Grassmannian spectral shooting and the stability of multi-dimensional travelling waves

Simon J.A. Malham
Veerle Ledoux (Ghent),
Jitse Niesen (Leeds),
Vera Thümmler (D-fine)

Heriot–Watt University, Edinburgh, UK

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Spectral problems

Parabolic nonlinear systems on $\mathbb{R} \times \mathbb{T}$:

$$\partial_t U = B\,\Delta U + c\,\partial_x U + F(U),$$

Travelling wave U_c. Small perturbations U satisfy:

$$B\,\Delta U + c\,\partial_x U + \mathrm{D}F(U_c)U = \lambda U.$$

Two main solution approaches:

- *Projection.*
- *Shooting.*

Motivation

Workshop at AIM, Palo Alto, May 2005:

> *Stability criteria for multi-dimensional waves and patterns.*

Organised by: Chris Jones, Yuri Latushkin, Bob Pego, Björn Sandstede and Arnd Scheel.

Setup

On $\mathbb{R}$: $\qquad B\,\Delta U + c\,\partial_x U + \mathrm{D}F(U_c)U = \lambda U$

$$\Leftrightarrow \qquad Y' = A(x;\lambda)\,Y$$

Assume for $\lambda \in \Omega \subseteq \mathbb{C}$:

- Exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$;
- Same Morse index k (Sandstede 2002).

Matching condition (Alexander, Gardner and Jones 1990):

$$D(\lambda) \equiv \mathrm{e}^{-\int_0^x \mathrm{Tr} A(\xi;\lambda)\,\mathrm{d}\xi} \det\big(Y^-(x;\lambda)\;\; Y^+(x;\lambda)\big).$$

Numerical issues

- Computational domain.
- Different exponential growth rates.
- Polynomial complexity.
- Flow singularities?!
- Where to match?
- Retaining analyticity.
- How to project transversely.

Outline

Stiefel and Grassmann manifolds

- Stiefel manifold:

$$\mathbb{V}(n,k) = \{k\text{-frames centred at the origin}\}.$$

- Grassmann manifold:

$$\mathrm{Gr}(n,k) = \{k\text{-dimensional subspaces of } \mathbb{C}^n\}.$$

- Trivial fibre bundle:

$$\mathbb{V}(n,k) \cong \mathrm{Gr}(n,k) \times \mathrm{GL}(k)$$

- Projection $\pi\colon k$-frame $\mapsto$ spanning k-plane.

Representation

Coordinate patches $\mathbb{U}_{\mathfrak{i}}$: multi-index $\mathfrak{i} = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$.

Example: $\mathbb{U}_{\{1,\ldots,k\}}$ uniquely represented by:

$$y_{\mathfrak{i}^\circ} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \hat{y}_{k+1,1} & \hat{y}_{k+1,2} & \cdots & \hat{y}_{k+1,k} \\ \hat{y}_{k+2,1} & \hat{y}_{k+2,2} & \cdots & \hat{y}_{k+2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{y}_{n,1} & \hat{y}_{n,2} & \cdots & \hat{y}_{n,k} \end{pmatrix}.$$

Local coordinate chart $\varphi_{\mathfrak{i}} \colon \mathbb{U}_{\mathfrak{i}} \to \mathbb{C}^{(n-k)k}$ given by $\varphi_{\mathfrak{i}} \colon y_{\mathfrak{i}^\circ} \mapsto \hat{y}$.

Grassmannian flows

$$Y' = A(x, Y)\, Y.$$

Substitute decomposition $Y = y_{\mathfrak{i}^\circ} u_{\mathfrak{i}}$:

$$y'_{\mathfrak{i}^\circ} u_{\mathfrak{i}} + y_{\mathfrak{i}^\circ} u'_{\mathfrak{i}} = (A_{\mathfrak{i}} + A_{\mathfrak{i}^\circ} y_{\mathfrak{i}^\circ})\, u_{\mathfrak{i}},$$

Project onto $\mathfrak{i}$th and $\mathfrak{i}^\circ$th rows:

$$\hat{y}' = c + d\,\hat{y} - \hat{y}(a + b\,\hat{y}) \qquad \text{and} \qquad u'_{\mathfrak{i}} = (a + b\,\hat{y})\, u_{\mathfrak{i}},$$

where $a = A_{\mathfrak{i}\times\mathfrak{i}}$, $b = A_{\mathfrak{i}\times\mathfrak{i}^\circ}$, $c = A_{\mathfrak{i}^\circ\times\mathfrak{i}}$ and $d = A_{\mathfrak{i}^\circ\times\mathfrak{i}^\circ}$.

Linear vector field: $A = A(x)$ only?

Drury–Oja flow

Humpherys and Zumbrun; QR-decomposition of $Y \in \mathbb{V}(n,k)$:

$$
\begin{aligned}
Q' &= (I_n - QQ^\dagger)A(x)Q, \\
(\det R)' &= \mathrm{Tr}\big(Q^\dagger A(x) Q\big)(\det R).
\end{aligned}
$$

Corresponds to Riccati with $u' = -\hat{y}^\dagger (c + d\,\hat{y})\,u$.

Grassmannian Gaussian elimination method (GGEM)

$$
\begin{array}{ccccccccc}
\mathbb{C}^{(n-k)k} & \xrightarrow{\varphi_{\mathrm{i}}^{-1}} & \mathbb{U}_{\mathrm{i}} & \xrightarrow{\mathrm{id}} & \mathbb{V}(n,k) & \xrightarrow{(\Lambda_{Y_0})^*} & \mathrm{GL}(n) & \xrightarrow{\log} & \mathfrak{gl}(n) \\
\downarrow & & \downarrow \scriptstyle{GGEM} & & \downarrow \scriptstyle{RK} & & \downarrow & & \scriptstyle{Magnus} \downarrow \\
\mathbb{C}^{(n-k)k} & \xleftarrow[\varphi_{\mathrm{i}'}]{} & \mathbb{U}_{\mathrm{i}'} & \xleftarrow[\mathrm{QOGE}]{} & \mathbb{V}(n,k) & \xleftarrow[\Lambda_{Y_0}]{} & \mathrm{GL}(n) & \xleftarrow[\exp]{} & \mathfrak{gl}(n)
\end{array}
$$

Quasi-optimal Gaussian elimination (QOGE)

GE with *free* stepwise max pivot, generates: $Y_{m+1} = y_{\mathfrak{i}^\circ} L$.

$$\begin{pmatrix} * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & * & \cdots & * \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ * & * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ * & * & * & * & \cdots & * \end{pmatrix}$$

Applications (planar fronts)

- $D(\lambda) \equiv \mathrm{e}^{-\int_0^x \mathrm{Tr} A(\xi;\lambda)\,\mathrm{d}\xi} \det\big(Y^-(x;\lambda)\ Y^+(x;\lambda)\big)$.
- $\det\big(Y^-\ Y^+\big) = \det\begin{pmatrix} y_{\mathrm{i}_-^\circ} & y_{\mathrm{i}_+^\circ}\end{pmatrix} \cdot \det\ u_{\mathrm{i}_-} \cdot \det\ u_{\mathrm{i}_+}$.
- $D(\lambda; x_*) \equiv \det\begin{pmatrix} y_{\mathrm{i}_-^\circ} & y_{\mathrm{i}_+^\circ}\end{pmatrix} \cdot \det L^- \cdot \det L^+$.
- Exponentially scale GGEM.

Boussinesq system

PDE: $u_{tt} = (1 - c^2)\, u_{xx} + 2c\, u_{xt} - u_{xxxx} - (u^2)_{xx}$.

Solitary waves: $\bar{u}(x) \equiv \frac{3}{2}(1 - c^2)\mathrm{sech}^2\big(\frac{1}{2}\sqrt{1 - c^2}\, x\big)$.

Stable when $1/2 < |c| < 1$ and unstable when $|c| < 1/2$.

$$A(x;\lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\lambda^2 - 2\bar{u}'' & 2\lambda c - 4\bar{u}' & (1 - c^2) - 2\bar{u} & 0 \end{pmatrix}.$$

Boussinesq: GGEM

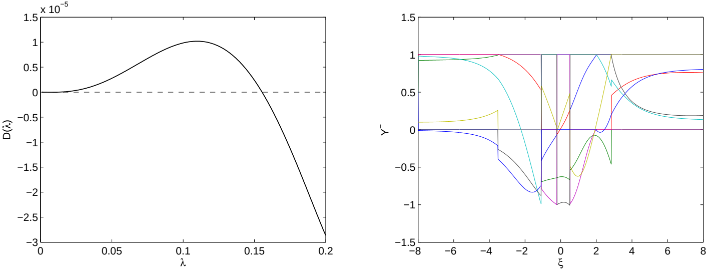


Figure: Evans function with GGEM-RK and $x_* = 8$ (left panel). Entries of $y_\mathfrak{i}$ for $\lambda = 0.15543141$ (right panel).

Boussinesq: Evans function vs matching point

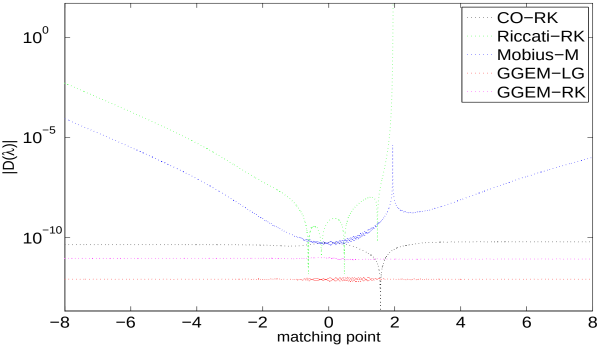


Figure: $|D(\lambda)|$ for λ equal to the eigenvalue for different matching points. The number of steps in the equidistant mesh was $N = 512$.

Boussinesq: error vs matching point

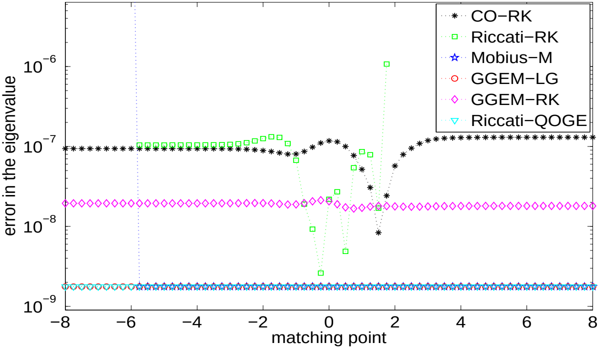


Figure: Error in the eigenvalue for different choices of the matching point: $N = 512$.

Autocatalytic fronts

$$\partial_t u = \delta \Delta u + c\partial_x u - uv^m,$$
$$\partial_t v = \Delta v + c\partial_x v + uv^m.$$

$$A(x;\lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda/\delta + \bar{v}^m/\delta & m\bar{u}\bar{v}^{m-1}/\delta & -c/\delta & 0 \\ -\bar{v}^m & \lambda - m\bar{u}\bar{v}^{m-1} & 0 & -c \end{pmatrix},$$

Autocatalytic fronts: Evans contours

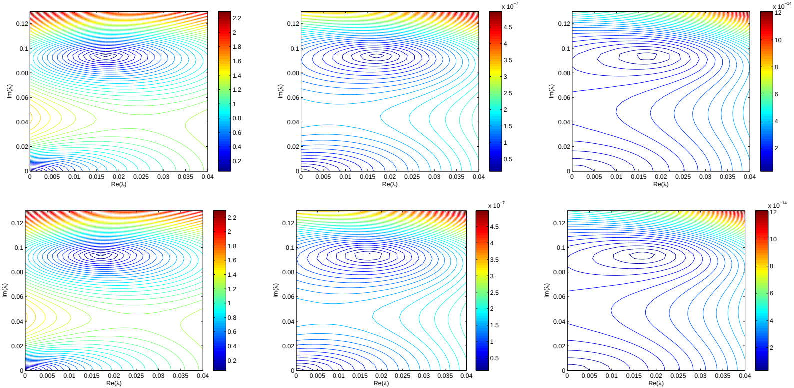


Figure: Contour lines of $|D(\lambda)|$ for $\delta = 0.1$ and $m = 9$ using the CO-RK and GGEM-LG (order: top three down to bottom three), matching at positions $x_* = -8, 0, +8$ (left to right).

Autocatalytic fronts: error vs matching point

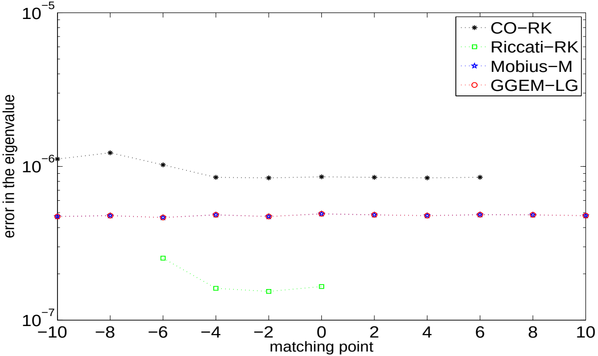


Figure: Error in the eigenvalue when $\delta = 0.1$ and $m = 9$: $N = 256$.

Transverse Fourier basis

On $\mathbb{R} \times \mathbb{T}$ we have:

$$B\Delta U + c\,\partial_x U + \mathrm{D}F(U_c)U = \lambda U.$$

On the Fourier modes $k = -K, -K+1, \ldots, K$:

$$\partial_x \hat{U}_k = \hat{P}_k,$$
$$\partial_x \hat{P}_k = \lambda B^{-1}\hat{U}_k + (k/\tilde{L})^2\hat{U}_k - c\,B^{-1}\hat{P}_k - \sum_{\nu=-K}^{K} B^{-1}\hat{D}_{k-\nu}\hat{U}_\nu,$$

Large ODE system

$$\partial_x \begin{pmatrix} \hat{U} \\ \hat{P} \end{pmatrix} = \begin{pmatrix} O_{N(2K+1)} & I_{N(2K+1)} \\ \tilde{A}_3(\lambda) + \hat{A}_3(x) & -c\, B^{-1} \otimes I_{2K+1} \end{pmatrix} \begin{pmatrix} \hat{U} \\ \hat{P} \end{pmatrix}$$

with $E_k(\lambda) \equiv \lambda B^{-1} + (k/\tilde{L})^2 I_N$:

$$\tilde{A}_3(\lambda) = \begin{pmatrix} E_{-K}(\lambda) & O & \cdots & O \\ O & E_{-K+1}(\lambda) & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & \cdots & O & E_{+K}(\lambda) \end{pmatrix}$$

$$\hat{A}_3(x) = -B^{-1} \otimes \begin{pmatrix} \hat{D}_0 & \hat{D}_{-1} & \cdots & \hat{D}_{-2K} \\ \hat{D}_1 & \hat{D}_0 & \cdots & \hat{D}_{-2K+1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{D}_{2K} & \hat{D}_{2K-1} & \cdots & \hat{D}_0 \end{pmatrix}.$$

Computing travelling waves: freezing method

Substitute $U(x, y, t) = V(x - \gamma(t), y, t)$ into original PDE:

$$
\begin{aligned}
\partial_t V &= B\,\Delta V + \gamma'(t)\partial_x V + F(V), \\
0 &= \int_{\mathbb{R}\times\mathbb{T}} \big(\partial_x \hat{V}(x, y, t)\big)^{\mathrm{T}} \big(\hat{V}(x, y, t) - V(x, y, t)\big)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

(Developed by Beyn and Thümmler.)

Wrinkled fronts (cubic autocatalytic system)

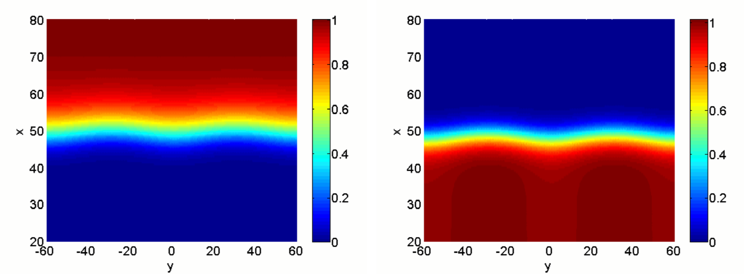
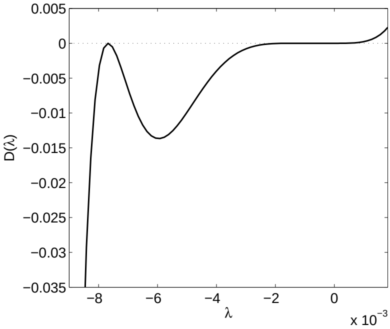
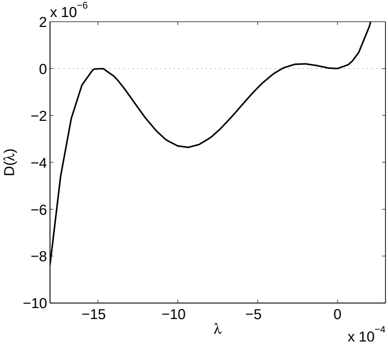


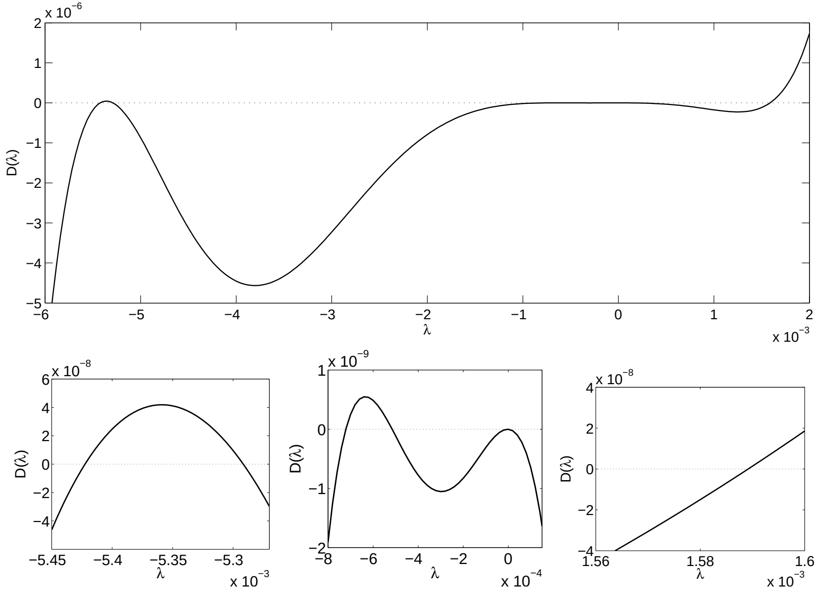
Figure: The wrinkled front for $\delta = 3$. Left panel: u component. Right panel: v component. Cut from domain $[-150, 150] \times [-60, 60]$, grid 300×240, FE rectangles.

Wrinkled front: Evans function for $\delta = 2.5$





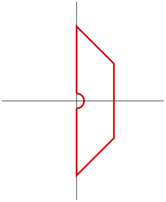
Wrinkled front: Evans function for $\delta = 3$



Wrinkled front: Eigenvalues for $\delta = 3$

K	Eigenvalues (Evans function)				
3	0.001609	−0.000026	−0.000781	−0.001296	−0.000670
4	0.001609	0.000002	−0.000001	−0.000519	−0.000670
5	0.001589	0.000002	−0.000001	−0.000519	−0.000720
6	0.001589	−0.000002	−0.000003	−0.000515	−0.000720
7	0.001589	−0.000002	−0.000003	−0.000515	−0.000721
8	0.001589	−0.000002	−0.000003	−0.000515	−0.000721
9	0.001589	−0.000002	−0.000003	−0.000515	−0.000721
⋮			⋮		
24	0.001589	−0.000002	−0.000003	−0.000515	−0.000721
	Eigenvalues (ARPACK)				
	0.001592	0.000000	0.000000	−0.000514	−0.000719

Wrinkled front: contour integration



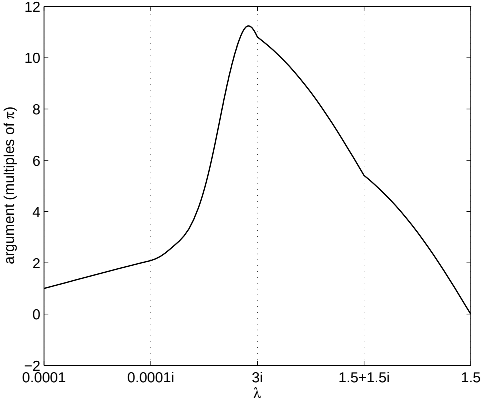
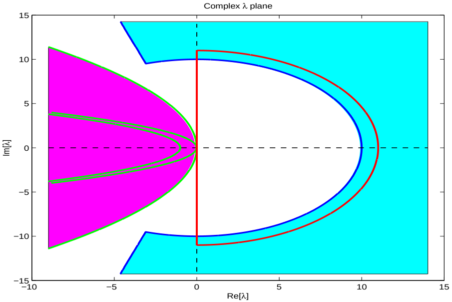


Figure: Left panel: contour. Right panel: $\arg\big(D(\lambda)\big)$ when λ transverses the top half. $\delta = 3$.

Future work

- Multiple sources of error: relative influence?
- Control theory: Lagrangian Grassmannian.
- Schubert cells.

Spectral plane



Boussinesq: Evans function

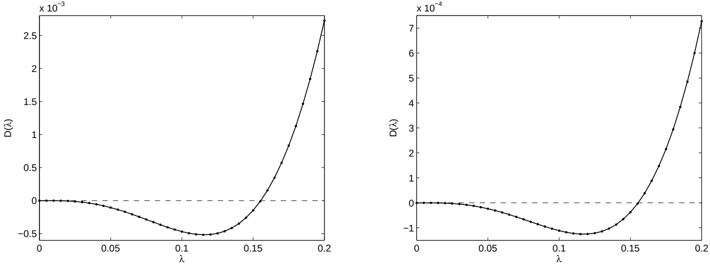


Figure: Wave speed $c = 0.4$. Left plot: Riccati-RK ($\mathfrak{i}^- = \{1, 2\}$ over $[-8, 0]$ and $\mathfrak{i}^+ = \{3, 4\}$ over $[0, 8]$). Right plot: CO-RK.

Boussinesq: eigenvalue error I

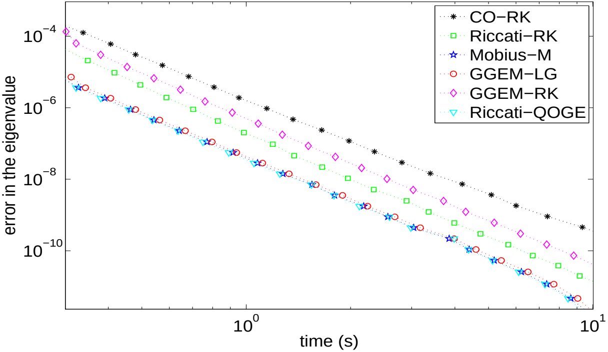


Figure: Error in the eigenvalue vs cputime, matching at $x_* = 0$.

Boussinesq: eigenvalue error II

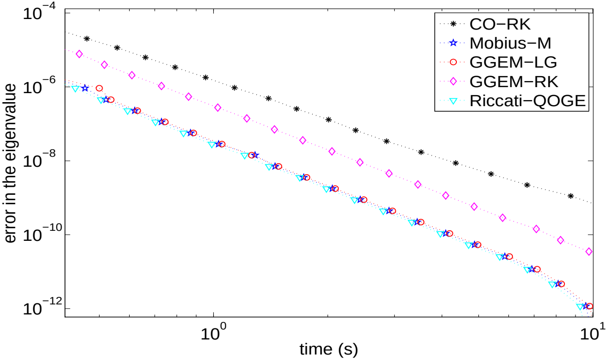


Figure: Error in the eigenvalue vs cputime, matching at $x_* = 8$.

Boussinesq: Evans function

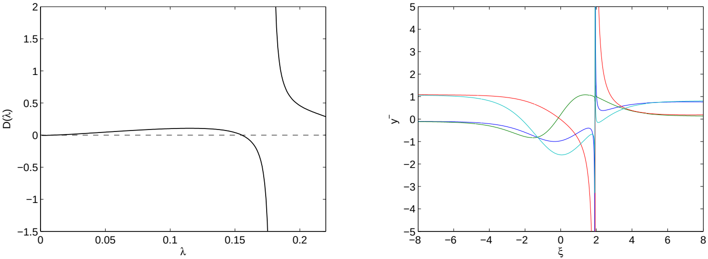


Figure: Left: Evans function with Möbius–Magnus: $x_* = +8$, Right: entries in $\hat{y}^-$ when $\lambda = 0.15543141$.

Autocatalytic fronts: Evans contours

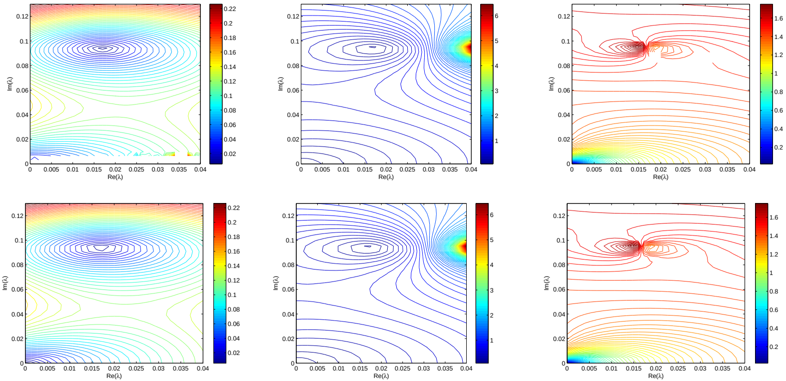


Figure: Contour lines of $|D(\lambda)|$ for $\delta = 0.1$ and $m = 9$ using the Riccati-RK, Möbius–Magnus (order: top three down to bottom three), matching at positions $x_* = -8, 0, +8$ (left to right).

Wrinkled front: same with Drury–Oja