

# Computing the Evans function using Grassmannians

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# Acknowledgments

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- ▶ **HOP programme (INI):** Arieh Iserles, Ernst Hairer

# Outline

- 1 Introduction
- 2 Evans function
- 3 Continuous orthogonalization
- 4 Grassmannian manifold
- 5 Examples

## Take home message:

*Shooting methods are about to take off!*

# Stability of travelling waves

$$U_t = BU_{\xi\xi} + cU_\xi + F(U)$$

- ▶ Travelling wave in moving frame:  $U(\xi, t) = U_c(\xi)$
- ▶ Perturbation ansatz:

$$U(\xi, t) = U_c(\xi) + \hat{U}(\xi) e^{\lambda t}$$

- ▶ Small perturbations satisfy:

$$\lambda \hat{U} = [B\partial_{\xi\xi} + cI\partial_\xi + DF(U_c(\xi))] \hat{U}$$

with  $\hat{U}(\xi) \rightarrow 0$  as  $\xi \rightarrow \pm\infty$ .

# Strategies

- ▶ **Projection:** Project the spectral equations onto a finite dimensional basis, which by construction satisfies the boundary conditions. Solve the matrix eigenvalue problem.
  - ▶ (+) Applies to structures of any dimension.
  - ▶ (-) Increasing accuracy or adaptation is costly and complicated; need to re-project onto the finer or adapted basis.
- ▶ **Shooting:** And matching.
  - ▶ (+) Much easier to fine-tune accuracy and adaptivity.
  - ▶ (-) Essentially one-dimensional method.

## Shooting reformulation

$$\lambda \hat{U} = [B \partial_{\xi\xi} + c I \partial_{\xi} + DF(U_c(\xi))] \hat{U}$$

Write  $Y = (\hat{U}, \hat{U}_{\xi})$  then

$$\begin{aligned} Y' &= A(\xi; \lambda) Y \\ Y &\rightarrow 0, \quad \xi \rightarrow \pm\infty \end{aligned}$$

where

$$A(\xi; \lambda) = \begin{pmatrix} 0 & I \\ B^{-1}(\lambda - DF(U_c(\xi))) & -c B^{-1} \end{pmatrix}$$

# The Evans function

## Limiting systems

$$A_{\pm}(\lambda) = \lim_{\xi \rightarrow \pm\infty} A(\xi, \lambda)$$

- ▶ Assume  $A_-$  has a  $k$ -dimensional **unstable manifold**
- ▶  $A_+$  has an  $(2n - k)$ -dimensional **stable manifold**
- ▶ Look for intersection under the “evolution” of the BVP

Wronskian:

$$\begin{aligned} D(\lambda) &= e^{-\int_0^{\xi} \text{Tr} A(x, \lambda) dx} \cdot \det \left( Y_1^- \cdots Y_k^- \quad Y_{k+1}^+ \cdots Y_{2n-k}^+ \right) \\ &= e^{-\int_0^{\xi} \text{Tr} A(x, \lambda) dx} \cdot \underbrace{\left( Y_1^- \wedge \cdots \wedge Y_k^- \right)}_{U_-(\xi; \lambda)} \wedge \underbrace{\left( Y_{k+1}^+ \wedge \cdots \wedge Y_{2n-k}^+ \right)}_{U_+(\xi; \lambda)} \end{aligned}$$

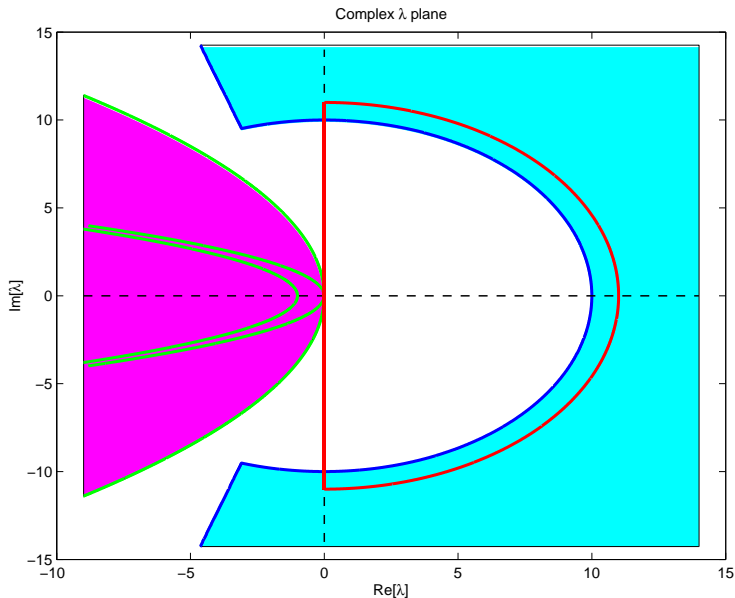
(Prefactor ensures  $\xi$ -independence.)

# Properties of the Evans function

(Evans, 1975; Alexander, Gardner & Jones, 1990)

- ▶ Zeros correspond to eigenvalues
- ▶ Analytic to the right of the essential spectrum
- ▶ Can use argument principle to determine number of zeros in the right half plane

# Spectrum of the linearized travelling wave operator



## Numerical issues

$$Y' = A(\xi; \lambda) Y$$

- ▶ Rescale by expected exponential growth, i.e.  $Y = e^{\mu x} Z$ :

$$Z' = (A(\xi; \lambda) - \mu I) Z$$

- ▶ Computing second unstable basis function numerically unstable
- ▶ Resolution: integrate exterior products  $U \in \bigwedge^k \mathbb{C}^n$

$$D(\lambda) = e^{-\int_0^\xi \text{Tr} A(x, \lambda) dx} \cdot \underbrace{\left( Y_1^- \wedge \cdots \wedge Y_k^- \right)}_{U_-(\xi; \lambda)} \wedge \underbrace{\left( Y_{k+1}^+ \wedge \cdots \wedge Y_{2n-k}^+ \right)}_{U_+(\xi; \lambda)}$$

# Multi-dimensional stability

- ▶  $\dim\left(\bigwedge^k \mathbb{C}^n\right) = \frac{n!}{k!(n-k)!}$
- ▶ AIM meeting May 2005: Stability criteria for multi-dimensional waves and patterns (C.K.R.T. Jones, Y. Latushkin, B. Sandstede)
- ▶ Humpherys & Zumbrun (2006): Continuous orthogonalization
- ▶ Drury–Oja flow (Drury, Davey, Bridges–Reich)

# Idea came from

- ▶ Schiff and Shnider:

Möbius projection:  $GL(n) \rightarrow \text{Grassmannian}$

- ▶ Greenberg and Marletta: minimal information for Evans function

# Principle fibre bundle

$$\pi: \mathbb{V}(n, k) \rightarrow \text{Gr}(n, k)$$

- ▶ base space  $\text{Gr}(n, k)$
- ▶  $\forall y \in \text{Gr}(n, k): \pi^{-1}(y)$  homeomorphic to fibre space  $\text{GL}(k)$
- ▶ projection map  $\pi$ : natural quotient map sending each  $k$ -frame centered at the origin to the  $k$ -plane it spans

# Stiefel and Grassmannian manifolds

$$\begin{pmatrix} Y_{11} & \cdots & Y_{1k} \\ \vdots & & \vdots \\ Y_{n1} & \cdots & Y_{nk} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ y_{k+1,1} & y_{k+1,2} & \cdots & y_{k+1,k} \\ y_{k+2,1} & y_{k+2,2} & \cdots & y_{k+2,k} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,1} & y_{n,2} & \cdots & y_{n,k} \end{pmatrix}$$

# Grassmannian manifold $\text{Gr}(n, k)$

- ▶  $Y = [Y_1 Y_2 \cdots Y_k]$
- ▶ Natural decomposition  $\mathbb{V}(n, k) \cong \text{Gr}(n, k) \times \text{GL}(k)$ :

$$Y = \begin{pmatrix} U \\ L \end{pmatrix} = \begin{pmatrix} I_k \\ y \end{pmatrix} U$$

where  $y = LU^{-1} \in \mathbb{C}^{(n-k) \times k}$  and  $U \in \text{GL}(k)$ .

- ▶ Coordinatization of  $\text{Gr}(n, k)$  implicitly chosen.

# Tangent space decomposition

$\mathbb{V}(n, k) = \text{Gr}(n, k) \times \text{GL}(k) \Rightarrow$  induced decomposition

$$\mathbb{T}_Y \mathbb{V}(n, k) = \mathbb{H}_Y \oplus \mathbb{V}_Y$$

$$V = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathbb{T}_Y \mathbb{V}(n, k)$$

$$\mathbb{H}_Y = \left\{ V_h = \begin{pmatrix} O_k \\ \eta \end{pmatrix} : \eta \in \mathbb{C}^{(n-k) \times k} \right\}$$

$$\mathbb{V}_Y = \left\{ V_\perp = \begin{pmatrix} \xi \\ O \end{pmatrix} : \xi \in \mathfrak{gl}(k) \right\}$$

## Riccati flow

$$V(x, Y) = \begin{pmatrix} a(x, Y) & b(x, Y) \\ c(x, Y) & d(x, Y) \end{pmatrix} Y, \quad Y \in \mathbb{V}(n, k)$$

$$V_h = \begin{pmatrix} O_k & O_{k \times (n-k)} \\ -y & I_{n-k} \end{pmatrix} V = \begin{pmatrix} O_k & \\ c + d y - y(a + b y) & \end{pmatrix} U$$

$$V_\perp = \begin{pmatrix} I_k & O_{k \times (n-k)} \\ y & O_{n-k} \end{pmatrix} V = \begin{pmatrix} I_k \\ y \end{pmatrix} (a + b y) U$$

$\Rightarrow$  coupled flow on  $\text{Gr}(n, k) \times \text{GL}(k)$

$$y' = c + d y - y(a + b y)$$

$$U' = (a + b y) U$$

# Continuous orthogonalization

- ▶  $Y = [Y_1 Y_2 \cdots Y_k]$
- ▶ Polar decomposition:  $Y = \Omega \alpha$
- ▶ Stiefel manifold  $\mathbb{V}(n, k)$ :  $\Omega^* \Omega = I$
- ▶ Substitute ansatz into  $Y' = AY$ :

$$\Omega' = (I - \Omega \Omega^*) A \Omega$$

- ▶ With slaved linear ODE for  $\gamma \equiv \det \alpha$ .

## Push forward the linear vector field to $\text{Gr}(n, k)$

$$Y' = A(x)Y$$

$$\begin{pmatrix} U \\ L \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} U \\ L \end{pmatrix}$$

$$\begin{aligned} \Rightarrow y' &= (LU^{-1})' \\ &= L'U^{-1} - LU^{-1}U'U^{-1} \\ &= (cU + dL)U^{-1} - y(aU + bL)U^{-1} \\ &= c + dy - y(a + by) \end{aligned}$$

# Evans function

$$\begin{aligned} D(\lambda) &\equiv \det(Y^- Y^+) \\ &= \det \begin{pmatrix} I_k & y^+ \\ y^- & I_{n-k} \end{pmatrix} \cdot \det U^- \cdot \det L^+ \end{aligned}$$

# Boussinesq system I

$$u_{tt} = u_{xx} - u_{xxxx} - (u^2)_{xx},$$

$$\bar{u}(\xi) = \frac{3}{2}(1 - c^2)\operatorname{sech}^2\left(\frac{1}{2}\sqrt{1 - c^2}\xi\right)$$

waves stable when  $1/2 < |c| < 1$  and unstable when  $|c| < 1/2$ .

## Boussinesq system II

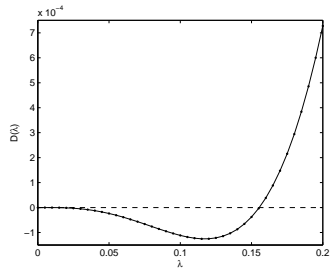
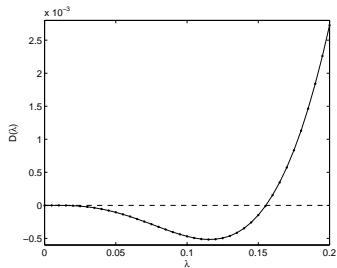


Figure: Riccati (left) and continuous orthogonalization (right).

# Autocatalytic fronts

$$u_t = \delta u_{xx} + cu_x - uv^m$$

$$v_t = v_{xx} + cv_x + uv^m$$

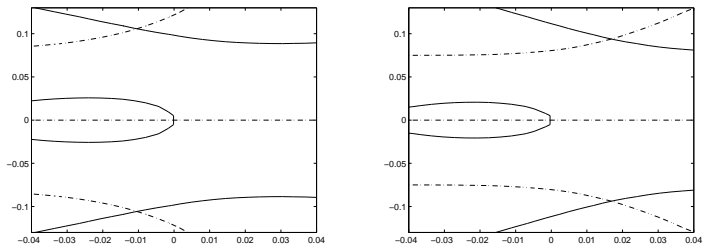


Figure:  $\delta = 0.1$ :  $m = 8$  (left) and  $m = 9$  (right).

# Ekman boundary layer I

$$\begin{aligned}u_t + uu_x + vv_y + ww_z + \frac{1}{R_o} p_x - \frac{2}{R_o} v &= \frac{1}{R_e} (u_{xx} + u_{yy}) + \frac{1}{R_o} u_{zz} \\v_t + uv_x + vv_y + wv_z + \frac{1}{R_o} p_y + \frac{2}{R_o} u &= \frac{1}{R_e} (v_{xx} + v_{yy}) + \frac{1}{R_o} v_{zz} \\w_t + uw_x + vw_y + ww_z + \frac{1}{R_o E_k} p_z &= \frac{1}{R_e} (w_{xx} + w_{yy}) + \frac{1}{R_o} w_{zz}\end{aligned}$$

Set  $R_e = R_o$ ,  $E_k = 1$ .

Base state:

$$V(z) = \cos(\epsilon)(1 - \exp(-z) \cos(z)) + \sin(\epsilon) \exp(-z) \sin(z)$$

$$U(z) = -\sin(\epsilon)(1 - \exp(-z) \cos(z)) + \cos(\epsilon) \exp(-z) \sin(z)$$

# Ekman boundary layer II

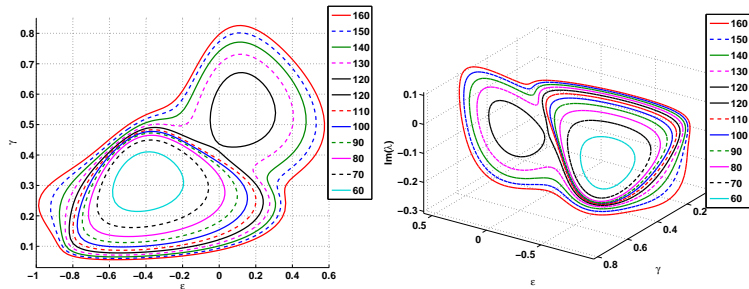


Figure: Neutral curves for rigid wall,  $Re$  fixed (values indicated).

## Future work: multi-d I

Cylindrical domain  $\mathbb{R} \times \mathbb{T}$ :

$$\partial_t U = B \Delta U + c \partial_x U + F(U)$$

$$B \Delta U + c \partial_x U + DF(U_c(x, y))U = \lambda U$$

## Multi-d II

$$U(x, y) = \sum_{k=-\infty}^{\infty} \hat{U}_k(x) e^{iky/\tilde{L}}$$

$$DF(U_c(x, y)) = \sum_{k=-\infty}^{\infty} \hat{D}_k(x) e^{iky/\tilde{L}}$$

## Multi-d II

$$\partial_x \hat{U}_k = \hat{P}_k$$

$$\partial_x \hat{P}_k = \lambda B^{-1} \hat{U}_k + (k/\tilde{L})^2 \hat{U}_k - c B^{-1} \hat{P}_k - \sum_{\nu=-K}^K B^{-1} \hat{D}_{k-\nu} \hat{U}_\nu$$

$$\partial_x \begin{pmatrix} \hat{U} \\ \hat{P} \end{pmatrix} = \begin{pmatrix} O_{2(2K+1)} & I_{2(2K+1)} \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} \hat{U} \\ \hat{P} \end{pmatrix}$$

# Multi-d III

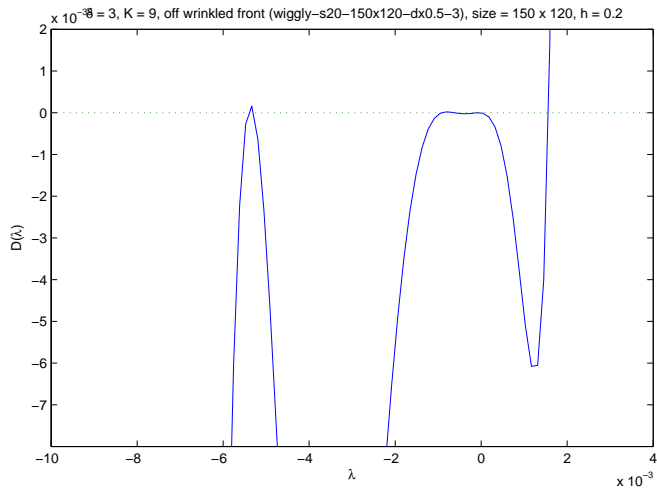


Figure: Evans function along real  $\lambda$ -axis