

# Grassmannian spectral shooting and the stability of multi-dimensional travelling waves

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# Spectral problems

Parabolic nonlinear systems on  $\mathbb{R} \times \mathbb{T}$ :

$$\partial_t U = B \Delta U + c \partial_x U + F(U),$$

Travelling wave  $U_c$ . Small perturbations  $U$  satisfy:

$$B \Delta U + c \partial_x U + DF(U_c)U = \lambda U.$$

Two main solution approaches:

- ▶ *Projection.*
- ▶ *Shooting.*

# Motivation

Workshop at AIM, Palo Alto, May 2005:

*Stability criteria for multi-dimensional waves and patterns.*

Organised by: Chris Jones, Yuri Latushkin, Bob Pego, Björn Sandstede and Arnd Scheel.

# Setup

$$\text{On } \mathbb{R}: \quad B \Delta U + c \partial_x U + DF(U_c)U = \lambda U$$

$$\Leftrightarrow \quad Y' = A(x; \lambda) Y$$

Assume for  $\lambda \in \Omega \subseteq \mathbb{C}$ :

- ▶ Exponential dichotomies on  $\mathbb{R}^-$  and  $\mathbb{R}^+$ ;
- ▶ Same Morse index  $k$  (Sandstede 2002).

Matching condition (Alexander, Gardner and Jones 1990):

$$D(\lambda) \equiv e^{-\int_0^x \text{Tr} A(\xi; \lambda) d\xi} \det(Y^-(x; \lambda) Y^+(x; \lambda)).$$

# Numerical issues

- ▶ Computational domain.
- ▶ Different exponential growth rates.
- ▶ Polynomial complexity.
- ▶ Flow singularities?!
- ▶ Where to match?
- ▶ Retaining analyticity.
- ▶ How to project transversely.

# Outline

- 1 Grassmann manifolds and flows
- 2 Patch evolution (GGEM)
- 3 Applications (planar fronts)
- 4 Transverse Fourier projection
- 5 Application (wrinkled fronts)
- 6 Future work

# Stiefel and Grassmann manifolds

- ▶ Stiefel manifold:

$$\mathbb{V}(n, k) = \{k\text{-frames centred at the origin}\}.$$

- ▶ Grassmann manifold:

$$\text{Gr}(n, k) = \{k\text{-dimensional subspaces of } \mathbb{C}^n\}.$$

- ▶ Fibre bundle:

$$\text{Gr}(n, k) \cong \mathbb{V}(n, k)/\text{GL}(k)$$

- ▶ Projection  $\pi: k\text{-frame} \mapsto \text{spanning } k\text{-plane}.$

# Representation

Coordinate patches  $\mathbb{U}_i$ : multi-index  $i = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ .

Example:  $\mathbb{U}_{\{1, \dots, k\}}$  uniquely represented by:

$$y_{i^\circ} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \hat{y}_{k+1,1} & \hat{y}_{k+1,2} & \cdots & \hat{y}_{k+1,k} \\ \hat{y}_{k+2,1} & \hat{y}_{k+2,2} & \cdots & \hat{y}_{k+2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{y}_{n,1} & \hat{y}_{n,2} & \cdots & \hat{y}_{n,k} \end{pmatrix}.$$

Local coordinate chart  $\varphi_i: \mathbb{U}_i \rightarrow \mathbb{C}^{(n-k)k}$  given by  $\varphi_i: y_{i^\circ} \mapsto \hat{y}$ .

# Grassmannian flows

$$Y' = A(x, Y) Y.$$

Substitute decomposition  $Y = y_{i^\circ} u_i$ :

$$y'_{i^\circ} u_i + y_{i^\circ} u'_i = (A_i + A_{i^\circ} y_{i^\circ}) u_i,$$

Project onto  $i$ th and  $i^\circ$ th rows:

$$\hat{y}' = c + d \hat{y} - \hat{y}(a + b \hat{y}) \quad \text{and} \quad u'_i = (a + b \hat{y}) u_i,$$

where  $a = A_{i \times i}$ ,  $b = A_{i \times i^\circ}$ ,  $c = A_{i^\circ \times i}$  and  $d = A_{i^\circ \times i^\circ}$ .

Linear vector field:  $A = A(x)$  only?

# Drury–Oja flow

Humpherys and Zumbrun;  $QR$ -decomposition of  $Y \in \mathbb{V}(n, k)$ :

$$\begin{aligned}Q' &= (I_n - QQ^\dagger)A(x)Q, \\(\det R)' &= \text{Tr}(Q^\dagger A(x)Q)(\det R).\end{aligned}$$

Corresponds to Riccati with  $u' = -\hat{y}^\dagger(c + d\hat{y})u$ .

# Grassmannian Gaussian elimination method (GGEM)

$$\begin{array}{ccccccc}
 \mathbb{C}^{(n-k)k} & \xrightarrow{\varphi_i^{-1}} & \mathbb{U}_i & \xrightarrow{\text{id}} & \mathbb{V}(n, k) & \xrightarrow{(\Lambda_{Y_0})^*} & \text{GL}(n) & \xrightarrow{\log} & \mathfrak{gl}(n) \\
 \downarrow & & \downarrow \text{GGEM} & & \downarrow \text{RK} & & \downarrow & & \downarrow \text{Magnus} \\
 \mathbb{C}^{(n-k)k} & \xleftarrow{\varphi_{i'}} & \mathbb{U}_{i'} & \xleftarrow{\text{QOGE}} & \mathbb{V}(n, k) & \xleftarrow{\Lambda_{Y_0}} & \text{GL}(n) & \xleftarrow{\exp} & \mathfrak{gl}(n)
 \end{array}$$

# Quasi-optimal Gaussian elimination (QOGE)

GE with *free* stepwise max pivot, generates:  $Y_{m+1} = y_{i^{\circ}} L$ .

$$\begin{pmatrix} * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & * & \cdots & * \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ * & * & * & * & \cdots & * \\ * & * & * & * & \cdots & * \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ * & * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ * & * & * & * & \cdots & * \end{pmatrix}$$

# Applications (planar fronts)

- ▶  $D(\lambda) \equiv e^{-\int_0^x \text{Tr} A(\xi; \lambda) d\xi} \det(Y^-(x; \lambda) Y^+(x; \lambda)).$
- ▶  $\det(Y^- Y^+) = \det \begin{pmatrix} y_{i_-}^{\circ} & y_{i_+}^{\circ} \end{pmatrix} \cdot \det u_{i_-} \cdot \det u_{i_+}.$
- ▶  $D(\lambda; x_*) \equiv \det \begin{pmatrix} y_{i_-}^{\circ} & y_{i_+}^{\circ} \end{pmatrix} \cdot \det L^- \cdot \det L^+.$
- ▶ Exponentially scale GGEM.

# Boussinesq system

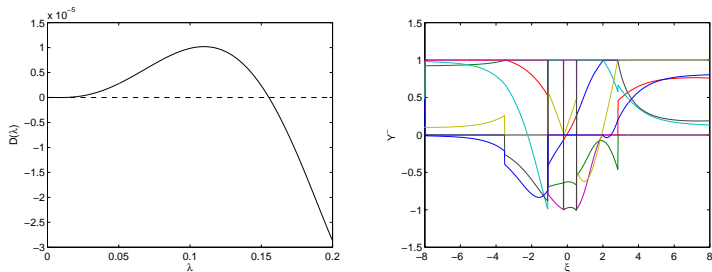
$$\text{PDE: } u_{tt} = (1 - c^2) u_{xx} + 2c u_{xt} - u_{xxxx} - (u^2)_{xx}.$$

$$\text{Solitary waves: } \bar{u}(x) \equiv \frac{3}{2}(1 - c^2) \operatorname{sech}^2\left(\frac{1}{2}\sqrt{1 - c^2} x\right).$$

Stable when  $1/2 < |c| < 1$  and unstable when  $|c| < 1/2$ .

$$A(x; \lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\lambda^2 - 2\bar{u}'' & 2\lambda c - 4\bar{u}' & (1 - c^2) - 2\bar{u} & 0 \end{pmatrix}.$$

# Boussinesq: GGEM



**Figure:** Evans function with GGEM-RK and  $x_* = 8$  (left panel). Entries of  $y_i$  for  $\lambda = 0.15543141$  (right panel).

# Boussinesq: Evans function vs matching point

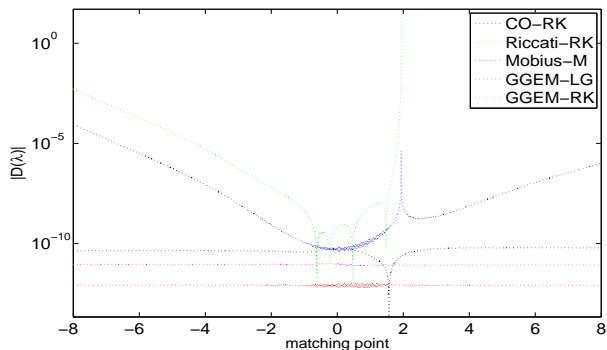


Figure:  $|D(\lambda)|$  for  $\lambda$  equal to the eigenvalue for different matching points. The number of steps in the equidistant mesh was  $N = 512$ .

# Boussinesq: error vs matching point

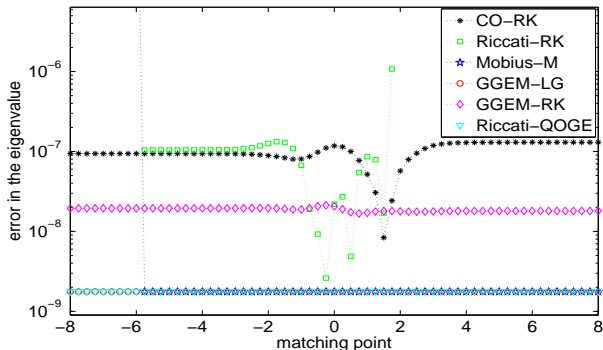


Figure: Error in the eigenvalue for different choices of the matching point:  $N = 512$ .

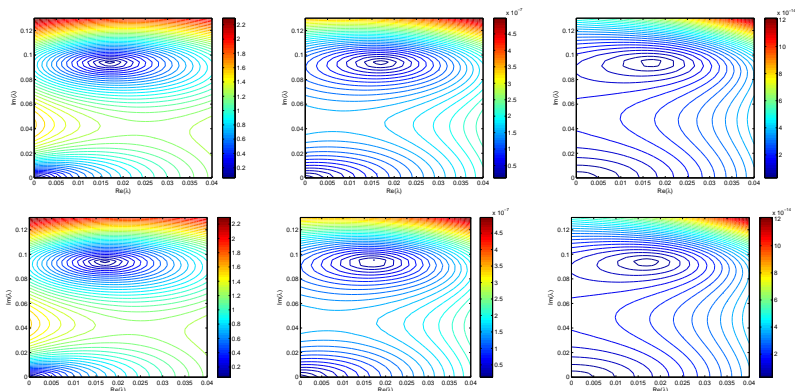
# Autocatalytic fronts

$$\partial_t u = \delta \Delta u + c \partial_x u - uv^m,$$

$$\partial_t v = \Delta v + c \partial_x v + uv^m.$$

$$A(x; \lambda) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda/\delta + \bar{v}^m/\delta & m\bar{u}\bar{v}^{m-1}/\delta & -c/\delta & 0 \\ -\bar{v}^m & \lambda - m\bar{u}\bar{v}^{m-1} & 0 & -c \end{pmatrix},$$

# Autocatalytic fronts: Evans contours



**Figure:** Contour lines of  $|D(\lambda)|$  for  $\delta = 0.1$  and  $m = 9$  using the CO-RK and GGEM-LG (order: top three down to bottom three), matching at positions  $x_* = -8, 0, +8$  (left to right).

# Autocatalytic fronts: error vs matching point

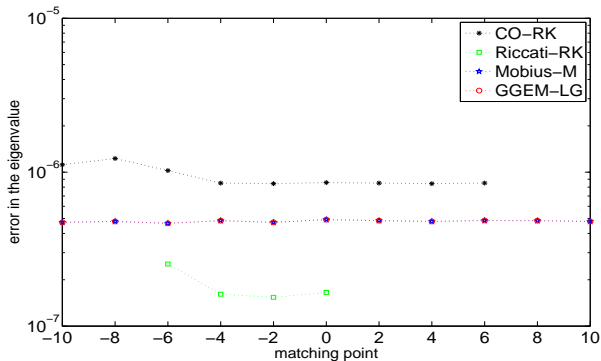


Figure: Error in the eigenvalue when  $\delta = 0.1$  and  $m = 9$ :  $N = 256$ .

# Transverse Fourier basis

On  $\mathbb{R} \times \mathbb{T}$  we have:

$$B\Delta U + c \partial_x U + DF(U_c)U = \lambda U.$$

On the Fourier modes  $k = -K, -K + 1, \dots, K$ :

$$\partial_x \hat{U}_k = \hat{P}_k,$$

$$\partial_x \hat{P}_k = \lambda B^{-1} \hat{U}_k + (k/\tilde{L})^2 \hat{U}_k - c B^{-1} \hat{P}_k - \sum_{\nu=-K}^K B^{-1} \hat{D}_{k-\nu} \hat{U}_\nu,$$

# Large ODE system

$$\partial_x \begin{pmatrix} \hat{U} \\ \hat{P} \end{pmatrix} = \begin{pmatrix} O_{N(2K+1)} & I_{N(2K+1)} \\ \tilde{A}_3(\lambda) + \hat{A}_3(x) & -c B^{-1} \otimes I_{2K+1} \end{pmatrix} \begin{pmatrix} \hat{U} \\ \hat{P} \end{pmatrix}$$

with  $E_k(\lambda) \equiv \lambda B^{-1} + (k/\tilde{L})^2 I_N$ :

$$\tilde{A}_3(\lambda) = \begin{pmatrix} E_{-K}(\lambda) & O & \cdots & O \\ O & E_{-K+1}(\lambda) & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & \cdots & O & E_{+K}(\lambda) \end{pmatrix}$$
$$\hat{A}_3(x) = -B^{-1} \otimes \begin{pmatrix} \hat{D}_0 & \hat{D}_{-1} & \cdots & \hat{D}_{-2K} \\ \hat{D}_1 & \hat{D}_0 & \cdots & \hat{D}_{-2K+1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{D}_{2K} & \hat{D}_{2K-1} & \cdots & \hat{D}_0 \end{pmatrix}.$$

# Computing travelling waves: freezing method

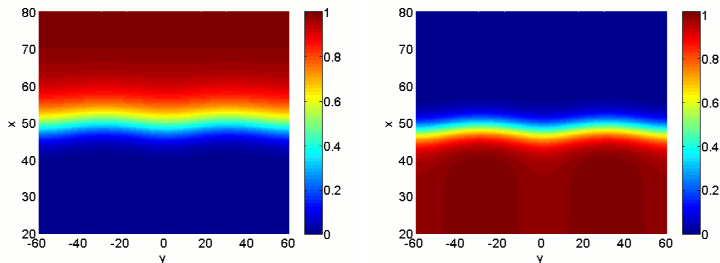
Substitute  $U(x, y, t) = V(x - \gamma(t), y, t)$  into original PDE:

$$\partial_t V = B \Delta V + \gamma'(t) \partial_x V + F(V),$$

$$0 = \int_{\mathbb{R} \times \mathbb{T}} (\partial_x \hat{V}(x, y, t))^T (\hat{V}(x, y, t) - V(x, y, t)) dx dy.$$

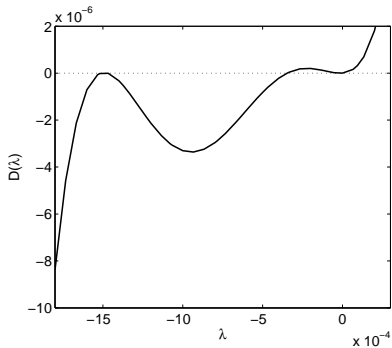
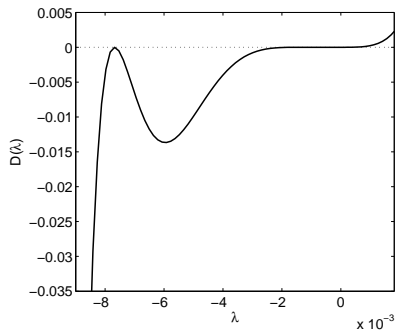
(Developed by Beyn and Thümmler.)

# Wrinkled fronts (cubic autocatalytic system)

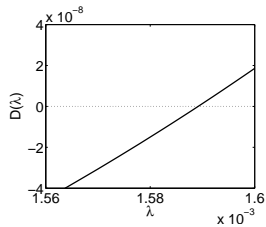
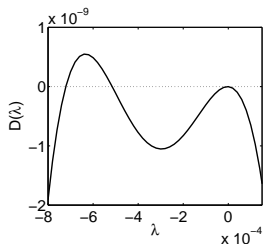
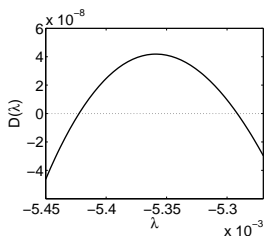
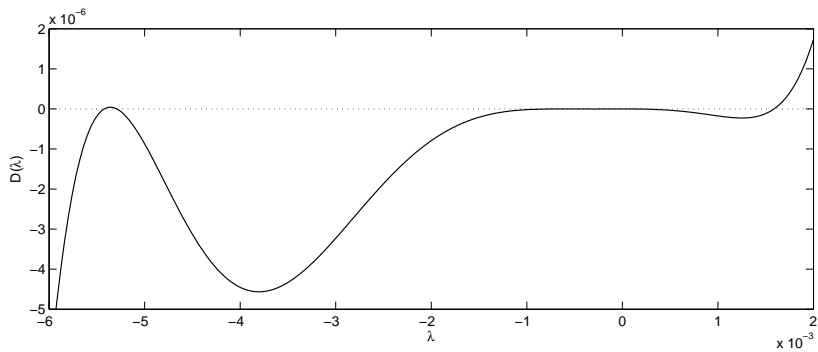


**Figure:** The wrinkled front for  $\delta = 3$ . Left panel:  $u$  component. Right panel:  $v$  component. Cut from domain  $[-150, 150] \times [-60, 60]$ , grid  $300 \times 240$ , FE rectangles.

# Wrinkled front: Evans function for $\delta = 2.5$



# Wrinkled front: Evans function for $\delta = 3$



## Wrinkled front: Eigenvalues for $\delta = 3$

$K$	Eigenvalues (Evans function)				
3	0.001609	-0.000026	-0.000781	-0.001296	-0.000670
4	0.001609	0.000002	-0.000001	-0.000519	-0.000670
5	0.001589	0.000002	-0.000001	-0.000519	-0.000720
6	0.001589	-0.000002	-0.000003	-0.000515	-0.000720
7	0.001589	-0.000002	-0.000003	-0.000515	-0.000721
8	0.001589	-0.000002	-0.000003	-0.000515	-0.000721
9	0.001589	-0.000002	-0.000003	-0.000515	-0.000721
$\vdots$			$\vdots$		
24	0.001589	-0.000002	-0.000003	-0.000515	-0.000721
	Eigenvalues (ARPACK)				
	0.001592	0.000000	0.000000	-0.000514	-0.000719

# Wrinkled front: contour integration

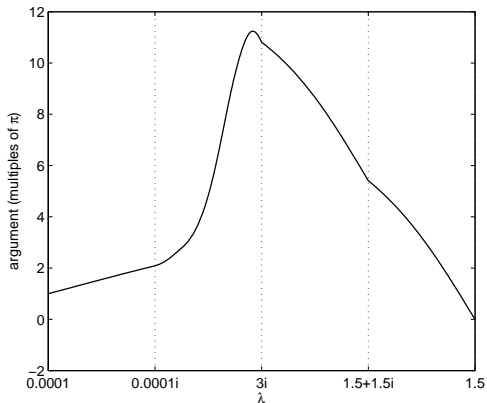
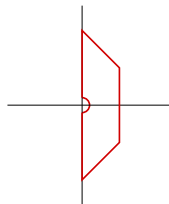
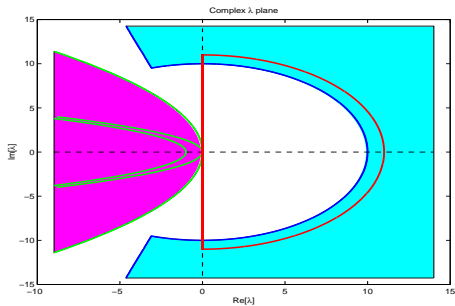


Figure: Left panel: contour. Right panel:  $\arg(D(\lambda))$  when  $\lambda$  transverses the top half.  $\delta = 3$ .

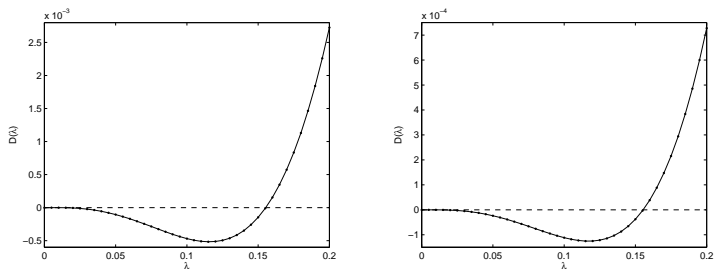
# Future work

- ▶ Multiple sources of error: relative influence?
- ▶ Control theory: Lagrangian Grassmannian.
- ▶ Schubert cells.

# Spectral plane



# Boussinesq: Evans function



**Figure:** Wave speed  $c = 0.4$ . Left plot: Riccati-RK ( $i^- = \{1, 2\}$  over  $[-8, 0]$  and  $i^+ = \{3, 4\}$  over  $[0, 8]$ ). Right plot: CO-RK.

# Boussinesq: eigenvalue error I

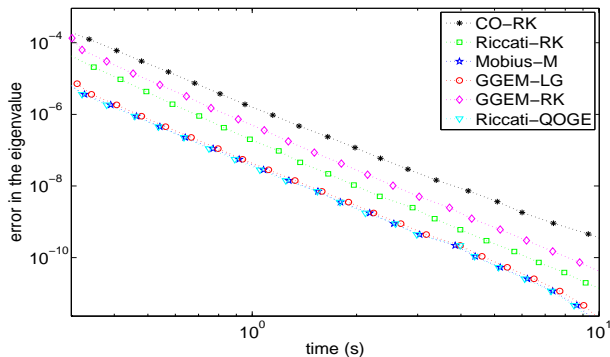


Figure: Error in the eigenvalue vs cputime, matching at  $x_* = 0$ .

## Boussinesq: eigenvalue error II

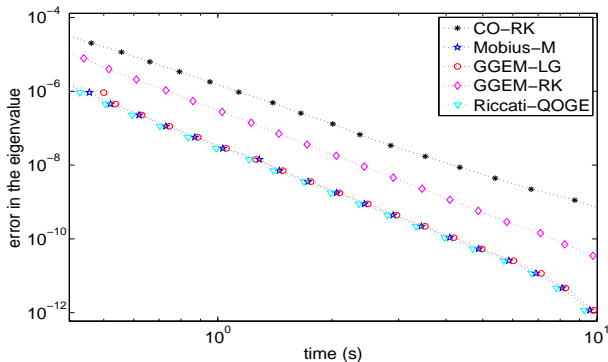
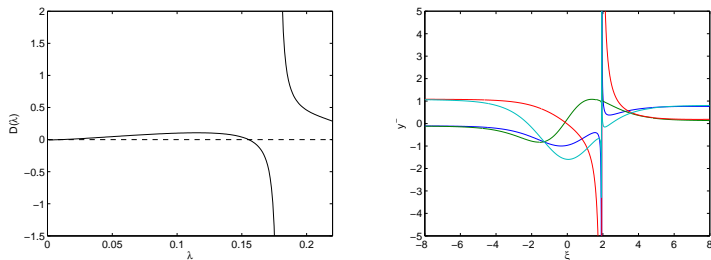


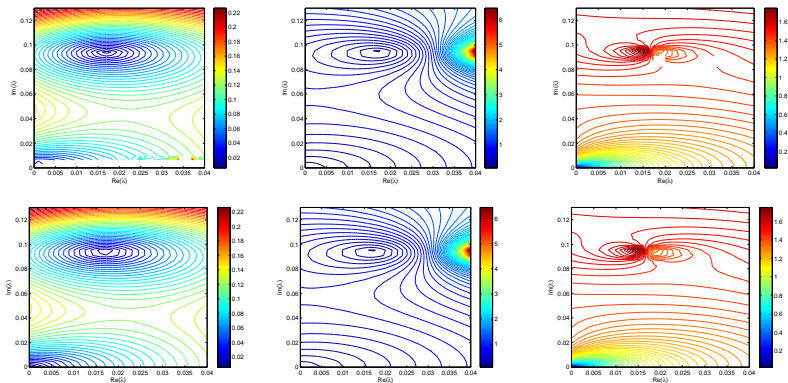
Figure: Error in the eigenvalue vs cputime, matching at  $x_* = 8$ .

# Boussinesq: Evans function



**Figure:** Left: Evans function with Möbius–Magnus:  $x_* = +8$ , Right: entries in  $\hat{y}^-$  when  $\lambda = 0.15543141$ .

# Autocatalytic fronts: Evans contours



**Figure:** Contour lines of  $|D(\lambda)|$  for  $\delta = 0.1$  and  $m = 9$  using the Riccati-RK, Möbius-Magnus (order: top three down to bottom three), matching at positions  $x_* = -8, 0, +8$  (left to right).

# Wrinkled front: same with Drury–Oja

