

Lie group stochastic integrators

Simon J.A. Malham

Work in collaboration with:

Anke Wiese & Gabriel Lord

Heriot-Watt University, Edinburgh, UK

La Trobe: 21st July 2006

Introduction

$$dy_t = V_0(y_t) dt + V_1(y_t) dW_t^1 + V_2(y_t) dW_t^2$$

Introduction

$$dy_t = V_0(y_t) dt + V_1(y_t) dW_t^1 + V_2(y_t) dW_t^2$$

- 1 Stochastic Taylor series
- 2 Exponential Lie series
- 3 Castell–Gaines method
- 4 Quadrature & efficiency
- 5 Lie group stochastic integrators
- 6 Conclusions

Introduction

Take home messages:

For numerical SDE methods that are conditioned on two driving Wiener paths, order 1 methods provide the same accuracy as higher order methods for a given computational effort.

Stochastic Lie group integration methods can preserve the evolution of an SDE solution in a smooth manifold.

Introduction

$$dy_t = V_0(y_t) dt + V_1(y_t) dW_t^1 + V_2(y_t) dW_t^2$$

$$y_t = y_0 + \int_0^t V_0(y_\tau) d\tau + \int_0^t V_1(y_\tau) dW_\tau^1 + \int_0^t V_2(y_\tau) dW_\tau^2$$

Introduction

$$dy_t = V_0(y_t) dt + V_1(y_t) dW_t^1 + V_2(y_t) dW_t^2$$

$$y_t = y_0 + \int_0^t V_0(y_\tau) d\tau + \int_0^t V_1(y_\tau) dW_\tau^1 + \int_0^t V_2(y_\tau) dW_\tau^2$$

$$W_t - W_s \sim N(0, \sqrt{t - s})$$

Abstract SDE

$$y_t = y_0 + \int_0^t V_0(y_\tau) d\tau + \int_0^t V_1(y_\tau) dW_\tau^1 + \int_0^t V_2(y_\tau) dW_\tau^2$$

Abstract SDE

$$y_t = y_0 + \int_0^t V_0(y_\tau) d\tau + \int_0^t V_1(y_\tau) dW_\tau^1 + \int_0^t V_2(y_\tau) dW_\tau^2$$

$$y = y_0 + K_0 \circ V_0 \circ y + K_1 \circ V_1 \circ y + K_2 \circ V_2 \circ y$$

Abstract SDE

$$y_t = y_0 + \int_0^t V_0(y_\tau) d\tau + \int_0^t V_1(y_\tau) dW_\tau^1 + \int_0^t V_2(y_\tau) dW_\tau^2$$

$$y = y_0 + K_0 \circ V_0 \circ y + K_1 \circ V_1 \circ y + K_2 \circ V_2 \circ y$$

$$y = y_0 + K \circ V \circ y$$

Stochastic Taylor series

$$y = y_0 + K \circ V \circ y$$

Stochastic Taylor series

$$y = y_0 + K \circ V \circ y$$

$$f \circ y = f \circ y_0 + K \circ V \circ f \circ y$$

Stochastic Taylor series

$$y = y_0 + K \circ V \circ y$$

$$f \circ y = f \circ y_0 + K \circ V \circ f \circ y$$

$$y = y_0 + K \circ V \circ y_0 + K \circ K \circ V \circ V \circ y$$

Stochastic Taylor series

$$y = y_0 + K \circ V \circ y$$

$$f \circ y = f \circ y_0 + K \circ V \circ f \circ y$$

$$y = y_0 + K \circ V \circ y_0 + K \circ K \circ V \circ V \circ y$$

$$y = y_0 + K \circ V \circ y_0 + K \circ K \circ V \circ V \circ y_0 \\ + K \circ K \circ K \circ V \circ V \circ V \circ y$$

Stochastic Taylor series

$$y_t = y_0 + K \circ V \circ y_0 + K^2 \circ V^2 \circ y_0 + K^3 \circ V^3 \circ y_0 + \dots$$

Stochastic Taylor series

$$y_t = y_0 + K \circ V \circ y_0 + K^2 \circ V^2 \circ y_0 + K^3 \circ V^3 \circ y_0 + \dots$$

- ▶ *Peano–Baker series, Feynman–Dyson path ordered exponential, Chen–Fleiss series or stochastic Taylor series*

Stochastic Taylor series

$$y_t = y_0 + K \circ V \circ y_0 + K^2 \circ V^2 \circ y_0 + K^3 \circ V^3 \circ y_0 + \dots$$

- ▶ *Peano–Baker series, Feynman–Dyson path ordered exponential, Chen–Fleiss series or stochastic Taylor series*

Flow-map:

$$y_t = \varphi_t \circ y_0$$

Stochastic Taylor series

$$y_t = y_0 + K \circ V \circ y_0 + K^2 \circ V^2 \circ y_0 + K^3 \circ V^3 \circ y_0 + \dots$$

- ▶ *Peano–Baker series, Feynman–Dyson path ordered exponential, Chen–Fleiss series or stochastic Taylor series*

Flow-map:

$$y_t = \varphi_t \circ y_0$$

$$\varphi_t = \text{Id} + K \circ V + K^2 \circ V^2 + K^3 \circ V^3 + \dots$$

Stochastic Taylor expansion

$$y_t \approx (\text{Id} + \varphi_{1/2} + \varphi_1 + \cdots) \circ y_0,$$

Stochastic Taylor expansion

$$y_t \approx (\text{Id} + \varphi_{1/2} + \varphi_1 + \cdots) \circ y_0,$$

$$\varphi_{1/2} = J_1 V_1 + J_2 V_2 + J_0 V_0 + J_{11} V_1^2 + J_{22} V_2^2,$$

Stochastic Taylor expansion

$$y_t \approx (\text{Id} + \varphi_{1/2} + \varphi_1 + \cdots) \circ y_0,$$

$$\varphi_{1/2} = J_1 V_1 + J_2 V_2 + J_0 V_0 + J_{11} V_1^2 + J_{22} V_2^2,$$

$$\varphi_1 = J_{12} V_2 V_1 + J_{21} V_1 V_2.$$

Stochastic Taylor expansion

$$y_t \approx (\text{Id} + \varphi_{1/2} + \varphi_1 + \cdots) \circ y_0,$$

$$\varphi_{1/2} = J_1 V_1 + J_2 V_2 + J_0 V_0 + J_{11} V_1^2 + J_{22} V_2^2,$$

$$\varphi_1 = J_{12} V_2 V_1 + J_{21} V_1 V_2.$$

$$K_i K_j \circ V_j V_i \circ y_0 \equiv \underbrace{\int_0^t \int_0^{\tau_1} dW_{\tau_2}^i dW_{\tau_1}^j}_{J_{ij}} V_j V_i \circ y_0$$

Exponential Lie series

$$y_t = \exp(\psi_t) \circ y_0,$$

$$\psi_t \equiv \ln \varphi_t = \psi_{1/2} + \psi_1 + \dots$$

Exponential Lie series

$$y_t = \exp(\psi_t) \circ y_0,$$

$$\psi_t \equiv \ln \varphi_t = \psi_{1/2} + \psi_1 + \dots$$

$$\psi_{1/2} = J_1 V_1 + J_2 V_2 + J_0 V_0,$$

Exponential Lie series

$$y_t = \exp(\psi_t) \circ y_0,$$

$$\psi_t \equiv \ln \varphi_t = \psi_{1/2} + \psi_1 + \dots$$

$$\psi_{1/2} = J_1 V_1 + J_2 V_2 + J_0 V_0,$$

$$\psi_1 = \frac{1}{2}(J_{21} - J_{12})[V_1, V_2],$$

Exponential Lie series

$$y_t = \exp(\psi_t) \circ y_0,$$

$$\psi_t \equiv \ln \varphi_t = \psi_{1/2} + \psi_1 + \dots$$

$$\psi_{1/2} = J_1 V_1 + J_2 V_2 + J_0 V_0,$$

$$\psi_1 = \frac{1}{2}(J_{21} - J_{12})[V_1, V_2],$$

- ▶ *Magnus* 1954, *Kunita* 1980, *Ben Arous* 1989, *Castell* 1993, *Burrage* 1999.

Numerical SDE schemes

General integrators

- ▶ Euler-Maruyama and Milstein methods
- ▶ Runge–Kutta type methods (*Kloeden & Platen* 1999)
- ▶ Exponential Lie series methods (*Burrage* 1999; *Misawa* 2001; *P-C. Moan* 2004)
- ▶ Castell–Gaines method

Castell–Gaines method

Truncated exponential Lie series across $[t_n, t_{n+1}]$:

$$\hat{\psi}_{t_n, t_{n+1}} = J_1 V_1 + J_2 V_2 + J_0 V_0 + \frac{1}{2}(J_{21} - J_{12})[V_1, V_2].$$

Castell–Gaines method

Truncated exponential Lie series across $[t_n, t_{n+1}]$:

$$\hat{\psi}_{t_n, t_{n+1}} = J_1 V_1 + J_2 V_2 + J_0 V_0 + \frac{1}{2}(J_{21} - J_{12})[V_1, V_2].$$

Approximate solution:

$$y_{t_{n+1}} \approx \exp(\hat{\psi}_{t_n, t_{n+1}}) \circ y_{t_n}.$$

Castell–Gaines method

Truncated exponential Lie series across $[t_n, t_{n+1}]$:

$$\hat{\psi}_{t_n, t_{n+1}} = J_1 V_1 + J_2 V_2 + J_0 V_0 + \frac{1}{2}(J_{21} - J_{12})[V_1, V_2].$$

Approximate solution:

$$y_{t_{n+1}} \approx \exp(\hat{\psi}_{t_n, t_{n+1}}) \circ y_{t_n}.$$

Castell–Gaines:

$$u'(\tau) = \hat{\psi}_{t_n, t_{n+1}} \circ u(\tau)$$

across $\tau \in [0, 1]$ **with** $u(0) = y_{t_n}$ **gives** $u(1) \approx y_{t_{n+1}}$.

Castell–Gaines method

Truncated exponential Lie series across $[t_n, t_{n+1}]$:

$$\hat{\psi}_{t_n, t_{n+1}} = J_1 V_1 + J_2 V_2 + J_0 V_0 + \frac{1}{2}(J_{21} - J_{12})[V_1, V_2].$$

Approximate solution:

$$y_{t_{n+1}} \approx \exp(\hat{\psi}_{t_n, t_{n+1}}) \circ y_{t_n}.$$

Castell–Gaines:

$$u'(\tau) = \hat{\psi}_{t_n, t_{n+1}} \circ u(\tau)$$

across $\tau \in [0, 1]$ with $u(0) = y_{t_n}$ gives $u(1) \approx y_{t_{n+1}}$.

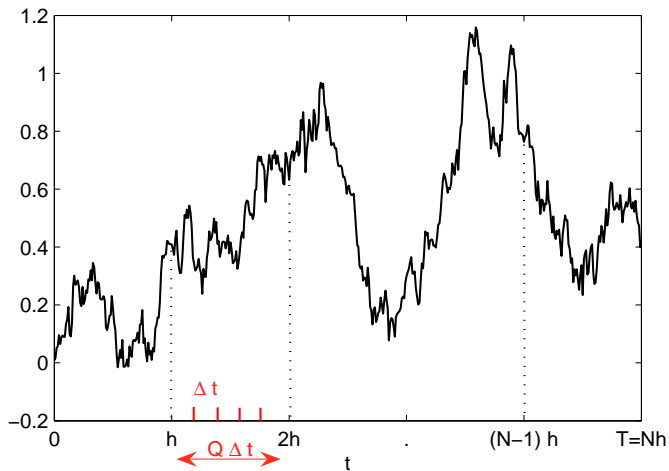
Need approximations for iterated integrals.

Accuracy

Uniformly accurate exponential Lie integrator:

$$\begin{aligned}\hat{\psi}^* &= J_1 V_1 + J_2 V_2 + J_0 V_0 + \frac{1}{2}(J_{21} - J_{12})[V_1, V_2] \\ &\quad + \frac{h^2}{12} ([V_1, [V_1, V_0]] + [V_2, [V_2, V_0]]) \\ &\quad + \frac{h^2}{12} (V_1[V_2, [V_2, V_1]] + V_2[V_1, [V_1, V_2]]) .\end{aligned}$$

Quadrature approximation



Quadrature approximation

Two inherent scales:

- ▶ Quadrature scale Δt —the *smallest scale* on which the Wiener paths W^1 and W^2 are generated;
- ▶ Time-step scale $h = Q\Delta t$ —on which the SDE is stepped forward.

Quadrature approximation

Two inherent scales:

- ▶ Quadrature scale Δt —the *smallest scale* on which the Wiener paths W^1 and W^2 are generated;
- ▶ Time-step scale $h = Q\Delta t$ —on which the SDE is stepped forward.

Practical approach:

Quadrature approximation

Two inherent scales:

- ▶ Quadrature scale Δt —the *smallest scale* on which the Wiener paths W^1 and W^2 are generated;
- ▶ Time-step scale $h = Q\Delta t$ —on which the SDE is stepped forward.

Practical approach:

- ▶ $J_{12}(t_n, t_{n+1}) \equiv \int_{t_n}^{t_{n+1}} (W_\tau^1 - W_{t_n}^1) dW_\tau^2$

Quadrature approximation

Two inherent scales:

- ▶ Quadrature scale Δt —the *smallest scale* on which the Wiener paths W^1 and W^2 are generated;
- ▶ Time-step scale $h = Q\Delta t$ —on which the SDE is stepped forward.

Practical approach:

- ▶ $J_{12}(t_n, t_{n+1}) \equiv \int_{t_n}^{t_{n+1}} (W_\tau^1 - W_{t_n}^1) dW_\tau^2$
- ▶ **Filtration:** $\mathcal{F}_Q = \{W_{t_n+q\Delta t}^i : \text{all } i, n, q\}$.

Quadrature approximation

Two inherent scales:

- ▶ Quadrature scale Δt —the *smallest scale* on which the Wiener paths W^1 and W^2 are generated;
- ▶ Time-step scale $h = Q\Delta t$ —on which the SDE is stepped forward.

Practical approach:

- ▶ $J_{12}(t_n, t_{n+1}) \equiv \int_{t_n}^{t_{n+1}} (W_\tau^1 - W_{t_n}^1) dW_\tau^2$
- ▶ Filtration: $\mathcal{F}_Q = \{W_{t_n+q\Delta t}^i : \text{all } i, n, q\}$.
- ▶ **Basic idea:** $J \rightarrow \mathbb{E}(J | \mathcal{F}_Q(n))$.

Quadrature approximation

Two inherent scales:

- ▶ Quadrature scale Δt —the *smallest scale* on which the Wiener paths W^1 and W^2 are generated;
- ▶ Time-step scale $h = Q\Delta t$ —on which the SDE is stepped forward.

Practical approach:

- ▶ $J_{12}(t_n, t_{n+1}) \equiv \int_{t_n}^{t_{n+1}} (W_\tau^1 - W_{t_n}^1) dW_\tau^2$
- ▶ Filtration: $\mathcal{F}_Q = \{W_{t_n+q\Delta t}^i : \text{all } i, n, q\}$.
- ▶ Basic idea: $J \rightarrow \mathbb{E}(J | \mathcal{F}_Q(n))$.
- ▶ **Polygonal volumes.**

Quadrature approximation

Two inherent scales:

- ▶ Quadrature scale Δt —the *smallest scale* on which the Wiener paths W^1 and W^2 are generated;
- ▶ Time-step scale $h = Q\Delta t$ —on which the SDE is stepped forward.

Practical approach:

- ▶ $J_{12}(t_n, t_{n+1}) \equiv \int_{t_n}^{t_{n+1}} (W_\tau^1 - W_{t_n}^1) dW_\tau^2$
- ▶ Filtration: $\mathcal{F}_Q = \{W_{t_n+q\Delta t}^i : \text{all } i, n, q\}$.
- ▶ Basic idea: $J \rightarrow \mathbb{E}(J | \mathcal{F}_Q(n))$.
- ▶ Polygonal volumes.
- ▶ **Wiktorsson 2001; Stump & Hill 2005.**

Quadrature approximation II

$$\begin{aligned}\mathbb{E}\left(\left|J_{12}(t_n, t_{n+1}) - \hat{J}_{12}(t_n, t_{n+1})\right|^2\right) \\ &= \sum_{q=0}^{Q-1} \mathbb{E}\left(\text{Var}[J_{12}(\tau_q, \tau_{q+1}) | \mathcal{F}_Q]\right) \\ &= \mathcal{O}((\Delta t)^2 Q) \\ &= \mathcal{O}(h^2/Q).\end{aligned}$$

Quadrature approximation II

$$\begin{aligned}\mathbb{E}\left(\left|J_{12}(t_n, t_{n+1}) - \hat{J}_{12}(t_n, t_{n+1})\right|^2\right) \\ &= \sum_{q=0}^{Q-1} \mathbb{E}\left(\text{Var}\left[J_{12}(\tau_q, \tau_{q+1}) \mid \mathcal{F}_Q\right]\right) \\ &= \mathcal{O}((\Delta t)^2 Q) \\ &= \mathcal{O}(h^2/Q).\end{aligned}$$

Clark & Cameron 1980

Global error vs computational effort

Global error vs computational effort

Proof.

$$\mathcal{E} = K_Q(M) \frac{h^{\frac{1}{2}}}{\sqrt{Q}}.$$

$$Q = \frac{1}{h^{2M-1}} \Rightarrow \mathcal{E} = K_{\mathcal{E}} h^M.$$

$$\mathcal{U} = (k_{\mathcal{U}} Q + \text{eval}) N = k_{\mathcal{U}} \frac{1}{h^{2M-1}} \frac{T}{h} = \mathcal{O}(\mathcal{E}^{-2}).$$

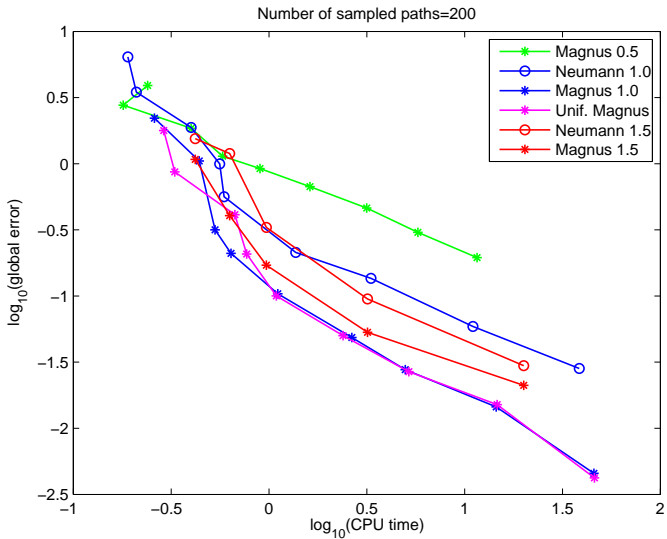
□

Linear system

$$V_i(y) = a_i y.$$

$$a_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{51}{200} \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{pmatrix},$$

Linear system



Lie group action

Munthe-Kaas (1999): $y_t = \Lambda(S_t, y_0)$

Lie group action

Munthe-Kaas (1999): $y_t = \Lambda(S_t, y_0)$

- ▶ $\Lambda(\text{Id}, y) = y$ **and** $\Lambda(R, \Lambda(S, y)) = \Lambda(RS, y)$.

Lie group action

Munthe-Kaas (1999): $y_t = \Lambda(S_t, y_0)$

- ▶ $\Lambda(\text{Id}, y) = y$ and $\Lambda(R, \Lambda(S, y)) = \Lambda(RS, y)$.
- ▶ $R(\tau) \in \mathcal{G}$: $R(0) = S$ and $R'(0) = a$. If $y = \Lambda(S, y_0) \Leftrightarrow y_0 = \Lambda(S^{-1}, y)$:

$$\begin{aligned}(\partial_1 \Lambda)(S, y_0) \circ a &\equiv \partial_\tau \Lambda(R(\tau), y_0) \Big|_{\tau=0} \\ &= \partial_\tau \Lambda(R(\tau), \Lambda(S^{-1}, y)) \Big|_{\tau=0} \\ &= \partial_\tau \Lambda(R(\tau)S^{-1}, y) \Big|_{\tau=0} \\ &= (\partial_1 \Lambda)(\text{Id}, y) \circ (aS^{-1}).\end{aligned}$$

Pullback to the Lie group I

$$y_t = y_0 + \sum_{i=0}^d \int_0^t V_i(\tau, y_\tau) dW_\tau^i$$

Pullback to the Lie group I

$$y_t = y_0 + \sum_{i=0}^d \int_0^t V_i(\tau, y_\tau) dW_\tau^i$$

$$\begin{aligned}\Lambda(S_t, y_0) &= \Lambda(\text{Id}, y_0) + \int_0^t (\partial_1 \Lambda)(S_\tau, y_0) \circ dS_\tau \\ &= \Lambda(\text{Id}, y_0) + \int_0^t (\partial_1 \Lambda)(\text{Id}, y_\tau) \circ (dS_\tau \cdot S_\tau^{-1})\end{aligned}$$

Pullback to the Lie group \mathbb{I}

$$y_t = y_0 + \sum_{i=0}^d \int_0^t V_i(\tau, y_\tau) dW_\tau^i$$

$$\begin{aligned}\Lambda(S_t, y_0) &= \Lambda(\text{Id}, y_0) + \int_0^t (\partial_1 \Lambda)(S_\tau, y_0) \circ dS_\tau \\ &= \Lambda(\text{Id}, y_0) + \int_0^t (\partial_1 \Lambda)(\text{Id}, y_\tau) \circ (dS_\tau \cdot S_\tau^{-1})\end{aligned}$$

$$\int_0^t (\partial_1 \Lambda)(\text{Id}, y_\tau) \circ (dS_\tau \cdot S_\tau^{-1}) = \sum_{i=0}^d \int_0^t V_i(\tau, y_\tau) dW_\tau^i.$$

Pullback to the Lie group I

$$y_t = y_0 + \sum_{i=0}^d \int_0^t V_i(\tau, y_\tau) dW_\tau^i$$

$$\begin{aligned}\Lambda(S_t, y_0) &= \Lambda(\text{Id}, y_0) + \int_0^t (\partial_1 \Lambda)(S_\tau, y_0) \circ dS_\tau \\ &= \Lambda(\text{Id}, y_0) + \int_0^t (\partial_1 \Lambda)(\text{Id}, y_\tau) \circ (dS_\tau \cdot S_\tau^{-1})\end{aligned}$$

$$\int_0^t (\partial_1 \Lambda)(\text{Id}, y_\tau) \circ (dS_\tau \cdot S_\tau^{-1}) = \sum_{i=0}^d \int_0^t V_i(\tau, y_\tau) dW_\tau^i.$$

$$S_t = \text{Id} + \sum_{i=0}^d \int_0^t ((\partial_1 \Lambda)(\text{Id}, y_\tau))^{-1} \circ (V_i(\tau, y_\tau)) S_\tau dW_\tau^i,$$

Pullback to the Lie group \mathbb{I}

$$S_t = \text{Id} + \sum_{i=0}^d \int_0^t ((\partial_1 \Lambda)(\text{Id}, y_\tau))^{-1} \circ (V_i(\tau, y_\tau)) S_\tau dW_\tau^i,$$

Pullback to the Lie group II

$$S_t = \text{Id} + \sum_{i=0}^d \int_0^t ((\partial_1 \Lambda)(\text{Id}, y_\tau))^{-1} \circ (V_i(\tau, y_\tau)) S_\tau dW_\tau^i,$$

$$V_i(t, y) = (\partial_1 \Lambda)(\text{Id}, y) \circ a_i(t, y),$$

Pullback to the Lie group II

$$S_t = \text{Id} + \sum_{i=0}^d \int_0^t ((\partial_1 \Lambda)(\text{Id}, y_\tau))^{-1} \circ (V_i(\tau, y_\tau)) S_\tau dW_\tau^i,$$

$$V_i(t, y) = (\partial_1 \Lambda)(\text{Id}, y) \circ a_i(t, y),$$

$$S_t = \text{Id} + \sum_{i=0}^d \int_0^t a_i(\tau, \Lambda(S_\tau, y_0)) S_\tau dW_\tau^i.$$

Examples

$$\mathcal{G} = \text{Diff}(\mathcal{M}), \quad \Lambda(S, y) = S \circ y$$
$$\implies y_t = S_t \circ y_0 \quad \text{with} \quad S_t \equiv \varphi_t.$$

Examples

$$\begin{aligned} \mathcal{G} &= \text{Diff}(\mathcal{M}), \quad \Lambda(S, y) = S \circ y \\ \implies y_t &= S_t \circ y_0 \quad \text{with} \quad S_t \equiv \varphi_t. \end{aligned}$$

$$V_i(t, y) = a_i(t, y) y \quad \text{with} \quad a_i \in \mathfrak{so}(n), \quad \mathcal{M} = \mathcal{G} = \text{SO}(n) :$$

$$\Lambda(S, y) = Sy \quad \implies \quad (\partial_1 \Lambda)(\text{Id}, y) \circ a = ay \quad \implies \quad y_t = S_t y_0.$$

Pullback to the Lie algebra

Pullback to the Lie algebra

$$S_t = \text{Id} + \sum_{i=0}^d \int_0^t a_i(\tau, \Lambda(S_\tau, y_0)) S_\tau dW_\tau^i.$$

Pullback to the Lie algebra

$$S_t = \text{Id} + \sum_{i=0}^d \int_0^t a_i(\tau, \Lambda(S_\tau, y_0)) S_\tau dW_\tau^i.$$

$$\exp(\sigma_t) = \text{Id} + \sum_{i=0}^d \int_0^t a_i(\tau, \Lambda(\exp(\sigma_\tau), y_0)) \exp(\sigma_\tau) dW_\tau^i.$$

Pullback to the Lie algebra

$$S_t = \text{Id} + \sum_{i=0}^d \int_0^t a_i(\tau, \Lambda(S_\tau, y_0)) S_\tau dW_\tau^i.$$

$$\exp(\sigma_t) = \text{Id} + \sum_{i=0}^d \int_0^t a_i(\tau, \Lambda(\exp(\sigma_\tau), y_0)) \exp(\sigma_\tau) dW_\tau^i.$$

$$\exp(\sigma_t) = \text{Id} + \int_0^t d\exp_{\sigma_\tau} \circ d\sigma_\tau \exp(\sigma_\tau).$$

Pullback to the Lie algebra

$$S_t = \text{Id} + \sum_{i=0}^d \int_0^t a_i(\tau, \Lambda(S_\tau, y_0)) S_\tau dW_\tau^i.$$

$$\exp(\sigma_t) = \text{Id} + \sum_{i=0}^d \int_0^t a_i(\tau, \Lambda(\exp(\sigma_\tau), y_0)) \exp(\sigma_\tau) dW_\tau^i.$$

$$\exp(\sigma_t) = \text{Id} + \int_0^t \text{dexp}_{\sigma_\tau} \circ d\sigma_\tau \exp(\sigma_\tau).$$

$$\sigma_t = \sum_{i=0}^d \int_0^t \text{dexp}_{\sigma_\tau}^{-1} \circ a_i(\tau, \Lambda(\exp \sigma_\tau, y_0)) dW_\tau^i,$$

Stochastic Taylor Munthe-Kaas method

$$\sigma_t = \sum_{i=0}^d \int_0^t \text{dexp}_{\sigma_\tau}^{-1} \circ a_i(\tau, \Lambda(\exp \sigma_\tau, y_0)) \, dW_\tau^i,$$

Stochastic Taylor Munthe-Kaas method

$$\sigma_t = \sum_{i=0}^d \int_0^t \text{dexp}_{\sigma_\tau}^{-1} \circ a_i(\tau, \Lambda(\exp \sigma_\tau, y_0)) \, dW_\tau^i,$$

All operations closed in the Lie algebra.

Stochastic Taylor Munthe-Kaas method

$$\sigma_t = \sum_{i=0}^d \int_0^t \text{dexp}_{\sigma_\tau}^{-1} \circ a_i(\tau, \Lambda(\exp \sigma_\tau, y_0)) \, dW_\tau^i,$$

All operations closed in the Lie algebra.

Algorithm:

Stochastic Taylor Munthe-Kaas method

$$\sigma_t = \sum_{i=0}^d \int_0^t \text{dexp}_{\sigma_\tau}^{-1} \circ a_i(\tau, \Lambda(\exp \sigma_\tau, y_0)) \, dW_\tau^i,$$

All operations closed in the Lie algebra.

Algorithm:

$$\hat{\sigma}_{t_n, t_{n+1}} = (J_0 v_0 + J_1 v_1 + J_2 v_2 + \frac{1}{2} J_1^2 v_1^2 + J_{12} v_2 v_1 + J_{21} v_1 v_2 + \frac{1}{2} J_2^2 v_2^2) \circ O.$$

Stochastic Taylor Munthe-Kaas method

$$\sigma_t = \sum_{i=0}^d \int_0^t \text{dexp}_{\sigma_\tau}^{-1} \circ a_i(\tau, \Lambda(\exp \sigma_\tau, y_0)) \, dW_\tau^i,$$

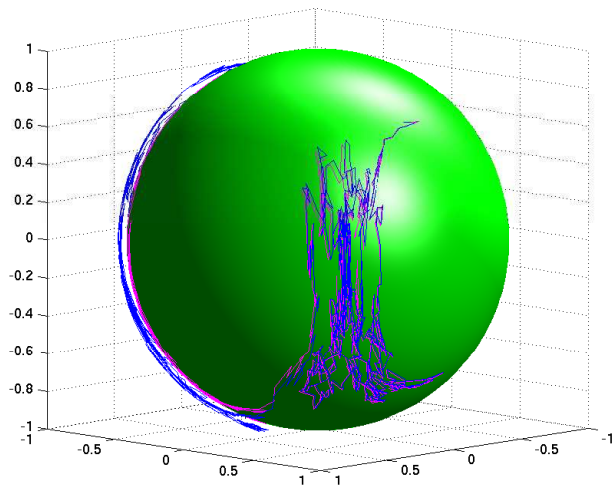
All operations closed in the Lie algebra.

Algorithm:

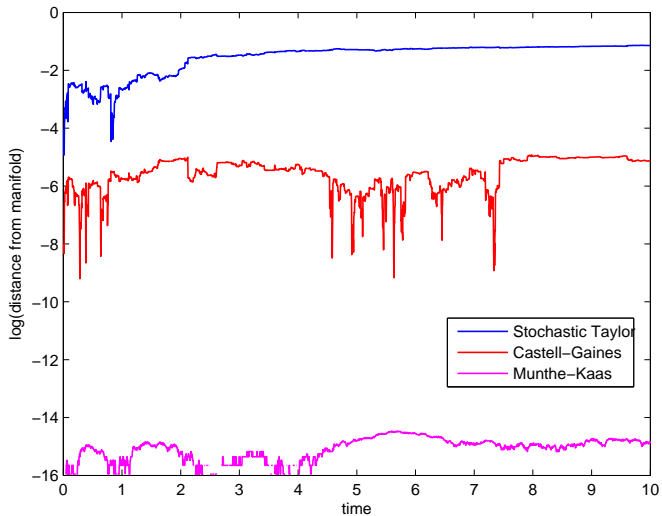
$$\hat{\sigma}_{t_n, t_{n+1}} = (J_0 v_0 + J_1 v_1 + J_2 v_2 + \frac{1}{2} J_1^2 v_1^2 + J_{12} v_2 v_1 + J_{21} v_1 v_2 + \frac{1}{2} J_2^2 v_2^2) \circ O.$$

$$y_{t_{n+1}} \approx \Lambda(\exp \hat{\sigma}_{t_n, t_{n+1}}, y_{t_n})$$

Rigid body



Rigid body



Conclusions

Take home messages:

For numerical SDE methods that are conditioned on two driving Wiener paths, order 1 methods provide the same accuracy as higher order methods for a given computational effort.

Stochastic Lie group integration methods can preserve the evolution of an SDE solution in a smooth manifold.

Conclusions

Take home messages:

For numerical SDE methods that are conditioned on two driving Wiener paths, order 1 methods provide the same accuracy as higher order methods for a given computational effort.

Stochastic Lie group integration methods can preserve the evolution of an SDE solution in a smooth manifold.

Future directions:

- ▶ Variable step scheme (*Gaines & Lyons 1997*).
- ▶ Zakai equation: Markov chain filters.
- ▶ Backwards SDEs

Accuracy I

For order 1/2 schemes:

$$R^{\text{els}} = \frac{1}{2}(J_{21} - J_{12})[V_1, V_2] \circ y_0;$$

$$R^{\text{st}} = R^{\text{els}} + \underbrace{\frac{1}{2}(J_{21} + J_{12})}_{\hat{R}}(V_1 V_2 + V_2 V_1) \circ y_0.$$

Local error:

$$\begin{aligned}\mathbb{E}((R^{\text{st}})^T R^{\text{st}}) &= \mathbb{E}((R^{\text{els}} + \hat{R})^T (R^{\text{els}} + \hat{R})) \\ &= \mathbb{E}((R^{\text{els}})^T R^{\text{els}}) + \mathbb{E}(\hat{R}^T \hat{R}) \\ &\quad + \underbrace{\mathbb{E}(\hat{R}^T R^{\text{els}})}_{=0} + \underbrace{\mathbb{E}((R^{\text{els}})^T \hat{R})}_{=0}.\end{aligned}$$

Quadrature approximation

Quadrature		\hat{J}_{12}	\hat{J}_{112}	\hat{J}_{120}	\hat{J}_{1112}
$\mathcal{E}^{\text{loc}}(n)$		$h/Q^{1/2}$	$h^{3/2}/Q^{1/2}$	$h^2/Q^{1/2}$	$h^2/Q^{1/2}$
\mathcal{U}	$\mathcal{O}(h^{3/2})$	h^{-1}
	$\mathcal{O}(h^2)$	h^{-2}	h^{-1}
	$\mathcal{O}(h^{5/2})$	h^{-3}	h^{-2}	h^{-1}	h^{-1}

Global error I

Global interval: $[0, T] = \cup_{n=0}^{N-1} [t_n, t_{n+1}]$, $t_n = nh$.

Global error I

Global interval: $[0, T] = \cup_{n=0}^{N-1} [t_n, t_{n+1}]$, $t_n = nh$.

Local remainder across $[t_n, t_{n+1}]$:

$$R = \exp(\psi) - \exp(\hat{\psi}) = \exp(\hat{\psi} + \rho) - \exp(\hat{\psi}) = \rho + \mathcal{O}(\psi\rho).$$

Global error I

Global interval: $[0, T] = \cup_{n=0}^{N-1} [t_n, t_{n+1}]$, $t_n = nh$.

Local remainder across $[t_n, t_{n+1}]$:

$$R = \exp(\psi) - \exp(\hat{\psi}) = \exp(\hat{\psi} + \rho) - \exp(\hat{\psi}) = \rho + \mathcal{O}(\psi\rho).$$

Stochastic Taylor or exponential Lie series:

$$y_{t_n, t_{n+1}} = \hat{y}_{t_n, t_{n+1}} + R_{t_n, t_{n+1}}.$$

Global error II

$$\begin{aligned}\mathcal{E}^2 &\equiv \mathbb{E} \left\| \left(\prod_{n=N-1}^0 \varphi_{t_n, t_{n+1}} - \prod_{n=N-1}^0 \hat{\varphi}_{t_n, t_{n+1}} \right) \circ y_0 \right\|^2 \\ &= \mathbb{E} \left\| \underbrace{\left(\sum_{n=0}^{N-1} \varphi_{t_{n+1}, t_N} R_{t_n, t_{n+1}} \hat{\varphi}_{t_0, t_n} \right)}_{\mathcal{R}} \circ y_0 \right\|^2.\end{aligned}$$

Recall: exponential Lie series (order 1)

$$y_t = \exp(\psi_t) \circ y_0, \quad \psi_t = \psi_{1/2} + \psi_1 + \psi_{3/2} + \cdots$$

Recall: exponential Lie series (order 1)

$$y_t = \exp(\psi_t) \circ y_0, \quad \psi_t = \psi_{1/2} + \psi_1 + \psi_{3/2} + \cdots$$

$$\psi_{1/2} = J_1 V_1 + J_2 V_2 + J_0 V_0,$$

Recall: exponential Lie series (order 1)

$$y_t = \exp(\psi_t) \circ y_0, \quad \psi_t = \psi_{1/2} + \psi_1 + \psi_{3/2} + \cdots$$

$$\psi_{1/2} = J_1 V_1 + J_2 V_2 + J_0 V_0,$$

$$\psi_1 = \frac{1}{2}(J_{21} - J_{12})[V_1, V_2],$$

Recall: exponential Lie series (order 1)

$$y_t = \exp(\psi_t) \circ y_0, \quad \psi_t = \psi_{1/2} + \psi_1 + \psi_{3/2} + \dots$$

$$\psi_{1/2} = J_1 V_1 + J_2 V_2 + J_0 V_0,$$

$$\psi_1 = \frac{1}{2}(J_{21} - J_{12})[V_1, V_2],$$

$$\begin{aligned} \psi_{3/2} = & \frac{1}{2}(J_{10} - J_{01})[V_0, V_1] + \frac{1}{2}(J_{20} - J_{02})[V_0, V_2] \\ & + (J_{112} - \frac{1}{2}J_1 J_{12} + \frac{1}{12}J_1^2 J_2) [V_1, [V_1, V_2]] \\ & + (J_{221} - \frac{1}{2}J_2 J_{21} + \frac{1}{12}J_2^2 J_1) [V_2, [V_2, V_1]] \\ & + (J_{110} - \frac{1}{2}J_1 J_{10} + \frac{1}{12}J_1^2 J_0) [V_1, [V_1, V_0]] \\ & + (J_{220} - \frac{1}{2}J_2 J_{20} + \frac{1}{12}J_2^2 J_0) [V_2, [V_2, V_0]]. \end{aligned}$$

Global error III

$$\mathcal{R} = \sum_{n=0}^{N-1} \sum_{\alpha} (\varphi_{t_{n+1}, t_N} V_{\alpha} \varphi_{t_0, t_n} \circ y_0) J_{\alpha}(t_n, t_{n+1}).$$

$$\mathbb{E}(\mathcal{R}^T \mathcal{R})$$

$$\begin{aligned} &= \sum_{n=0}^{N-1} \sum_{\alpha, \beta} \mathbb{E} \left((\varphi_{t_{n+1}, t_N} V_{\alpha} \varphi_{t_0, t_n} \circ y_0)^T (\varphi_{t_{n+1}, t_N} V_{\beta} \varphi_{t_0, t_n} \circ y_0) \right) \\ &\quad \cdot \mathbb{E}(J_{\alpha}(t_n, t_{n+1}) J_{\beta}(t_n, t_{n+1})) \\ &+ \sum_{n \neq m} \sum_{\alpha, \beta} \mathbb{E} \left((\varphi_{t_{n+1}, t_N} V_{\alpha} \varphi_{t_0, t_n} \circ y_0)^T (\varphi_{t_{m+1}, t_N} V_{\beta} \varphi_{t_0, t_m} \circ y_0) \right) \\ &\quad \cdot \mathbb{E}(J_{\alpha}(t_n, t_{n+1})) \mathbb{E}(J_{\beta}(t_m, t_{m+1})). \end{aligned}$$

Stochastic Riccati system

$$S(t) = I + \sum_{i=0}^d \int_0^t f_i(\tau, S(\tau)) dW_i(\tau).$$

$$f_i(t, S) = S(t)A_i(t)S(t) + B_i(t)S(t) + S(t)C_i(t) + D_i(t).$$

- ▶ Stochastic linear-quadratic optimal control.
- ▶ eg. mean-variance hedging in finance (*Bobrovnytska & Schweizer 2004; Kohlmann & Tang 2003*).

Riccati II

If $\mathbb{A}_i(t) \equiv \begin{pmatrix} B_i(t) & D_i(t) \\ -A_i(t) & -C_i(t) \end{pmatrix}$ and $\mathbb{U} = \begin{pmatrix} U \\ V \end{pmatrix}$ satisfies

$$\mathbb{U}(t) = \mathbb{I} + \sum_{i=0}^d \int_0^t \mathbb{A}_i(\tau) \mathbb{U}(\tau) dW_i(\tau),$$

then $S = UV^{-1}$ solves the Riccati system.

$$A_0 = \begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}, \quad C_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad D_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix},$$

$$D_1 = a_1 \quad \text{and} \quad D_2 = a_2.$$

Riccati III

Kloeden & Platen:

$$\begin{aligned} S_{n+1} &= S_n + f(S_n)h + D_1 J_1 + D_2 J_2 \\ &+ \frac{h}{4} (f(Y_1^+) + f(Y_1^-) + f(Y_2^+) + f(Y_2^-) - 4f(S_n)) \\ &+ \frac{1}{2\sqrt{h}} ((f(Y_1^+) - f(Y_1^-))J_{10} + (f(Y_2^+) - f(Y_2^-))J_{20}) , \end{aligned}$$

$$Y_j^\pm = S_n + \frac{h}{2} f(S_n) \pm D_j \sqrt{h}$$

$$f(S) = SA_0S + B_0S + SC_0 + D_0 .$$

Riccati IV

