

# Efficient strong integrators for linear stochastic systems

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# Introduction

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- 1 Neumann & Magnus series
- 2 Global error: Neumann vs Magnus
- 3 Quadrature & efficiency
- 4 Numerical experiments
  - ▶ Linear SDE
  - ▶ Nonlinear Riccati SDE
- 5 Conclusions

# Introduction

## Take home message:

*For a given computational effort, a modified order 1 Magnus integrator is the most accurate of all numerical methods.*

## Neumann series

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- ▶ *Peano–Baker series, Feynman–Dyson path ordered exponential, Chen–Fleiss series or Neumann series*

# Magnus series

$$S(t) = \exp(\sigma(t)) ,$$

$$\begin{aligned}\sigma(t) &= \ln(I + K \circ I + K^2 \circ I + \dots) \\ &= K \circ I + K^2 \circ I - \frac{1}{2}(K \circ I)^2 + \dots ,\end{aligned}$$

- ▶ *Magnus* 1954, *Kunita* 1980, *Ben Arous* 1989, *Burrage* 1999.

# Integral operators

$$\begin{aligned}K \circ I &= K_0 \circ I + K_1 \circ I + K_2 \circ I \\ &= \int_0^t A_0(\tau) d\tau + \int_0^t A_1(\tau) dW_1(\tau) + \int_0^t A_2(\tau) dW_2(\tau)\end{aligned}$$

$$\begin{aligned}K^2 \circ I &= (K_0 + K_1 + K_2)^2 \circ I \\ &= (K_0^2 + K_0K_1 + K_1K_0 + K_1^2 + K_1K_2 + K_2K_1 + \dots) \circ I\end{aligned}$$

$$K_i K_j \circ I \equiv \int_0^t A_i(\tau_1) \int_0^{\tau_1} A_j(\tau_2) dW_j(\tau_2) dW_i(\tau_1)$$

## Constant coefficient, non-commutative case

$$K_j K_i \circ I \equiv a_j a_i \underbrace{\int_0^t \int_0^{\tau_1} dW_i(\tau_2) dW_j(\tau_1)}_{J_{ij}}$$

$$K_k K_j K_i \circ I \equiv a_k a_j a_i \underbrace{\int_0^t \int_0^{\tau_1} \int_0^{\tau_2} dW_i(\tau_3) dW_j(\tau_2) dW_k(\tau_1)}_{J_{ijk}}$$

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Special non-commutative cases:

$$S(t) = I + a_1 \cdot \int_0^t S(\tau) dW_1(\tau) + \int_0^t S(\tau) dW_2(\tau) \cdot a_2,$$

$$S(t) = \exp(a_1 W_1(t)) \cdot \exp(a_2 W_2(t)).$$

## Neumann expansion (order 3/2)

$$S^{\text{neu}}(t) \approx I + S_{1/2} + S_1 + S_{3/2} + \cdots,$$

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$$\begin{aligned} S_{3/2} = & a_0 a_1 J_{10} + a_1 a_0 J_{01} + a_0 a_2 J_{20} + a_2 a_0 J_{02} \\ & + a_1^3 J_{111} + a_2 a_1^2 J_{112} + a_1 a_2 a_1 J_{121} + a_2^2 a_1 J_{122} \\ & + a_1^2 a_2 J_{211} + a_2 a_1 a_2 J_{212} + a_1 a_2^2 J_{221} + a_2^3 J_{222} \\ & + a_0^2 J_{00} + a_1^2 a_0 J_{011} + a_1 a_0 a_1 J_{101} + a_0 a_1^2 J_{110} + a_2^2 a_0 J_{022} \\ & + a_2 a_0 a_2 J_{202} + a_0 a_2^2 J_{220} + a_1^4 J_{1111} + a_2^2 a_1^2 J_{1122} \\ & + a_2 a_1 a_2 a_1 J_{1212} + a_1 a_2 a_1 a_2 J_{2121} + a_1^2 a_2^2 J_{2211} + a_2^4 J_{2222}. \end{aligned}$$

## Relations between Stratonovich integrals

$$J_{121} = J_1 J_{12} - 2J_{112} ,$$

$$J_{122} = J_2 J_{12} - \frac{1}{2} J_1 J_2^2 + J_{221} ,$$

$$J_{210} = J_0 J_{21} - J_2 J_{01} + J_{012} ,$$

$$J_{102} = J_1 J_{02} - J_{021} - J_{012} .$$

- ▶ Shuffle algebras: *Gaines* 1995; *Kawski* 2001.

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# Numerical SDE schemes

## General integrators

- ▶ Euler-Maruyama and Milstein methods
- ▶ Runge–Kutta type methods (*Kloeden & Platen* 1999)
- ▶ Magnus (*Burrage* 1999; *Misawa* 2001; *P-C. Moan* 2004)
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## Magnus integrators

- ▶ Simple.
- ▶ **Low dimensional computationally favourable basis**

$$\{J_0, J_1, J_2, J_{12}, J_{01}, J_{02}, J_{112}, J_{221}, J_{110}, J_{220}\} .$$

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**Neumann or Magnus:**

$$S(t_n, t_{n+1}) = \underbrace{\hat{S}(t_n, t_{n+1})}_{\text{approx}} + R(t_n, t_{n+1}).$$

## Global error II

$$\begin{aligned}\mathcal{E}^2 &\equiv \sup_{\|Y_0\|=1} \mathbb{E} \left\| \left( \prod_{n=N-1}^0 S(t_n, t_{n+1}) - \prod_{n=N-1}^0 \hat{S}(t_n, t_{n+1}) \right) Y_0 \right\|^2 \\ &= \sup_{\|Y_0\|=1} \mathbb{E} \left\| \underbrace{\left( \sum_{n=0}^{N-1} \hat{S}(t_{n+1}, t_N) R(t_n, t_{n+1}) \hat{S}(t_0, t_n) \right)}_{\mathcal{R}} Y_0 \right\|^2.\end{aligned}$$

$$\prod_{n=N-1}^0 (\hat{S}(t_n, t_{n+1}) + R(t_n, t_{n+1})) - \prod_{n=N-1}^0 \hat{S}(t_n, t_{n+1})$$

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## Global error III

$$A_\alpha = [a_1, [a_1, a_2]] \quad \text{and} \quad J_\alpha = J_{112} - \frac{1}{2}J_1 J_{12} + \frac{1}{12}J_1^2 J_2.$$

$$\mathcal{R} = \sum_{n=0}^{N-1} \sum_{\alpha} (\hat{S}(t_{n+1}, t_N) A_\alpha \hat{S}(t_0, t_n)) J_\alpha(t_n, t_{n+1}).$$

$$\begin{aligned} & \mathbb{E}(\mathcal{R}^T \mathcal{R}) \\ &= \sum_{n=0}^{N-1} \sum_{\alpha, \beta} \mathbb{E} \left( (\hat{S}(t_{n+1}, t_N) A_\alpha \hat{S}(t_0, t_n))^T (\hat{S}(t_{n+1}, t_N) A_\beta \hat{S}(t_0, t_n)) \right) \\ & \quad \cdot \mathbb{E}(J_\alpha(t_n, t_{n+1}) J_\beta(t_n, t_{n+1})) \\ &+ \sum_{n \neq m} \sum_{\alpha, \beta} \mathbb{E} \left( (\hat{S}(t_{n+1}, t_N) A_\alpha \hat{S}(t_0, t_n))^T (\hat{S}(t_{m+1}, t_N) A_\beta \hat{S}(t_0, t_m)) \right) \\ & \quad \cdot \mathbb{E}(J_\alpha(t_n, t_{n+1})) \mathbb{E}(J_\beta(t_m, t_{m+1})). \end{aligned}$$

# Magnus more accurate than Neumann?

For order 1/2 schemes:

$$R^{\text{mag}} = \frac{1}{2}[a_1, a_2](J_{21} - J_{12});$$

$$R^{\text{neu}} = R^{\text{mag}} + \underbrace{\frac{1}{2}(a_1 a_2 + a_2 a_1)(J_{21} + J_{12})}_{\hat{R}}.$$

Local error:

$$\begin{aligned}\mathbb{E}((R^{\text{neu}})^T R^{\text{neu}}) &= \mathbb{E}((R^{\text{mag}} + \hat{R})^T (R^{\text{mag}} + \hat{R})) \\ &= \mathbb{E}((R^{\text{mag}})^T R^{\text{mag}}) + \mathbb{E}(\hat{R}^T \hat{R}) \\ &\quad + \underbrace{\mathbb{E}(\hat{R}^T R^{\text{mag}})}_{=0} + \underbrace{\mathbb{E}((R^{\text{mag}})^T \hat{R})}_{=0}.\end{aligned}$$

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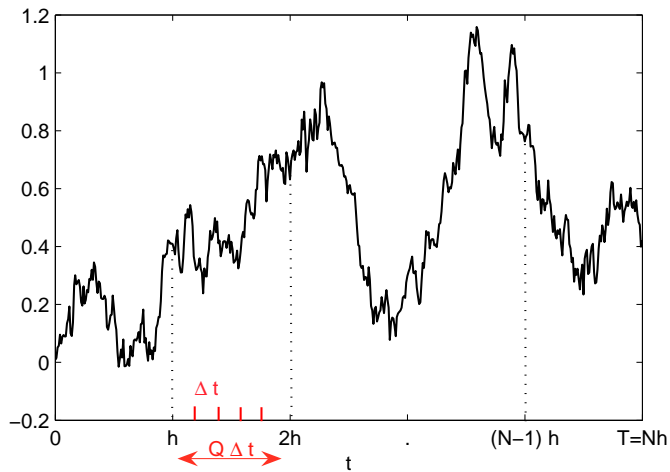
For order 1 schemes:

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**Uniformly accurate Magnus integrator:**

$$\begin{aligned}\sigma_1 &= a_1 J_1 + a_2 J_2 + a_0 J_0 + \frac{1}{2}[a_1, a_2](J_{21} - J_{12}) \\ &\quad + \frac{h^2}{12}([a_1, [a_1, a_0]] + [a_2, [a_2, a_0]]) \\ &\quad + \frac{h^2}{12}(a_1[a_2, [a_2, a_1]] + a_2[a_1, [a_1, a_2]]).\end{aligned}$$

# Quadrature approximation



# Quadrature approximation

## Two inherent scales:

- ▶ Wiener-discrete-path scale  $\Delta t$ —the *smallest scale* on which the Wiener paths  $W_1(t)$  and  $W_2(t)$  are generated;
- ▶ Time-step scale  $h = Q\Delta t$ —on which the SDE is stepped forward.

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- ▶ **Basic idea:**  $J \rightarrow \mathbb{E}(J | \mathcal{F}_Q(n))$ .

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- ▶ **Polygonal volumes.**

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- ▶ Polygonal volumes.
- ▶ **Stump & Hill 2005.**

## Quadrature approximation II

$$\hat{J}_{12}(t_n, t_{n+1}) = \frac{1}{2} \sum_{q=0}^{Q-1} \left( W_1(\tau_{q+1}) + W_1(\tau_q) - 2W_1(\tau_0) \right) \Delta W_2(t_q).$$

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$$\begin{aligned} & \mathbb{E} \left( |J_{12}(t_n, t_{n+1}) - \hat{J}_{12}(t_n, t_{n+1})|^2 \right) \\ &= \mathbb{E} \left( \left| \sum_{q=0}^{Q-1} (J_{12}(\tau_q, \tau_{q+1}) - \frac{1}{2} \Delta W_1(\tau_q) \Delta W_2(\tau_q)) \right|^2 \right) \\ &= \sum_{q=0}^{Q-1} \mathbb{E} \left( \text{Var} [J_{12}(\tau_q, \tau_{q+1}) | \mathcal{F}_Q] \right) \\ &= \mathcal{O}((\Delta t)^2 Q) \\ &= \mathcal{O}(h^2/Q). \end{aligned}$$

# Quadrature approximation IV

Quadrature		$\hat{J}_{12}$	$\hat{J}_{112}$	$\hat{J}_{120}$	$\hat{J}_{1112}$
$\mathcal{E}^{\text{loc}}(n)$		$h/Q^{1/2}$	$h^{3/2}/Q^{1/2}$	$h^2/Q^{1/2}$	$h^2/Q^{1/2}$
$\mathcal{U}$	$\mathcal{O}(h^{3/2})$	<b><math>h^{-1}</math></b>	...	...	...
	$\mathcal{O}(h^2)$	$h^{-2}$	$h^{-1}$	...	...
	$\mathcal{O}(h^{5/2})$	<b><math>h^{-3}</math></b>	$h^{-2}$	$h^{-1}$	$h^{-1}$

# Global error vs computational effort

## Theorem

$$\mathcal{U} = \underbrace{(k_{\mathcal{U}} K_{\varepsilon}^2) \varepsilon^{-2}}_{\mathcal{U}^{\text{quad}}} + \underbrace{((c_M n^2 + c_E) K_{\varepsilon}^{1/M}) \varepsilon^{-1/M}}_{\mathcal{U}^{\text{eval}}}.$$

# Global error vs computational effort

## Theorem

$$\mathcal{U} = \underbrace{(k_{\mathcal{U}} K_{\mathcal{E}}^2) \mathcal{E}^{-2}}_{\mathcal{U}^{\text{quad}}} + \underbrace{((c_M n^2 + c_E) K_{\mathcal{E}}^{1/M}) \mathcal{E}^{-1/M}}_{\mathcal{U}^{\text{eval}}}.$$

Proof.

$$\mathcal{E} = K_Q(M) \frac{h^{\frac{1}{2}}}{\sqrt{Q}} + K_T(M) h^M.$$

$$Q = \frac{1}{h^{2M-1}} \Rightarrow \mathcal{E} = K_{\mathcal{E}} h^M.$$

$$\mathcal{U} = (k_{\mathcal{U}} Q + c_M n^2 + c_E) N.$$

□

## Flops: two Wiener processes

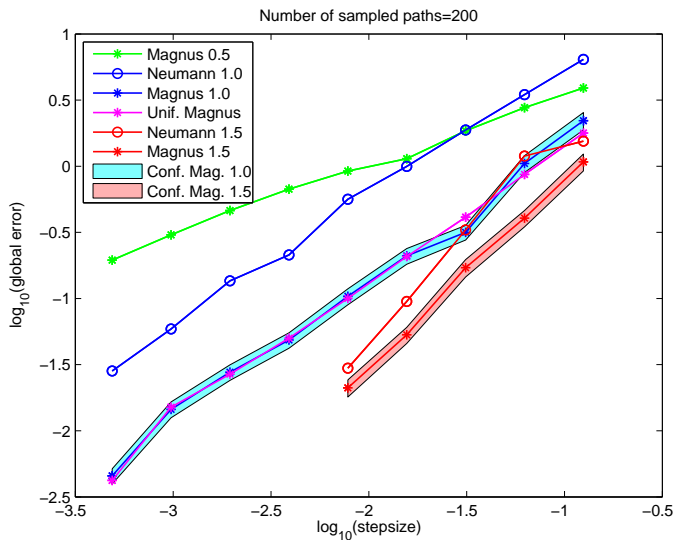
	For each path		
Order	Neumann	Magnus	Runge–Kutta
$\frac{1}{2}$	$9n^2$	$5n^2 + 6n^3$	...
1	$13n^2$	$7n^2 + 6n^3$	...
$1\frac{1}{2}$	$56n^2$	$19n^2 + 6n^3$	$20n^3 + 37n^2$

# Numerical simulations

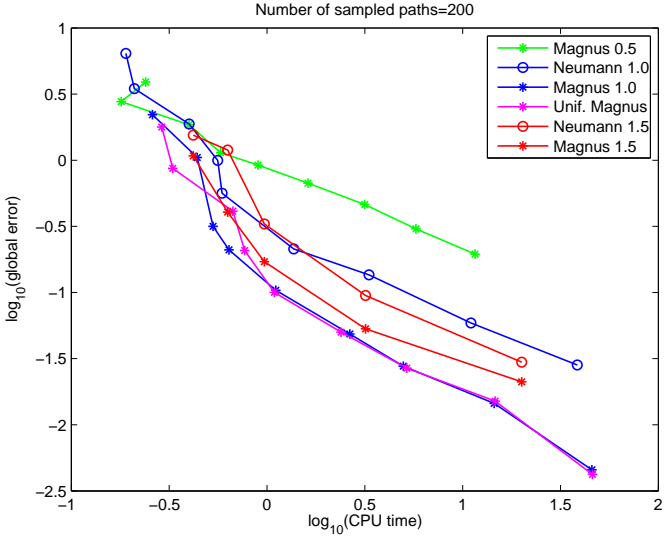
## Linear system

$$a_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{51}{200} \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{pmatrix},$$

# Linear system



# Linear system



# Stochastic Riccati system

$$S(t) = I + \sum_{i=0}^d \int_0^t f_i(\tau, S(\tau)) dW_i(\tau).$$

$$f_i(t, S) = S(t)A_i(t)S(t) + B_i(t)S(t) + S(t)C_i(t) + D_i(t).$$

- ▶ Stochastic linear-quadratic optimal control.
- ▶ eg. mean-variance hedging in finance (*Bobrovnytska & Schweizer 2004; Kohlmann & Tang 2003*).

## Riccati II

If  $\mathbb{A}_i(t) \equiv \begin{pmatrix} B_i(t) & D_i(t) \\ -A_i(t) & -C_i(t) \end{pmatrix}$  and  $\mathbb{U} = \begin{pmatrix} U \\ V \end{pmatrix}$  satisfies

$$\mathbb{U}(t) = \mathbb{I} + \sum_{i=0}^d \int_0^t \mathbb{A}_i(\tau) \mathbb{U}(\tau) dW_i(\tau),$$

then  $S = UV^{-1}$  solves the Riccati system.

$$A_0 = \begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}, \quad C_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad D_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix},$$

$D_1 = a_1$  and  $D_2 = a_2$ .

## Riccati III

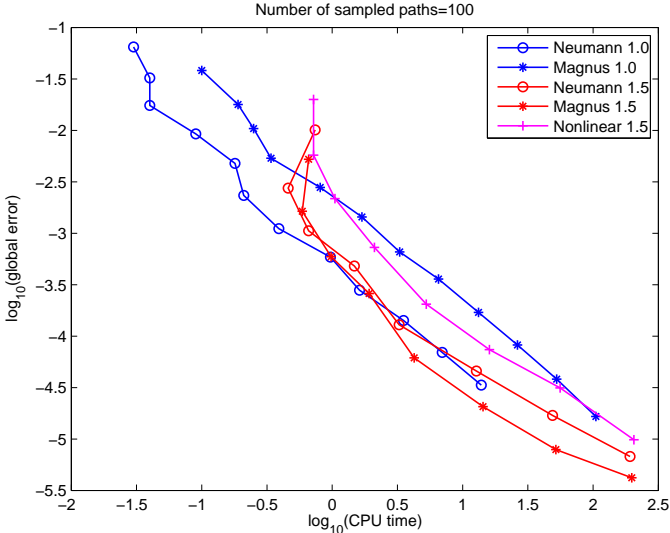
*Kloeden & Platen:*

$$\begin{aligned} S_{n+1} &= S_n + f(S_n)h + D_1 J_1 + D_2 J_2 \\ &+ \frac{h}{4} (f(Y_1^+) + f(Y_1^-) + f(Y_2^+) + f(Y_2^-) - 4f(S_n)) \\ &+ \frac{1}{2\sqrt{h}} ((f(Y_1^+) - f(Y_1^-))J_{10} + (f(Y_2^+) - f(Y_2^-))J_{20}) , \end{aligned}$$

$$Y_j^\pm = S_n + \frac{h}{2} f(S_n) \pm D_j \sqrt{h}$$

$$f(S) = SA_0S + B_0S + SC_0 + D_0 .$$

# Riccati IV



# Conclusions

## Take home message:

*For a given computational effort, a modified order 1 Magnus integrator is the most accurate of all numerical methods.*

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*For a given computational effort, a modified order 1 Magnus integrator is the most accurate of all numerical methods.*

## Future directions:

- ▶ Variable step scheme (*Gaines & Lyons 1997*).
- ▶ Zakai equation: Markov chain filters.
- ▶ Lie-group preserving properties (*Castell & Gaines 1995*; *Burrage et al. 2004*; *Misawa 2001*; *Milstein et al. 2002*).
- ▶ Backwards SDEs
- ▶ Nonlinear SDEs