
High order stochastic integrators for linear systems

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November 11th, 2005

Outline

- Introduction: linear SDEs
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Introduction: linear Stratonovich system

$$S(t) = I + \int_0^t A_0(\tau) S(\tau) dW_0(\tau) + \int_0^t A_1(\tau) S(\tau) dW_1(\tau) + \int_0^t A_2(\tau) S(\tau) dW_2(\tau)$$

$$S = I + K_0 \circ S + K_1 \circ S + K_2 \circ S$$

$$S = I + K \circ S$$

$$K \equiv K_0 + K_1 + K_2 \quad \text{and} \quad W_0(t) \equiv t$$

Neumann series

$$S = I + K \circ S$$

$$(I - K) \circ S = I$$

$$S = (I - K)^{-1} \circ I$$

$$S = (I + K + K^2 + K^3 + K^4 + \dots) \circ I$$

$$S(t) = I + K \circ I + K^2 \circ I + K^3 \circ I + K^4 \circ I + \dots$$

(*Peano–Baker series, Feynman–Dyson path ordered exponential or Chen–Fleiss series*)

Magnus series

$$S(t) = \exp(\sigma(t)) ,$$

$$\sigma(t) = \ln(I + K \circ I + K^2 \circ I + \dots)$$

$$= K \circ I + K^2 \circ I - \frac{1}{2}(K \circ I)^2 + \dots ,$$

(Magnus 1954, Kunita 1980, Ben Arous 1989, Burrage 1999)

Integral operators

$$\begin{aligned} \mathbf{K} \circ I &= \mathbf{K}_0 \circ I + \mathbf{K}_1 \circ I + \mathbf{K}_2 \circ I \\ &= \int_0^t A_0(\tau) dW_0(\tau) + \int_0^t A_1(\tau) dW_1(\tau) + \int_0^t A_2(\tau) dW_2(\tau) \end{aligned}$$

$$\begin{aligned} \mathbf{K}^2 \circ I &= (\mathbf{K}_0 + \mathbf{K}_1 + \mathbf{K}_2)^2 \circ I \\ &= (\mathbf{K}_0^2 + \mathbf{K}_0\mathbf{K}_1 + \mathbf{K}_1\mathbf{K}_0 + \mathbf{K}_1^2 + \mathbf{K}_1\mathbf{K}_2 + \mathbf{K}_2\mathbf{K}_1 \\ &\quad + \mathbf{K}_2^2 + \mathbf{K}_2\mathbf{K}_0 + \mathbf{K}_0\mathbf{K}_2) \circ I \end{aligned}$$

$$\mathbf{K}_i\mathbf{K}_j \circ I \equiv \int_0^t A_i(\tau_1) \int_0^{\tau_1} A_j(\tau_2) dW_j(\tau_2) dW_i(\tau_1)$$

Constant coefficient, non-commutative case

$$\mathbf{K}_i \mathbf{K}_j \circ I \equiv a_i a_j \underbrace{\int_0^t \int_0^{\tau_1} dW_j(\tau_2) dW_i(\tau_1)}_{J_{ji}}$$

$$S^{\text{neu}}(t) = \sum_{n=0}^{\infty} \sum_{p \in \mathbb{P}_n} a_{p_n} \cdots a_{p_1} J_{p_1 \cdots p_n}(t),$$

Special non-commutative cases:

$$S(t) = I + a_1 \cdot \int_0^t S(\tau) dW_1(\tau) + \int_0^t S(\tau) dW_2(\tau) \cdot a_2,$$

$$S(t) = \exp(a_1 W_1(t)) \cdot \exp(a_2 W_2(t)).$$

Neumann expansion (order 2)

$$S^{\text{neu}}(t) \approx I + S_{1/2} + S_1 + S_{3/2} + S_2,$$

$$S_{1/2} = a_1 J_1 + a_2 J_2 + a_0 J_0 + a_1^2 J_{11} + a_2^2 J_{22},$$

$$S_1 = a_2 a_1 J_{12} + a_1 a_2 J_{21},$$

$$\begin{aligned} S_{3/2} = & a_0^2 J_{00} + a_0 a_1 J_{10} + a_1 a_0 J_{01} + a_0 a_2 J_{20} + a_2 a_0 J_{02} \\ & + a_1^3 J_{111} + a_2 a_1^2 J_{112} + a_1 a_2 a_1 J_{121} + a_2^2 a_1 J_{122} \\ & + a_1^2 a_2 J_{211} + a_2 a_1 a_2 J_{212} + a_1 a_2^2 J_{221} + a_2^3 J_{222} \\ & + a_1^2 a_0 J_{011} + a_1 a_0 a_1 J_{101} + a_0 a_1^2 J_{110} + a_2^2 a_0 J_{022} \\ & + a_2 a_0 a_2 J_{202} + a_0 a_2^2 J_{220} + a_1^4 J_{1111} + a_2^2 a_1^2 J_{1122} \\ & + a_2 a_1 a_2 a_1 J_{1212} + a_1 a_2 a_1 a_2 J_{2121} + a_1^2 a_2^2 J_{2211} + a_2^4 J_{2222}, \end{aligned}$$

$$\begin{aligned} S_2 = & a_0 a_1 a_2 J_{210} + a_0 a_2 a_1 J_{120} + a_1 a_0 a_2 J_{201} + a_1 a_2 a_0 J_{021} \\ & + a_2 a_0 a_1 J_{102} + a_2 a_1 a_0 J_{012} + a_2 a_1^3 J_{1112} + a_1^2 a_2 a_1 J_{1121} \\ & + a_1^2 a_2 a_1 J_{1211} + a_1 a_2^2 a_1 J_{1221} + a_2^3 a_1 J_{1222} + a_2 a_1^3 J_{2111} \\ & + a_2 a_1^2 a_2 J_{2112} + a_2^2 a_1 a_2 J_{2122} + a_2 a_1 a_2^2 J_{2212} + a_1 a_2^3 J_{2221}. \end{aligned}$$

Relations between Stratonovich integrals

$$J_{121} = J_1 J_{12} - 2J_{112},$$

$$J_{122} = J_2 J_{12} - \frac{1}{2} J_1 J_2^2 + J_{221},$$

$$J_{210} = J_0 J_{21} - J_2 J_{01} + J_{012},$$

$$J_{102} = J_1 J_{02} - J_{021} - J_{012},$$

$$J_{1121} = J_{112} J_1 - 3J_{1112},$$

$$J_{1211} = \frac{1}{2} J_1^2 J_{12} - 2J_{112} J_1 + 3J_{1112},$$

$$J_{1212} = \frac{1}{2} J_{12}^2 - 2J_{1122},$$

$$J_{1221} = (J_2 J_{12} - \frac{1}{2} J_1 J_2^2 + J_{221}) J_1 - \frac{1}{2} J_{12}^2,$$

$$J_{1222} = J_{122} J_2 - \frac{1}{2} J_{12} J_2^2 + \frac{1}{6} J_1 J_2^3 - J_{2221}.$$

(Kloeden & Platen 1999; Gaines 1995; Kowski 2001)

Magnus expansion (order 2)

$$S^{\text{mag}}(t) = \exp(\sigma(t)), \quad \sigma(t) = s_{1/2} + s_1 + s_{3/2} + s_2 + \dots$$

$$s_{1/2} = a_1 J_1 + a_2 J_2 + a_0 J_0,$$

$$s_1 = \frac{1}{2}[a_1, a_2](J_{21} - J_{12}),$$

$$\begin{aligned} s_{3/2} = & \frac{1}{2}[a_0, a_1](J_{10} - J_{01}) + \frac{1}{2}[a_0, a_2](J_{20} - J_{02}) \\ & + [a_1, [a_1, a_2]] \left(J_{112} - \frac{1}{2}J_1 J_{12} + \frac{1}{12}J_1^2 J_2 \right) \\ & + [a_2, [a_2, a_1]] \left(J_{221} - \frac{1}{2}J_2 J_{21} + \frac{1}{12}J_2^2 J_1 \right) \\ & + [a_1, [a_1, a_0]] \left(J_{110} - \frac{1}{2}J_1 J_{10} + \frac{1}{12}J_1^2 J_0 \right) \\ & + [a_2, [a_2, a_0]] \left(J_{220} - \frac{1}{2}J_2 J_{20} + \frac{1}{12}J_2^2 J_0 \right), \end{aligned}$$

$$\begin{aligned} s_2 = & + [a_2, [a_1, a_0]] \left(J_{120} + \frac{1}{2}J_1 J_{02} + \frac{1}{2}J_0 J_{21} - \frac{2}{3}J_0 J_1 J_2 \right) \\ & + [a_1, [a_2, a_0]] \left(J_{210} + \frac{1}{2}J_2 J_{01} + \frac{1}{2}J_0 J_{12} - \frac{2}{3}J_0 J_1 J_2 \right) \\ & - [a_1, [a_1, [a_1, a_2]]] \left(J_{1112} - \frac{1}{2}J_1 J_{112} + \frac{1}{12}J_1^2 J_{12} \right) \\ & - [a_2, [a_2, [a_2, a_1]]] \left(J_{2221} - \frac{1}{2}J_2 J_{221} + \frac{1}{12}J_2^2 J_{21} \right) \\ & + [a_1, [a_2, [a_1, a_2]]] \left(\frac{1}{24}J_1^2 J_2^2 - \frac{1}{2}J_2 J_{112} + \frac{1}{6}J_1 J_2 J_{21} - \frac{1}{2}J_1 J_{221} + J_{1122} \right). \end{aligned}$$

Numerical SDE schemes

General integrators

- Euler-Maruyama and Milstein methods
- Runge–Kutta type methods (Kloeden & Platen 1999)
- Magnus (Burrage 1999; Misawa 2001; P-C. Moan 2004)

Linear systems: Neumann \equiv stochastic Taylor \equiv Runge–Kutta

Magnus integrators

- Not overly complex (Burrage 1999)
- Low dimensional computationally favourable basis

$\{J_0, J_1, J_2, J_{12}, J_{01}, J_{02}, J_{112}, J_{221}, J_{110}, J_{220}, J_{120}, J_{210}, J_{1112}, J_{2221}, J_{1122}\}$.

- More accurate for the same computational expense for smaller systems

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Implementation: Global errors I

Global interval:

$$[0, T] = \cup_{n=0}^{N-1} [t_n, t_{n+1}], \quad t_n = nh$$

Global error:

$$\begin{aligned} \mathcal{E} &= \mathbb{E} \left\| \prod_{n=N-1}^0 S(t_n, t_{n+1}) - \prod_{n=N-1}^0 S^{\text{approx}}(t_n, t_{n+1}) \right\| \\ &= \mathcal{E}^{\text{trunc}} + \mathcal{E}^{\text{quad}}, \end{aligned}$$

Local error:

$$\begin{aligned} e^{\sigma(t_n, t_{n+1})} - e^{\sigma_M(t_n, t_{n+1})} &= e^{\sigma_M(t_n, t_{n+1}) + R_M(t_n, t_{n+1})} - e^{\sigma_M(t_n, t_{n+1})} \\ &= R_M(t_n, t_{n+1}) + \mathcal{O}(\sigma_M R_M). \end{aligned}$$

Global errors II

$$\begin{aligned}\mathcal{E} &= \mathbb{E} \left\| \prod_{n=N-1}^0 e^{\sigma(t_n, t_{n+1})} - \prod_{n=N-1}^0 e^{\sigma_M(t_n, t_{n+1})} \right\| \\ &= \mathbb{E} \left\| \sum_{n=0}^{N-1} R_M(t_n, t_{n+1}) \right\| \\ &\leq \sum_{\alpha} \|a_{\alpha}\| \mathbb{E} \left| \sum_{n=0}^{N-1} J_{\alpha}(t_n, t_{n+1}) \right|.\end{aligned}$$

Global errors III

$$\begin{aligned}\mathbb{E}(J_{120}) &= \mathbb{E}(J_1 J_{02}) = \mathbb{E}(J_0 J_{21}) = \mathbb{E}(J_0 J_1 J_2) = 0, \\ \mathbb{E}(J_{1112}) &= \mathbb{E}(J_1 J_{112}) = \mathbb{E}(J_1^2 J_{12}) = 0, \\ \mathbb{E}\left(\frac{1}{24} J_1^2 J_2^2 + \frac{1}{6} J_1 J_2 J_{21} + J_{1122}\right) &= \mathbb{E}(J_2 J_{112}) = \mathbb{E}(J_1 J_{221}) = 0, \\ \mathbb{E}\left(J_{110} - \frac{1}{2} J_1 J_{10} + \frac{1}{12} J_1^2 J_0\right) &= \frac{1}{12} h^2.\end{aligned}$$

$$\begin{aligned}\mathbb{E} \left| \sum_{n=0}^{N-1} J_\alpha(t_n, t_{n+1}) \right| &\leq \left(\mathbb{E} \sum_{n=0}^{N-1} (J_\alpha(t_n, t_{n+1}))^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{n=0}^{N-1} \text{Var}(J_\alpha(t_n, t_{n+1})) \right)^{\frac{1}{2}} \\ &= \sqrt{N} h^2 \\ &= \sqrt{T} h^{3/2}.\end{aligned}$$

Magnus more accurate than Neumann?

For 1/2 order schemes:

$$R^{\text{mag}}(t_n, t_{n+1}) = \frac{1}{2}[a_1, a_2](J_{21} - J_{12});$$

$$R^{\text{neu}}(t_n, t_{n+1}) = R^{\text{mag}}(t_n, t_{n+1}) + \underbrace{\frac{1}{2}(a_1 a_2 + a_2 a_1)(J_{21} + J_{12})}_{\hat{R}}.$$

Using Variance as error measure, for all initial data x_0 :

$$\begin{aligned} \mathbb{E}(\langle R^{\text{neu}} x_0, R^{\text{neu}} x_0 \rangle) &= \mathbb{E}(x_0^{\text{T}} (R^{\text{neu}})^{\text{T}} R^{\text{neu}} x_0) \\ &= \mathbb{E}(x_0^{\text{T}} (R^{\text{mag}} + \hat{R})^{\text{T}} (R^{\text{mag}} + \hat{R}) x_0) \\ &= x_0^{\text{T}} \left(\mathbb{E}((R^{\text{mag}})^2) + \mathbb{E}((\hat{R})^2) \right. \\ &\quad \left. + \underbrace{\mathbb{E}(\hat{R} \cdot R^{\text{mag}})}_{=0} + \underbrace{\mathbb{E}(R^{\text{mag}} \cdot \hat{R})}_{=0} \right) x_0. \end{aligned}$$

Quadrature approximation

Two inherent scales:

- Wiener-discrete-path scale Δt —the *smallest scale* on which the Wiener paths $W_1(t)$ and $W_2(t)$ are generated;
- Time-step scale $h = Q\Delta t$ —on which the SDE is stepped forward.

- $J_{12}(t_n, t_{n+1}) \equiv \int_{t_n}^{t_{n+1}} (W_1(\tau) - W_1(t_n)) dW_2(\tau)$

- Filtration: $\mathcal{F}_Q(n) = \{W_i(t_n + q \Delta t) : i = 1, 2; q = 0, \dots, Q\}$

- Basic idea: $J \rightarrow \mathbb{E}(J | \mathcal{F}_Q(n))$

- Polygonal volumes

Quadrature approximation II

$$\hat{J}_{12}(t_n, t_{n+1}) = \frac{1}{2} \sum_{q=0}^{Q-1} \left((W_1(\tau_{q+1}) - W_1(\tau_0)) + (W_1(\tau_q) - W_1(\tau_0)) \right) \Delta W_2(t_q);$$

$$\begin{aligned} \hat{J}_{112}(t_n, t_{n+1}) &= \frac{1}{6} \sum_{q=0}^{Q-1} \left(\left((W_1(\tau_{q+1}) - W_1(\tau_0))^2 \right. \right. \\ &\quad \left. \left. + (W_1(\tau_{q+1}) - W_1(\tau_0))(W_1(\tau_q) - W_1(\tau_0)) \right. \right. \\ &\quad \left. \left. + (W_1(\tau_q) - W_1(\tau_0))^2 \right) \Delta W_2(\tau_q) \right) \\ &\quad + \frac{1}{12} \Delta t (W_2(t_{n+1}) - W_2(t_n)); \end{aligned}$$

$$\begin{aligned} \hat{J}_{120}(t_n, t_{n+1}) &= \frac{1}{2} \Delta t \sum_{q=0}^{Q-1} (Q - q - 1) \left(\Delta W_1(\tau_q) + 2(W_1(\tau_q) - W_1(\tau_0)) \right) \Delta W_2(\tau_q) \\ &\quad + \frac{1}{6} \Delta t \sum_{q=0}^{Q-1} \left(\Delta W_1(\tau_q) + 3(W_1(\tau_q) - W_1(\tau_0)) \right) \Delta W_2(\tau_q). \end{aligned}$$

Quadrature approximation III

$$\begin{aligned}
\mathcal{E}_{12}(n) &\equiv \mathbb{E} \left(\left| J_{12}(t_n, t_{n+1}) - \hat{J}_{12}(t_n, t_{n+1}) \right| \right) \\
&= \mathbb{E} \left(\left| \sum_{q=0}^{Q-1} \left(J_{12}(\tau_q, \tau_{q+1}) - \frac{1}{2} \Delta W_1(\tau_q) \Delta W_2(\tau_q) \right) \right| \right) \\
&\leq \left(\sum_{q=0}^{Q-1} \mathbb{E} \left| J_{12}(\tau_q, \tau_{q+1}) - \frac{1}{2} \Delta W_1(\tau_q) \Delta W_2(\tau_q) \right|^2 \right)^{1/2} \\
&= \left(\sum_{q=0}^{Q-1} \mathbb{E} \left(\mathbb{E} \left(\left| J_{12}(\tau_q, \tau_{q+1}) - \frac{1}{2} \Delta W_1(\tau_q) \Delta W_2(\tau_q) \right|^2 \middle| \mathcal{F}_Q(n) \right) \right) \right)^{1/2} \\
&= \left(\sum_{q=0}^{Q-1} \mathbb{E} \left(\text{Var} \left(J_{12}(\tau_q, \tau_{q+1}) \middle| \mathcal{F}_Q(n) \right) \right) \right)^{1/2} \\
&= \left(\sum_{q=0}^{Q-1} \mathbb{E} \left(\frac{1}{6} \Delta t \left((\Delta W_1(\tau_q))^2 + (\Delta W_2(\tau_q))^2 \right) \right) \right)^{1/2} \\
&= \mathcal{O}(\Delta t \sqrt{Q}) \\
&= \mathcal{O}(h/\sqrt{Q}),
\end{aligned}$$

Quadrature approximation IV

Quadrature		\hat{J}_{12}	\hat{J}_{112}	\hat{J}_{120}	\hat{J}_{1112}	\hat{J}_{1122}
$\mathcal{E}^{\text{loc}}(n)$		$h/Q^{1/2}$	$h^{3/2}/Q^{1/2}$	$h^2/Q^{1/2}$	$h^2/Q^{1/2}$	$h^2/Q^{1/2}$
\mathcal{U}	$\mathcal{O}(h^{3/2})$	h^{-1}	\dots	\dots	\dots	\dots
	$\mathcal{O}(h^2)$	h^{-2}	h^{-1}	\dots	\dots	\dots
	$\mathcal{O}(h^{5/2})$	h^{-3}	h^{-2}	h^{-1}	h^{-1}	h^{-1}

Error vs effort

Locally

$$\begin{aligned} \mathcal{E}^{\text{loc}} \sim h^{3/2} \quad \text{and} \quad \mathcal{U} \sim h^{-1} &\implies \log \mathcal{E}^{\text{loc}} \sim -\frac{3}{2} \log \mathcal{U}, \\ \mathcal{E}^{\text{loc}} \sim h^2 \quad \text{and} \quad \mathcal{U} \sim h^{-2} &\implies \log \mathcal{E}^{\text{loc}} \sim -\log \mathcal{U}, \\ \mathcal{E}^{\text{loc}} \sim h^{5/2} \quad \text{and} \quad \mathcal{U} \sim h^{-3} &\implies \log \mathcal{E}^{\text{loc}} \sim -\frac{5}{6} \log \mathcal{U}. \end{aligned}$$

Globally

$$\mathcal{E}^{\text{quad}} = \mathcal{O}\left(\frac{h^\alpha}{\sqrt{Q}}\right) \sqrt{N} = \mathcal{O}\left(\frac{h^{\alpha-\frac{1}{2}}}{\sqrt{Q}} \sqrt{T}\right).$$

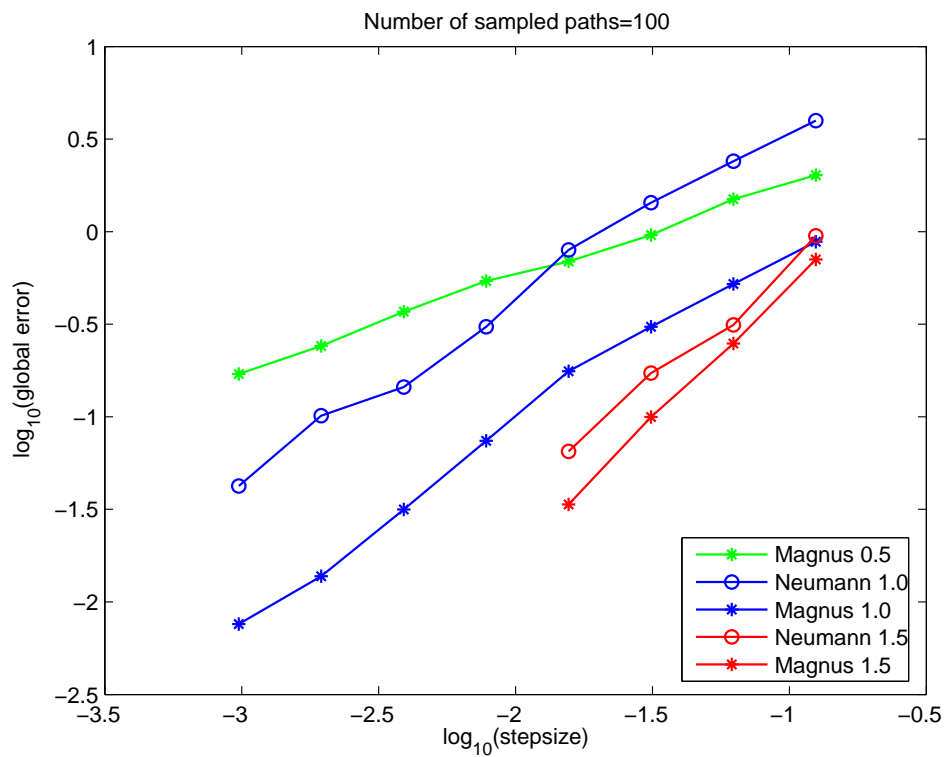
$$\mathcal{U} \sim QN.$$

Numerical simulations

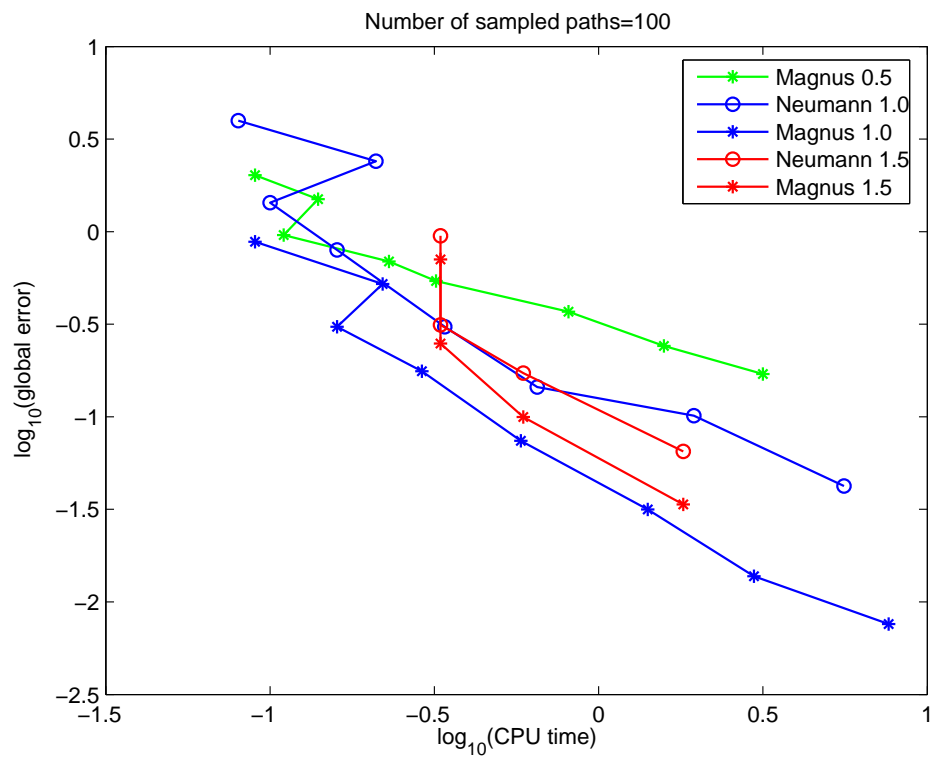
Linear system

$$a_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{51}{200} \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{pmatrix},$$

Linear system



Linear system



Flops: two Wiener processes

Order	For each path			Independent of the path	
	Neumann	Magnus	Runge–Kutta	Neumann	Magnus
$\frac{1}{2}$	$9n^2$	$5n^2 + 5n^3$	\dots	n/a	n/a
1	$13n^2$	$7n^2 + 5n^3$	\dots	$4n^3$	$2n^3$
$1\frac{1}{2}$	$63n^2$	$19n^2 + 5n^3$	$20n^3 + 37n^2$	$29n^3$	$14n^3$
2	$95n^2$	$29n^2 + 5n^3$	\dots	$45n^3$	$26n^3$

Stochastic Riccati system

$$S(t) = I + \sum_{i=0}^d \int_0^t (S(\tau)A_i(\tau)S(\tau) + B_i(\tau)S(\tau) + S(\tau)C_i(\tau) + D_i(\tau)) dW_i(\tau).$$

- Stochastic linear-quadratic optimal control.
- eg. mean-variance hedging in finance.
- Bobrovnytska & Schweizer 2004; Kohlmann & Tang 2003.

Riccati II

If $\mathbb{A}_i(t) \equiv \begin{pmatrix} B_i(t) & D_i(t) \\ -A_i(t) & -C_i(t) \end{pmatrix}$ and $\mathbb{U} = \begin{pmatrix} U \\ V \end{pmatrix}$ satisfies

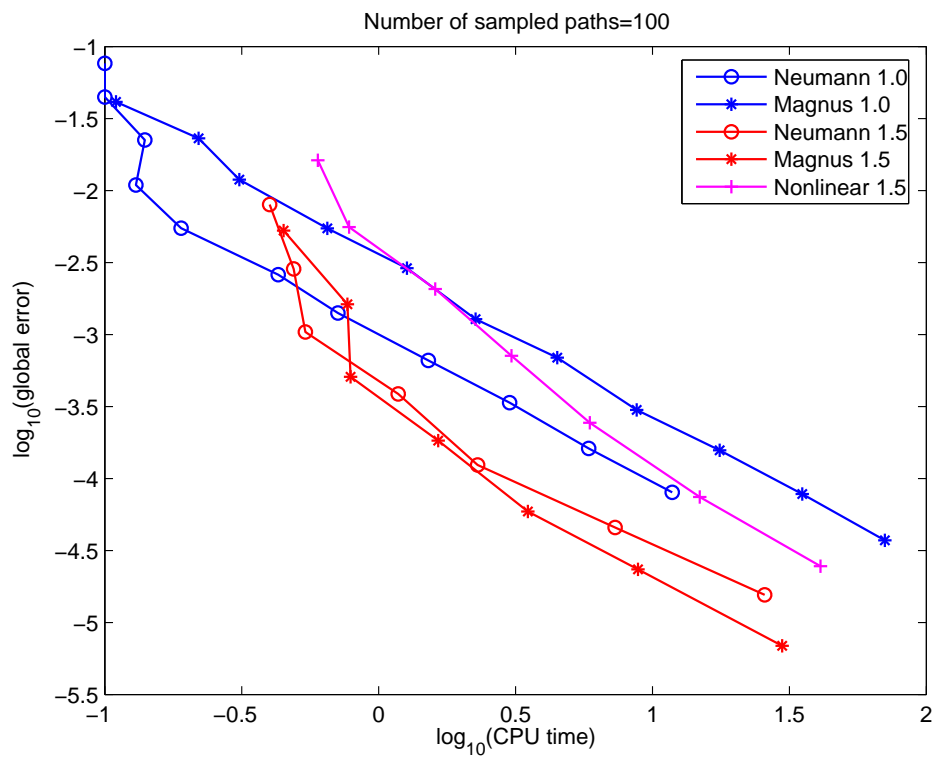
$$\mathbb{U}(t) = \mathbb{I} + \sum_{i=0}^d \int_0^t \mathbb{A}_i(\tau) \mathbb{U}(\tau) dW_i(\tau),$$

then $S = UV^{-1}$ solves the Riccati system.

$$A_0 = \begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}, \quad C_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad D_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix},$$

$$D_1 = a_1 \text{ and } D_2 = a_2.$$

Riccati III



Riccati IV

Kloeden & Platen:

$$\begin{aligned} S_{n+1} &= S_n + f(S_n)h + D_1 J_1 + D_2 J_2 \\ &\quad + \frac{h}{4} (f(Y_1^+) + f(Y_1^-) + f(Y_2^+) + f(Y_2^-) - 4f(S_n)) \\ &\quad + \frac{1}{2\sqrt{h}} ((f(Y_1^+) - f(Y_1^-)) J_{10} + (f(Y_2^+) - f(Y_2^-)) J_{20}) , \end{aligned}$$

$$Y_j^\pm = S_n + \frac{h}{2} f(S_n) \pm D_j \sqrt{h}$$

$$f(S) = SA_0S + B_0S + SC_0 + D_0 .$$

Conclusions

- Error vs time? Parallelization.
- Variable step scheme (Gaines & Lyons 1997).
- Zakai equation: Markov chain filters.
- Lie-group preserving properties (Castell & Gaines 1995; Burrage *et al.* 2004; Misawa 2001; Milstein *et al.* 2002).
- Driven by Lévy processes
- Backwards SDEs
- Nonlinear SDEs