-priori estimate (analogous to the *energy-type* estimates of Chapter 4) reveals,

$$\varepsilon \leq \nu^3 L^{-4}\mathcal{G}^2 \tag{6.105}$$

and so with $\lambda_{Ko} = (\nu^3/\varepsilon)^{1/4}$ we find

$$\left(\frac{L}{\lambda_{Ko}}\right)^2 \leq \frac{L^2\varepsilon^{1/2}}{\nu^{3/2}} \leq \mathcal{G}\ . \tag{6.106}$$

Temam [88] then proves (Chapter 6, Theorem 3.1) that *if m is defined as*

$$m - 1 < c\left(\frac{L}{\lambda_{Ko}}\right)^2 = c\left(\frac{\varepsilon}{\nu^3}\right)^{1/2} L^2 \leq m \tag{6.107}$$

where c is a dimensionless constant, then

1. *the m-dimensional volume element in H is exponentially decaying in the phase space as* $t \to \infty$,
2. *if* $\mathcal{A}$ *denotes the* $d = 2$ *Navier-Stokes attractor (that we know exists) then*
   - $d_H(\mathcal{A}) \leq m$
   - $d_M(\mathcal{A}) \leq 2m$

   where $d_H(\mathcal{A})$ *and* $d_M(\mathcal{A})$ *are (respectively) the Hausdorff and fractal dimensions of the attractor* $\mathcal{A}$.

Note: We can replace m in (6.107) in the result above by a larger $m = m_1$ given by

$$m_1 - 1 < c\mathcal{G} \leq m_1 \tag{6.108}$$

and so we are able to make an estimate of the form

$$\mathcal{N}_{Ko} \sim \left(\frac{L}{\lambda_{Ko}}\right)^2 \sim \mathcal{G}\ . \tag{6.109}$$

We can improve on this result via the enstrophy disipation η.

We indentify the enstrophy dissipation flux as

$$\eta = 2\nu L^{-2}\langle H_2\rangle \tag{6.110}$$

and another simple a-priori estimate gives us,

$$\eta \leq \nu^3 L^{-6}\mathcal{G}^2 \tag{6.111}$$

and so with $\ell = \lambda_{Kr} = (\nu^3/\eta)^{1/6}$ we find

$$\left(\frac{L}{\lambda_{Kr}}\right)^2 = \left(\frac{\eta L^6}{\nu^3}\right)^{1/3} \leq \mathcal{G}^{2/3} . \tag{6.112}$$

Temam [88] then proves (Chapter 6, Theorem 3.2) that *if* m_2 *is defined as*

$$m_2 - 1 < c\left(\frac{L}{\lambda_{Kr}}\right)^2 \left(1 + \log\left(\frac{L}{\lambda_{Kr}}\right)\right)^{1/3} \leq m_2 \tag{6.113}$$

then

1. *the* m_2*-dimensional volume element in* V *is exponentially decaying in the phase space as* $t \to \infty$,
2. *the global attractor* $\mathcal{A}$ *for the* $d = 2$ *Navier-Stokes equations is such that*
   - $d_H(\mathcal{A}) \leq m_2$
   - $d_M(\mathcal{A}) \leq 2m_2$.

Further we can replace m_2 in (6.113) of this result (using (6.112)) by a larger $m = m_3$ given by

$$m_3 - 1 < c\mathcal{G}^{2/3}(1 + \log\mathcal{G})^{1/3} \leq m_3 \tag{6.114}$$

and so we can make an estimate of the form

$$\mathcal{N}_{Kr} \sim \left(\frac{L}{\lambda_{Kr}}\right)^2 \left(1 + \log\left(\frac{L}{\lambda_{Kr}}\right)\right)^{1/3} \sim c\mathcal{G}^{2/3}(1 + \log\mathcal{G})^{1/3} . \tag{6.115}$$

This result is also reproduced by Doering and Gibbon [25].

6.7.2 Determining Modes

In Foias et al. [31] (1983) and Foias et al. [17] (1985) the concept of *determining modes* is introduced for the two and three dimensional (respectively) space-periodic Navier-Stokes equations.

Definition 6.7.1 *Suppose*

$$u_1 = \sum_{j=1}^{\infty} a_j(t)\omega_j \quad \textit{and} \quad u_2 = \sum_{j=1}^{\infty} b_j(t)\omega_j$$

are two solutions to the Navier-Stokes equations (4.1)-(4.4) where $\{\omega_1, \omega_2, \ldots\}$ *are a set of divergence-free eigenfunctions spanning* H *(for example the eigenfunctions of the operator* A*).*

Then a finite set of M *modes* $\{\omega_1, \ldots, \omega_M\}$ *are said to be* determining *if the condition*

$$\sum_{i=1}^{M} |a_j(t) - b_j(t)| \to 0 \quad as \quad t \to \infty \tag{6.116}$$

is necessary and sufficient *for*

$$\int_{\Omega_L} |u_1(x,t) - u_2(x,t)|\, dx \to 0 \quad as \quad t \to \infty . \tag{6.117}$$

However, as Oliver [72] points out, such a set of modes does not guarantee the long-term behaviour of the system can be represented by a set of Ordinary Differential Equations for the modes $\{\omega_1, \ldots, \omega_M\}$ – which would be the case via the inertial manifold concept.

Further, length scales which we compute via the number of determining modes are larger than those calculated by other means. Indeed, Foias et al. [31] estimate the number of determining modes for $d = 2$ to be bounded by $c\mathcal{G}(1 + \log \mathcal{G})^{1/2}$, and also in a later paper (Foias et al. [17], 1985) they prove that for the three dimensional Navier-Stokes equations the attractor dimension is always majorised by the number of determining modes.

6.8 Conclusions

Thus we have given a rigorous definition for the number of relevant modes via a Fourier splitting argument from which we defined two sets of minimum length scales, $\ell_{N,r}$ and $\mathcal{L}_{N,r}$.

If we assumed our flow to be laminar, we showed that the scales we derived corresponded to the classical Kolmogorov dissipation scale. However, without this assumption, the estimates for our scales have an extra term which reveals how inertial range intermittency might be accounted for.

We subsequently argued for the relative suitability of the scales $\ell_{N,r}$ and $\mathcal{L}_{N,r}$ and why we expected the number of degrees of freedom associated with alternative estimates for length scales might be smaller (i.e. the corresponding length scales would be larger). We then provided some alternative estimates for minimum length scales, in particular for the two dimensional case for which there exists a wide range of such estimates. In summary, the various estimates we have are

Estimate of	$d = 2$
Number of Determining modes	$\sim c\mathcal{G}(1 + \log \mathcal{G})^{1/2}$
Degrees of Freedom from $d_H(\mathcal{A})$	$\sim c\mathcal{G}^{2/3}(1 + \log \mathcal{G})^{1/3}$
Degrees of Freedom from $\ell_{N,r}^{-2} = \langle \kappa_{N,r}^2 \rangle$	$\sim c\mathcal{G}(1 + \log \mathcal{G})^{1/2}$
Degrees of Freedom from $\mathcal{L}_{N,r}^{-2} = \kappa_{N,r}^2$	$\sim c\mathcal{G}^2$

much as we expected.

Chapter 7

The MagnetoHydrodynamics Equations.

In this last chapter, we introduce the MHD equations and derive some a-priori estimates for them which are analogous to those for the Navier-Stokes equations (which were outlined in Chapter 4). We set the stage for a further in-depth investigation.

7.1 Introduction to MHD

Let us suppose we have a conducting fluid material contained in a volume $\Omega_L = [0, L]^d$ but that this material is non-magnetic, that is to say that the permeability μ of the material is $4\pi \cdot 10^{-7}$ Henry per metre. Also let us suppose that this fluid has a characteristic permittivity ε, and electrical conductivity σ. Hence, both Maxwell's equations and a modified form of the Navier-Stokes equations apply to this fluid material.

Let us first examine Maxwell's equations. Let

- J = current density
- B = the magnetic field
- E = electric intensity (field)
- ρ_c = charge density
- ρ = mass density of fluid material.

Then, conservation of charge gives

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot J = 0 \tag{7.1}$$

and Maxwell's equations are

$$\text{curl} E = -\frac{\partial B}{\partial t} \tag{7.2}$$

$$\text{curl} H = J + \frac{\partial D}{\partial t} \tag{7.3}$$

$$\nabla \cdot D = \rho_c \tag{7.4}$$

$$\nabla \cdot B = 0 \tag{7.5}$$

where $D = \varepsilon E$ and $B = \mu H$.

With reference to Cowling [23] we can ignore Maxwell's displacement currents which means that the term $\partial \rho_c / \partial t$ in (7.1) is of order u^2/c^2 times the remaining term (u = fluid velocity and c = the velocity of light) and so we *neglect* this term i.e. *relativistic effects are ignored.* Thus (7.1)–(7.5) now become

$$\nabla \cdot J = 0 \tag{7.6}$$

$$\text{curl} B = \mu J \tag{7.7}$$

$$\text{curl} E = -\frac{\partial B}{\partial t} \tag{7.8}$$

$$\nabla \cdot B = 0 \, . \tag{7.9}$$

To proceed any further, we must now carefully establish exactly what sort of conducting fluids and conditions we have in mind [54, 43, 29]. We assume our fluid to be a conducting liquid or dense ionized gas (to which Ohm's Law is still applicable). We also assume that the characteristic distances and time intervals for the fluid motion concerned are very much larger than the mean free path and mean free time of the current carriers (electrons and ions). This means that currents are determined by self-induction rather than by electrical resistance (particle effects are ignored) and also that there is no separation of charge (the fluid is effectively electrically neutral – to a good approximation).

Hence if J' and E' are the current density and electric field (respectively) measured in the rest frame of the medium/fluid; with J and E those measured relative to the laboratory; then Ohm's Law holds (σ = electrical conductivity)

$$J' \; = \; \sigma E' \, . \tag{7.10}$$

If the fluid material has velocity $u(x,t)$, then the transformation between the two frames of reference is

$$J \; = \; J' + \rho_c u \tag{7.11}$$

and

$$E' \; = \; E + u \wedge B \tag{7.12}$$

where ρ_c is the charge density. The second term on the right-hand side of (7.11) is due to the convection of current (from motion of resultant charge) while the first is due to the conduction of electric current (whose effect we ignore under the assumptions outlined above). Hence for a fluid particle volume dV, and charge ρ_c, we have the following expression for the Lorentz force on the fluid particle

$$dF \; = \; \rho_c E' \, dV \; = \; \rho_c E \, dV + (\rho_c u) \wedge B \, dV \approx (J \wedge B) \, dV \tag{7.13}$$

which we could also obtain directly from the Maxwell Stress Tensor [54]. Note that the $\rho_c E$ term is ignored (fluid electrically neutral).

Let us now consider the fluid equations: we shall assume the fluid to be incompressible and to have uniform unit density ($\rho = 1$). Hence the equation of conservation of mass density gives

$$\text{div } u = 0 \, . \tag{7.14}$$

In the equation of motion there are three types of body (volume) forces

- $J \wedge B$ per unit volume of Electro-Magnetic origin

- Electrostatic force due to charge density (negligible in comparison and therefore ignored – as explained above)
- External body force per unit volume which we call f

Hence, our equation of motion is

$$u_t + (u \cdot \nabla)u = \nu\Delta u - \frac{1}{\rho}\nabla\mathcal{P} + f + \frac{1}{\rho}J \wedge B \,. \tag{7.15}$$

Thus, with σ spatially uniform, our governing equations are (7.6)–(7.9), (7.10) and (7.14), (7.15).

If we substitute (7.10) into (7.8) and use (7.7) we get

$$\frac{\partial B}{\partial t} = \mathrm{curl}(u \wedge B) + \eta\Delta B \tag{7.16}$$

where we have also made use of the identity $\mathrm{curl}\,(\mathrm{curl}B) = \nabla(\nabla \cdot B) - \Delta B$ and $\eta = (\sigma\mu)^{-1} =$ resistivity. The first term on the right-hand side of (7.16) is essentially a transport term and the second is due to a dissipation (leak) effect.

We can now use another identity, $\mathrm{curl}\,(u \wedge B) = (B \cdot \nabla)u - (u \cdot \nabla)B$ (for divergence-free u and B) in (7.16); as well as

$$J \wedge B = \frac{1}{\mu}\left[(B \cdot \nabla)B - \nabla\left(\frac{B^2}{2}\right)\right] \tag{7.17}$$

in (7.15) to get our full set of governing equations in the form

$$u_t + (u \cdot \nabla)u - \nu\Delta u - \frac{1}{\mu\rho}(B \cdot \nabla)B + \frac{1}{\rho}\nabla\left(\mathcal{P} + \frac{1}{2\mu}B^2\right) = f \tag{7.18}$$

$$B_t + (u \cdot \nabla)B - \eta\Delta B - (B \cdot \nabla)u = 0 \tag{7.19}$$

$$\nabla \cdot u = 0 \tag{7.20}$$

$$\nabla \cdot B = 0 \,. \tag{7.21}$$

Finally, we can transform these equations (in the usual way) into the non-dimensional form shown in the next section ($x \to x/L$, $u \to u/U$, etc.).

7.2 The Equations and Their Setting

Consider the MHD equations in the domain $\Omega_L \equiv [0, L]^d$ with periodic boundary conditions in the following non-dimensional form,

$$u_t + (u \cdot \nabla)\, u - Re^{-1}\,\Delta u - (\beta \cdot \nabla)\,\beta + \nabla(\mathcal{P} + \frac{1}{2}\beta^2) \;=\; g \tag{7.22}$$

$$\beta_t + (u \cdot \nabla)\,\beta - R_m^{-1}\,\Delta\beta - (\beta \cdot \nabla)\,u \;=\; 0 \tag{7.23}$$

$$\nabla \cdot u \;=\; 0 \tag{7.24}$$

$$\nabla \cdot \beta \;=\; 0 \tag{7.25}$$

where

- u = fluid velocity
- $\mathcal{P}$ = pressure
- $\beta = S^{1/2} B$ and B is the magnetic field
- g = external volume forces applied to the fluid which is a C^∞ function (time-independent)

(all quantities non-dimensionalised).

Further,

- Re = Reynold's Number
- R_m = Magnetic Reynold's Number
- $S > 0$ is related to the Alfvén Number

(again all non-dimensional quantities).

In addition to periodic boundary conditions we shall take $\int_{\Omega_L} g\,dx = 0$ so that

$$\int_{\Omega_L} u\,dx = \int_{\Omega_L} \beta\,dx = 0 \tag{7.26}$$

for the domain Ω_L for all time.

With L as a typical representative length, U a typical velocity, B_0 a typical magnetic field measure, μ = the magnetic permeability, σ = conductivity of the fluid and where we have assumed unit mass density, $\rho = 1$; then we define the magnetic Reynold's number as

$$R_m = LU\sigma\mu \tag{7.27}$$

and the Alfvén number is given by

$$A = \frac{U(\mu\rho)^{1/2}}{B_0} \tag{7.28}$$

The magnetic Reynold's number is directly analogous to the original Reynold's number for fluids, i.e. it gives a comparison between transport/forcing effects and dissipation effects for B, whereas, the ratio of inertial to magnetic stresses are typically of the order of the square of the Alfvén number.

From these definitions, $S = A^{-2}$ and so $S > 0$ is given by

$$S = \frac{M^2}{ReR_m} \equiv \frac{B_0^2}{\mu\rho U^2} \tag{7.29}$$

where M is the Hartmann Number (gives a comparison between magnetic viscous force and the ordinary viscous force per unit volume),

$$M = B_0 L \left(\frac{\sigma}{\rho\nu}\right)^{1/2}. \tag{7.30}$$

The non-dimensional form of (7.22) – (7.25) motivate us to define

Definition 7.2.1 *For* $N \geq 0$, $d = 2, 3$, $\beta_i = S^{1/2} B_i$ *and* $\zeta_i = (\mathrm{curl}\,\beta)_i$ *we define the functionals*

$$H_N = \sum_{i=1}^{d} \sum_{|n|=N} \int_{\Omega_L} |D^n u_i|^2 \, dx = |u|^2_{N,2,\Omega_L} \tag{7.31}$$

and

$$E_N = \sum_{i=1}^{d} \sum_{|n|=N} \int_{\Omega_L} |D^n \beta_i|^2 \, dx = |\beta|^2_{N,2,\Omega_L} \tag{7.32}$$

which we combine naturally to define

$$G_N = H_N + E_N \ . \tag{7.33}$$

7.3 The Energy Estimate

Consider

$$G_0 = H_0 + E_0 = \sum_{i=1}^{d} \int_{\Omega_L} |u_i|^2 \, dx + \sum_{i=1}^{d} \int_{\Omega_L} |\beta_i|^2 \, dx \tag{7.34}$$

which is a measure of the total energy of our system.

Theorem 7.3.1 *Assuming* u *and* β *to be smooth solutions of (7.22)-(7.25) then for* $d = 2, 3$, $\forall\, t > 0$, *with* $R = \max\{Re, R_m\}$, *we can make the a-priori estimate*

$$\frac{1}{2} \dot{G}_0 \ \leq \ -\frac{1}{R} G_1 + G_0^{1/2} \, \|g\|_2 \ . \tag{7.35}$$

Proof: Using that $\nabla \cdot u$ and $\nabla \cdot \beta$ are zero:

$$\frac{1}{2}\dot{H}_0 = -\frac{1}{R_e} H_1 + \sum_{i,j} \int_{\Omega_L} u_i \beta_j \beta_{i,j} \, dx + \sum_{i} \int_{\Omega_L} u_i g_i \, dx \tag{7.36}$$

and

$$\frac{1}{2}\dot{E}_0 = -\frac{1}{R_m} E_1 - \sum_{i,j} \int_{\Omega_L} u_i \beta_j \beta_{i,j} \, dx \tag{7.37}$$

which we can combine to give

$$\frac{1}{2}\dot{G}_0 \leq -\frac{1}{R} G_1 + G_0^{1/2} \|g\|_2 \tag{7.38}$$

where $R = \max\{Re, R_m\}$. □

Corollary 7.3.2 G_0 *is bounded (a-priori) for all time and in fact*

$$\overline{G_0} \leq \left[\frac{R\|g\|_2}{c} \right]^2 \ . \tag{7.39}$$

Proof: Simply apply Poincaré's inequality to the first term on the right-hand side of (7.35) and use Lemma 4.8.2. Then look for the absorbing ball to get the result. □

7.4 The MHD Ladder

We can prove the following theorem for the G_N's,

Theorem 7.4.1 (MHD Ladder Theorem) *Assuming u and β to be smooth solutions of (7.22)–(7.25), then for all $N \geq 1$, $t > 0$, $d = 2, 3$ and with $R = \max\{Re, R_m\}$ we can derive the a-priori estimates*

$$\frac{1}{2}\dot{G}_N \leq -\frac{1}{R}G_{N+1} + c\,(\|Du\|_\infty + \|D\beta\|_\infty)\,G_N + G_N^{1/2}\,\|D^N g\|_2 \,. \tag{7.40}$$

Proof: Using (7.22) – (7.25) directly, we see that we can write

$$\begin{aligned} \frac{1}{2}\dot{H}_N = & -\frac{1}{R_e}\sum\int(D^{n+1}u_i)^2 - \sum\int(D^n u_i)\,D^n(u_j u_{i,j}) \\ & + \sum\int(D^n u_i)\,D^n(\beta_j\beta_{i,j}) + \sum\int(D^n u_i)\,(D^n g_i) \end{aligned} \tag{7.41}$$

and

$$\begin{aligned} \frac{1}{2}\dot{E}_N = & -\frac{1}{R_m}\sum\int(D^{n+1}\beta_i)^2 - \sum\int(D^n\beta_i)\,D^n(u_j\beta_{i,j}) \\ & + \sum\int(D^n\beta_i)\,D^n(\beta_j u_{i,j}) \end{aligned} \tag{7.42}$$

which we can combine to get

$$\begin{aligned} \frac{1}{2}\dot{G}_N = & -\frac{1}{R_e}\sum\int(D^{n+1}u_i)^2 - \frac{1}{R_m}\sum\int(D^{n+1}\beta_i)^2 \\ & - \sum\int(D^n u_i)\,D^n(u_j u_{i,j}) + \sum\int(D^n u_i)\,D^n(\beta_j\beta_{i,j}) \\ & - \sum\int(D^n\beta_i)\,D^n(u_j\beta_{i,j}) + \sum\int(D^n\beta_i)\,D^n(\beta_j u_{i,j}) \\ & + \sum\int(D^n u_i)\,(D^n g_i)\,. \end{aligned} \tag{7.43}$$

Consider each of the non-linear terms

$$\begin{aligned} \mathrm{NL1} &= -\sum\int(D^n u_i)D^n(u_j u_{i,j}) \\ &= -\sum_{i,j,N}\int(D^n u_i)\,(\sum_{\ell\neq 0}C_\ell^n D^\ell u_j\, D^{n-\ell+1_j}u_i) \qquad |\ell| \leq N\,. \end{aligned} \tag{7.44}$$

Note that the $\ell = 0$ term in the Leibniz expansion is zero. Similarly,

$$\begin{aligned} \mathrm{NL2} &= -\sum\int(D^n\beta_i)D^n(u_j\beta_{i,j}) \\ &= -\sum_{i,j,N}\int(D^n\beta_i)\,(\sum_{\ell\neq 0}C_\ell^n D^\ell u_j\, D^{n-\ell}\beta_{i,j}) \qquad |\ell| \leq N\,. \end{aligned} \tag{7.45}$$

Now consider the two remaining non-linear terms together

$$\mathrm{NL34} = \sum\int(D^n u_i)D^n(\beta_j\beta_{i,j}) + \sum\int(D^n\beta_i)D^n(\beta_j u_{i,j}) \tag{7.46}$$

$$\begin{aligned} &= \sum_{i,j,N}\int(D^n u_i)\,(\sum_{\ell\neq 0}C_\ell^n D^\ell\beta_j\, D^{n-\ell}\beta_{i,j}) \\ &\quad + \sum_{i,j,N}\int(D^n\beta_i)\,(\sum_{\ell\neq 0}C_\ell^n D^\ell\beta_j\, D^{n-\ell}u_{i,j}) \end{aligned} \tag{7.47}$$

where $\ell \leq N$ and the two corresponding $\ell = 0$ terms cancel each other out.

Now consider (7.45)

$$\begin{aligned} \mathrm{NL2} &\leq E_N^{1/2}\left[\sum_{i,j,N}\sum_{\ell\neq 0}\int C_\ell^n (D^\ell u_j)^2 (D^{n-\ell}\beta_{i,j})^2\right]^{1/2} && (7.48)\\ &\leq E_N^{1/2}\left[\sum_{i,j}\sum_{\ell\neq 0} C_\ell^N \mathcal{A}_{i,j}^\ell\right]^{1/2} && (7.49)\end{aligned}$$

where

$$\mathcal{A}_{i,j}^\ell = \sum_N \int (D^\ell u_j)^2 (D^{n-\ell}\beta_{i,j})^2 \leq \|D^\ell u_j\|_p^2 \, \|D^{n-\ell+1_j}\beta_i\|_q^2 \tag{7.50}$$

where $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$.

Now let us consider the following pair of Gagliardo-Nirenberg inequalities

$$\begin{aligned} \|D^\ell u_j\|_p &\leq c\,\|D^n u_j\|_2^a \|Du_j\|_\infty^{1-a} && (7.51)\\ \|D^{n-\ell+1_j}\beta_i\|_q &\leq c\,\|D^n \beta_i\|_2^b \|D\beta_i\|_\infty^{1-b} && (7.52)\end{aligned}$$

where we require

$$\begin{aligned} \frac{1}{p} &= \tfrac{L-1}{d} + a\left(\tfrac{1}{2}-\tfrac{N-1}{d}\right) \quad 0 \leq \frac{L-1}{N-1} \leq a < 1 && (7.53)\\ \frac{1}{q} &= \tfrac{N-L}{d} + b\left(\tfrac{1}{2}-\tfrac{N-1}{d}\right) \quad 0 \leq \frac{N-L}{N-1} \leq b < 1\,. && (7.54)\end{aligned}$$

Now if we choose $ap = 2$ and $bq = 2$ then

$$a = \frac{L-1}{N-L} \quad \text{and} \quad b = \frac{N-L}{N-1} \tag{7.55}$$

with $1/p + 1/q = 1/2$.

For $2 \leq L \leq N-1$ we see that (7.53) & (7.54) are satisfied. (7.51) and (7.52) are also seen to be satisfied for $L = 1$ & $L = N$.

Hence,

$$\mathrm{NL2} \leq c\, E_N^{1/2} H_N^{a/2} \|Du\|_\infty^{1-a} E_N^{b/2} \|D\beta\|_\infty^{1-b}\,. \tag{7.56}$$

Similarly,

$$\mathrm{NL1} \leq c\, H_N \|Du\|_\infty\,, \tag{7.57}$$

$$\mathrm{NL34} \leq c\, H_N^{1/2} E_N^{1/2} \|D\beta\|_\infty + c\, E_N^{1/2} E_N^{a/2} \|D\beta\|_\infty^{1-a} H_N^{b/2} \|Du\|_\infty^{1-b}\,. \tag{7.58}$$

Hence, if $R = \max\{Re, R_m\}$ then (via Young's inequality and/or the definition $G_N = H_N + E_N$) we see that we get the result of the MHD Ladder Theorem. □

Further, let us construct a vector $f = \tau_0\, g$ where τ_0 is a characteristic time scale. Consequently we define

Definition 7.4.1

$$F_N = G_N + \|D^N f\|_2^2\,. \tag{7.59}$$

Then we can prove the following corollary,

Corollary 7.4.2 *For smooth solutions and $d = 2, 3$, $N \geq 1$, with $\lambda_0^{-2} = \frac{R}{2}\tau_0^{-1} + \lambda_f^{-2}$, we have*

$$\frac{1}{2}\dot{F}_N \leq -\frac{1}{R}F_{N+1} + \left[c(\|Du\|_\infty + \|D\beta\|_\infty) + \frac{1}{R}\lambda_0^{-2}\right]F_N \tag{7.60}$$

Proof: Note that

$$G_N^{1/2}\|D^N g\|_2 \leq \frac{1}{2}\tau_0^{-1}F_N \tag{7.61}$$

and so

$$\begin{aligned}
\frac{1}{2}\dot{F}_N \leq & -\frac{1}{R}G_{N+1} - \frac{1}{R}\|D^{N+1}f\|_2^2 \\
& + \left(c\|Du\|_\infty + c\|D\beta\|_\infty + \frac{1}{2}\tau_0^{-1}\right)F_N + \frac{1}{R}\|D^{N+1}f\|_2^2 \qquad (7.62)\\
\leq & -\frac{1}{R}F_{N+1} + \left(c\|Du\|_\infty + c\|D\beta\|_\infty + \frac{1}{2}\tau_0^{-1}\right)F_N \\
& + \frac{1}{R}\frac{\|D^{N+1}f\|_2^2}{\|D^N f\|_2^2}F_N \qquad (7.63)\\
\leq & -\frac{1}{R}F_{N+1} + \left(c\|Du\|_\infty + c\|D\beta\|_\infty + \frac{1}{2}\tau_0^{-1} + \frac{1}{R}\lambda_f^{-2}\right)F_N\,. \qquad (7.64)
\end{aligned}$$

and hence the result. □

Thus we can conclude results parallel to those proved from the Ladder Theorem for the Navier-Stokes equations in Chapter 4. i.e. for the $d = 2, 3$ MHD equations, provided we assume regular strong solutions for all finite-time intervals and that u and β are uniformly bounded in time in V, then there exist absorbing sets for u and β in all the $\dot{H}^n_{per}(\Omega_L)^d$, for all $n \in \mathbb{N}$ and hence we can deduce a C^∞ global attractor.

Chapter 8

Conclusions and Further Work

8.1 Research Presented in this Thesis

The following original work (indented and enumerated below) was presented in this thesis:

In Chapter 2 the general framework for the evolution equation was introduced, the semi-group of operators defined, and then a theorem was provided which would guarantee the existence of a global attractor when the existence of an absorbing set is known. Following some theory on linear operators, we outlined how we would proceed in order to show the regularity of solutions in the attractor.

Some elementary functional analysis for Lebesgue and Sobolev spaces followed in the first part of Chapter 3. Consequently:

1. With the help of a Poincaré type of inequality proved for mean-zero functions (on Ω_L) the equivalence of semi-norms and full-norms on $\dot{W}^{m,p}(\Omega_L)$ (where the dot indicates zero mean) for $1 \leq p < \infty$ was shown; which was also extended to $p = \infty$ for space-periodic smooth functions.
2. The Gagliardo-Nirenberg inequality was reproduced for functions with compact support, but with the multiplicative constants provided as far as was possible.
3. Two alternative versions of the Gagliardo-Nirenberg inequality were proved (via extension theorems):
   (a) The first for full Sobolev norms for functions with no specified boundary conditions on Ω_L.
   (b) The second for semi-norms for mean-zero functions on periodic boundary conditions on Ω_L. This result was needed and used repeatedly in the following chapters when calculating a-priori estimates.

After introducing the initial and boundary value Navier-Stokes problem we were to study, "weak" and "strong" solutions were also introduced and their relevance discussed. It was apparent that for the two dimensional Navier-Stokes problem, unique, regular solutions have been proved to exist for all time, but in the three dimensional case we must make some extra assumptions in order to prove regularity for all time:

4. An alternative (to that of Serrin) proof of the minimum assumptions sufficient for regularity (the solution assumed uniformly bounded in $L^{3+\epsilon}$) was proved via direct functional analytic techniques.

Subsequently, the non-well-posedness of the three dimensional problem was discussed and related to the existence of strong solutions. A 'Ladder Theorem' was then proved, which comprises of a set of a-priori estimates which we used to show the regularity of solutions in the attractor – provided we assumed that strong solutions existed for all finite-time intervals and were also uniformly bounded in time.

5. An interpolation inequality, important for the particular version of the Ladder Theorem outlined, was also proved.

Then in Chapter 5,

6. The 'Lattice Theorem' was proved – this provided a much more general set of a-priori estimates (previously unknown) for semi-norms, and which we hoped, could be used to improve on Serrin's result (the $L^{3+\epsilon}$ result mentioned). However, the estimates only reproduced (at the bottom 'rung') Serrin's result, and thus indicated that via the functional analytic techniques we have employed, we might not improve this result. These estimates also showed that u assumed bounded in $L^{3+\epsilon}$ uniformly in time was sufficient to prove the existence of a C^∞ global attractor.

Calculations for the a-priori estimates of the Lattice Theorem suggested the alternative assumptions which were possible (via lattice-like techniques) to prove the existence of strong solutions for all time – the $||\mathcal{P}||_{2(1+\delta)}$ result, which was subsequently improved:

7. The minimum assumptions sufficient for a C^∞ attractor, was then reduced to $||\mathcal{P}||_{15/8+\epsilon}$ via some further estimates, which crucially involved the Lattice Theorem.

Having investigated the question of regularity as far as seemed possible, we then turned our attention to defining a minimum length scale for a Navier-Stokes fluid flow.

In Chapter 6, we introduced the notion of turbulence and importantly, the Kolmogorov dissipation length scale. The limitations of Kolmogorov's scaling theory as well as several alternative theories were discussed, and we subsequently set about a rigorous mathematical derivation for an appropriate minimum length scale. A Fourier splitting argument provided us with an estimate for the number of 'relevant modes' we should consider – we defined our minimum length scale via the number of relevant modes.

Consequently, we introduce the wavenumbers $\kappa_{N,r}$ for which we proved a large set of a-priori estimates (specific to the Navier-Stokes equations). In particular:

8. An estimate for the $\kappa_{N,r}$'s allowed us to derive a result analogous to Ladyzhenskaya's result for small enough initial data, or large enough viscosity.

9. We were also able to provide an alternative proof (via the $\kappa_{N,r}$'s) of the Beale, Kato and Majda result concerning the breakdown of regularity occuring when $||\omega||_\infty$ becomes singular.

We then proved some further (and more important) a-priori estimates for the $\kappa_{N,r}$'s, in particular Corollary 6.5.4 and:

10. The estimate for $\overline{\kappa_{N,0}^2}$ in Theorem 6.5.5.

With these estimates we then showed how the Kolmogorov dissipation length applies to quiescent (laminar) flows and that the definition we provided for the minimum length scale via $\langle\kappa_{N,r}^2\rangle$ or $\overline{\kappa_{N,r}^2}$ incorporates a term which might account for inertial range intermittency. Subsequently, in the last part of Chapter 6, we compared estimates for length scales defined via alternative means (such as the attractor dimension) with estimates for the length scales we defined.

Lastly,

11. The MHD equations were derived and some a-priori estimates (including an 'MHD Ladder Theorem') were proved – from which we could deduce some results analogous to those proved for the Navier-Stokes equations.

8.2 Further Work

It seems (to the author ar least) that new results which can be achieved via the specific functional analytic techniques outlined are exhausted. Another, more subtle approach seems to be necessary.

There are some important questions which are left unanswered (particularly) in the $d = 3$ case:

1. Can we relax the assumptions necessary for any of the Gagliardo-Nirenberg inequalities we have proved, and also, find explicitly, the exact form of the multiplicative constants? These would be helpful for a whole range of evolutionary nonlinear PDE problems (such as the CGL equation) in addition to the Navier-Stokes equations. Dimensional analysis suggests that an improvement (sharpening) of this inequality is not possible. Can we prove the inequality in more general terms?

2. Can we prove the existence of regular strong solutions for all time for the three dimensional Navier-Stokes equations? This is an open and very crucial problem. Can we, for instance, improve on Serrin's result or the pressure results of Chapters 4 and 5? Can we improve the Lattice Theorem and also provide the multiplicative constants?

3. Further, the question of the existence of an 'Inertial Manifold' for the two dimensional Navier-Stokes equations (on periodic or other boundary conditions) must be addressed. This would have several important consequences, including resolving the 'best' length scale for two dimensional numerical simulations.

4. Is it possible to extend most of the results we have shown for the Navier-Stokes equations to the MHD system? Does a lattice-like structure exist? Can we prove a result analogous to Serrin's? What is the best estimate we can give for the attractor dimension – provided we can prove it exists without assumptions? Can we define an appropriate minimum length scale?

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