Universal optimal stochastic expansions

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Outline

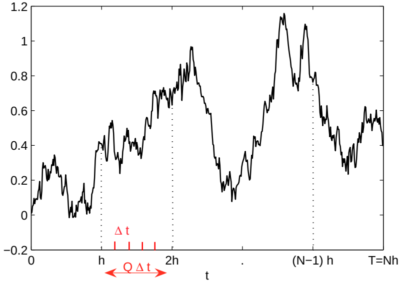
Stochastic differential equations

$$\mathrm{d}y_t = V_0(y_t)\,\mathrm{d}t + V_1(y_t)\,\mathrm{d}W_t^1 + \cdots + V_d(y_t)\,\mathrm{d}W_t^d$$

$$y_t = y_0 + \sum_{i=0}^{d} \int_0^t V_i(y_\tau)\,\mathrm{d}W_\tau^i$$

- Solution process: $y\colon \mathbb{R}_+ \to \mathbb{R}^N$
- Non-commuting vector fields: $V_i = \sum_{j=1}^n V_i^j(y)\partial_y$
- d-dimensional driving signal: $(W^1, \ldots, W^d)$
- Convention: $W_t^0 \equiv t$

Wiener process



- $W_t - W_s \sim \mathrm{N}(0, \sqrt{t-s})$
- Independent increments
- Continuous, potentially nowhere differentiable

Stochastic chain rule

$$y_t = y_0 + \sum_i \int_0^t V_i \circ y_\tau \, \mathrm{d}W_\tau^i$$

Itô lemma (stochastic chain rule) $\Rightarrow$

$$f \circ y_t = f \circ y_0 + \sum_j \int_0^t V_j \circ f \circ y_\tau \, \mathrm{d}W_\tau^j$$

E.g. choose $f = V_i \Rightarrow$

$$V_i \circ y_t = V_i \circ y_0 + \sum_j \int_0^t V_j \circ V_i \circ y_\tau \, \mathrm{d}W_\tau^j$$

$$y_t = y_0 + \sum_i \int_0^t \left(V_i \circ y_0 + \sum_j \int_0^{\tau_1} V_j \circ V_i \circ y_{\tau_2} \, \mathrm{d}W_{\tau_2}^j \right) \mathrm{d}W_{\tau_1}^i$$

Stochastic Taylor series

$$y_t = y_0 + \sum_i \int_0^t \mathrm{d}W^i_{\tau_1} V_i \circ y_0 + \sum_{i,j} \int_0^t \int_0^{\tau_1} V_j \circ V_i \circ y_{\tau_2} \, \mathrm{d}W^j_{\tau_2} \, \mathrm{d}W^i_{\tau_1}$$

Now choose $f = V_j \circ V_i \Rightarrow$

$$y_t = y_0 + \sum_i \underbrace{\int_0^t \mathrm{d}W^i_{\tau_1}}_{J_i(t)} V_i \circ y_0 + \sum_{i,j} \underbrace{\int_0^t \int_0^{\tau_1} \mathrm{d}W^j_{\tau_2} \, \mathrm{d}W^i_{\tau_1}}_{J_{ji}(t)} V_j \circ V_i \circ y_0 + \cdots$$

- Euler-Maruyama and Milstein methods
- Need approximations for iterated integrals for strong solution

Flow map

$$y_t = \varphi_t \circ y_0$$

Hence

$$\varphi_t = \mathrm{id} + \sum_i J_i(t) V_i + \sum_{i,j} J_{ji}(t) V_j \circ V_i + \sum_{i,j,k} J_{kji}(t) V_k \circ V_j \circ V_i + \cdots$$

i.e.

$$\varphi_t = \mathrm{id} + \sum_{w \in \mathbb{A}^+} J_w(t) V_w$$

Here $\mathbb{A}^+ = \big\{$non-empty words over $\mathbb{A} = \{0, 1, \ldots, d\}\big\}$

Remainders and local error

Basic idea

$$\varphi = \mathrm{id} + \sum_{w \in \mathbb{A}^+} J_w\, V_w$$

Suppose: $\varphi = \mathrm{id} + F(\psi) \Rightarrow$

$$\psi = F^{-1}(\varphi - \mathrm{id}) = \sum_{k=1}^{\infty} C_k\, (\varphi - \mathrm{id})^k = \sum_{w \in \mathbb{A}^+} K_w\, V_w$$

where

$$K_w = \sum_{k=1}^{|w|} C_k \sum_{\substack{u_1,\ldots,u_k \in \mathbb{A}^+ \\ u_1 u_2 \cdots u_k = w}} J_{u_1} J_{u_2} \cdots J_{u_k}(t)$$

Goal: choose best C_k

Basic idea cont'd

- $C_k = (-1)^k/(k+1)$ generates exponential Lie series
- Order $\frac{1}{2}$ Lie series more accurate
- Similarly for orders 1 and $\frac{3}{2}$ when assume $[V_i, V_j] = 0$ for $i, j \in \{1, \ldots, d\}$
- Counter-example for order $\frac{3}{2}$: $[V_i, V_j] \neq 0$
- Now *assume no drift*: $\mathbb{A} = \{1, \ldots, d\}$
- *Shuffle relations*: $J_u J_v = \sum_{w \in \mathrm{sh}(u,v)} J_w$

$$
\begin{aligned}
J_{a_1} J_{a_2} &= J_{a_1 a_2} + J_{a_2 a_1} \\
J_{a_1} J_{a_2 a_3} &= J_{a_1 a_2 a_3} + J_{a_2 a_1 a_3} + J_{a_2 a_3 a_1}
\end{aligned}
$$

Hopf algebra of words

$$\mathbb{A}^* = \textit{free monoid} \text{ on } \mathbb{A}$$
$$= \text{endow set words over } \mathbb{A} \text{ with } \textit{concat product}$$

$a_i \in \mathbb{A} \Rightarrow w = a_1 \ldots a_n \in \mathbb{A}$

$1 = \text{empty word} \Rightarrow 1 \cdot w = w \cdot 1 = w$

$\mathbb{K}$ = commutative ring with unit

$\mathbb{K}\langle\mathbb{A}\rangle$ = noncommutative polynomials/series on $\mathbb{A}$ over $\mathbb{K}$

Polynomial: $P = \sum_{w \in \mathbb{A}} (P, w) w$

(Reutenauer)

Concatenation and shuffle algebras

Concatenation product on $\mathbb{K}\langle\mathbb{A}\rangle$:

$$PQ = \sum_{u,v\in\mathbb{A}^*} (P,u)(Q,v)uv$$

Shuffle product: $u \sqcup\!\sqcup\, v$

Extended to $\mathbb{K}\langle\mathbb{A}\rangle$ by:

$$P \sqcup\!\sqcup\, Q = \sum_{u,v\in\mathbb{A}^*} (P,u)(Q,v)u \sqcup\!\sqcup\, v$$

Associative, distributive wrt addition, $1 \sqcup\!\sqcup\, w = w \sqcup\!\sqcup\, 1 = w$

Two bi-algebra structures: $(\mathbb{K}\langle\mathbb{A}\rangle, \eta, \varepsilon, c, \delta)$ and $(\mathbb{K}\langle\mathbb{A}\rangle, \eta, \varepsilon, s, \delta')$

Hopf algebra tensor product

Bialgebra with antipode $\alpha \in \mathrm{End}(\mathbb{K}\langle\mathbb{A}\rangle)$

Antipode here is linear signed reversal mapping:

$$\alpha \circ (a_1 \dots a_n) = (-1)^n a_n \dots a_1$$

Hence two Hopf algebra structures, consequently set:

$$\mathcal{H} = \mathbb{K}\langle\mathbb{A}\rangle \otimes \mathbb{K}\langle\mathbb{A}\rangle$$

$$(u \otimes x)(v \otimes y) = (u \sqcup\!\sqcup v) \otimes (xy)\,,$$

Concatenation-shuffle operator algebra

$$u_{n_1} \sqcup\!\sqcup\, u_{n_2} \sqcup\!\sqcup \ldots \sqcup\!\sqcup\, u_{n_k} = \zeta\big(\underbrace{c^{n_1} s c^{n_2} s c^{n_3} \ldots s c^{n_k}}_{b} \otimes \underbrace{u_{n_1} u_{n_2} \ldots u_{n_k}}_{w}\big)$$

$\mathbb{B} = \{c, s\} \longrightarrow \mathbb{B}^* \longrightarrow \mathbb{K}\langle\mathbb{B}\rangle$

Shuffle gluing product: $g : b_1 \otimes b_2 \mapsto b_1 s b_2$

Associative tensor algebra: $\mathcal{K} = \bigoplus_{n\geq 0} \mathbb{K}\langle\mathbb{B}\rangle_n \otimes \mathbb{K}\langle\mathbb{A}\rangle_{n+1}$

$$(b_1 \otimes u_1)(b_2 \otimes u_2) = (b_1 s b_2) \otimes (u_1 u_2)\,,$$

Homomorphism: $\zeta : \mathcal{K} \to \mathbb{K}\langle\mathbb{A}\rangle$

Pullback to Hopf algebra

$$\varphi = 1 \otimes 1 + \sum_{w \in \mathbb{A}^+} w \otimes w$$

$$\begin{aligned}
\Rightarrow \quad \psi &= \sum_{k \geq 1} C_k \big(\varphi - 1 \otimes 1\big)^k \\
&= \sum_{k \geq 1} C_k \left(\sum_{w \in \mathbb{A}^+} w \otimes w \right)^k \\
&= \sum_{k \geq 1} C_k \sum_{u_1, \ldots, u_k \in \mathbb{A}^+} (u_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup u_k) \otimes (u_1 \ldots u_k) \\
&= \sum_{w \in \mathbb{A}^*} \left(\sum_{k=1}^{|w|} C_k \sum_{\substack{u_1, \ldots, u_k \in \mathbb{A}^+ \\ w = u_1 \ldots u_k}} u_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup u_k \right) \otimes w
\end{aligned}$$

Coefficients in $\mathbb{K}\langle\mathbb{B}\rangle$

$w \in \mathbb{A}^+$ with $|w| = n+1$:

$$
\begin{aligned}
K \circ w &= \sum_{k=1}^{|w|} C_k \sum_{\substack{u_1,\ldots,u_k \in \mathbb{A}^+ \\ w = u_1 \ldots u_k}} u_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup u_k \\
&= \sum_{k=1}^{n+1} C_k \, \zeta\big((c^{n-(k-1)} \sqcup\!\sqcup s^{k-1}) \otimes w\big) \\
&= \sum_{k=0}^{n} C_{k+1} \, \zeta\big((c^{n-k} \sqcup\!\sqcup s^{k}) \otimes w\big) \\
&= \zeta\left(\left(\sum_{k=0}^{n} C_{k+1}(c^{n-k} \sqcup\!\sqcup s^{k})\right) \otimes w\right) .
\end{aligned}
$$

Sinh-log coefficients

Lemma

With

$$C_k = \begin{cases} 1\,, & k = 1\,, \\ \frac{1}{2}(-1)^{k+1}\,, & k \geq 2\,, \\ \frac{1}{2}(-1)^{n+1} + \epsilon\,, & k = n\,, \end{cases}$$

then

$$K = \tfrac{1}{2}\big(c^n - \alpha^n\big) + \epsilon\, s^n$$

where for $w \in \mathbb{K}\langle\mathbb{A}\rangle_{n+1}$*:* $\alpha^n \circ w \equiv \alpha \circ w$

Proof: step 1

Partial integration formula:

$$a_1 \dots a_{n+1} = (a_1 \dots a_n) ⧢ a_{n+1} - (a_1 \dots a_{n-1}) ⧢ (a_{n+1} a_n) + \cdots + (-1)^n a_{n+1} \dots a_1$$

$$c^n = -c^{n-1}s - c^{n-2}s\alpha - c^{n-3}s\alpha^2 - \cdots - \alpha^n$$

$$\alpha^n = -c^n - \sum_{k=0}^{n-1} c^k s \alpha^{n-k-1}$$

Proof: step 2

Antipode polynomial:

$$\alpha^n \equiv -(c - s)^n \,.$$

True for $n = 1, 2$, assume true for $k = 1, 2, \ldots, n-1$. Direct expansion $\Rightarrow$

$$(c - s)^n = c^n - \sum_{k=0}^{n-1} c^k s (c - s)^{n-k-1}$$

Use induction.

Proof: step 3

Recall:

$$K = \tfrac{1}{2}c^n + \tfrac{1}{2}\sum_{k=0}^{n}(-1)^k(c^{n-k} \sqcup\!\sqcup\, s^k) + \epsilon\, s^n$$
$$= \tfrac{1}{2}c^n + \tfrac{1}{2}(c-s)^n + \epsilon\, s^n$$
$$= \tfrac{1}{2}c^n - \tfrac{1}{2}\alpha^n + \epsilon\, s^n$$

Hence

$$K_w = \tfrac{1}{2}\big(J_w - J_{\alpha\circ w}\big) + \epsilon \prod_{i=1}^{n+1} J_{w_i}$$

Sinh-log remainder is smaller I

$$\|R^{\mathsf{st}} \circ y_0\|_{L^2}^2 = \|R^{\mathsf{sl}} \circ y_0\|_{L^2}^2 + E$$

where

$$E \equiv \mathbb{E}\left(\bar{R}\circ y_0\right)^{\mathrm{T}}\left(R^{\mathsf{sl}}\circ y_0\right)+\mathbb{E}\left(R^{\mathsf{sl}}\circ y_0\right)^{\mathrm{T}}\left(\bar{R}\circ y_0\right)+\mathbb{E}\left(\bar{R}\circ y_0\right)^{\mathrm{T}}\left(\bar{R}\circ y_0\right)$$

$$R^{\mathsf{sl}} = \sum_{\substack{w\in\mathbb{A}^+\\ |w|\geq n+1}} K_w\, V_w + \cdots$$

$$\bar{R} = \sum_{\substack{w\in\mathbb{A}^+\\ |w|=n+1}} \bar{J}_w\, V_w$$

where $\bar{J}_w = J_w - K_w$

Sinh-log remainder is smaller II

$$E = \sum_{\substack{u,v \in \mathbb{A}^+ \\ |u|=|v|=n+1}} \mathbb{E}\left(\bar{J}_u K_v + K_u \bar{J}_v + \bar{J}_u \bar{J}_v\right)(V_u \circ y_0)^T (V_v \circ y_0)$$

$$\begin{aligned}
\bar{J}_u K_v &+ K_u \bar{J}_v + \bar{J}_u \bar{J}_v \\
&= \epsilon^0 \left(\tfrac{1}{2}(J_u J_v - J_{\alpha \circ u} J_{\alpha \circ v}) + \tfrac{1}{4}(J_u + J_{\alpha \circ u})(J_v + J_{\alpha \circ v}) \right) \\
&\quad - \epsilon^1 \left(\tfrac{1}{2}(J_u - J_{\alpha \circ u}) \prod_{i=1}^{n+1} J_{v_i} + \tfrac{1}{2}(J_v - J_{\alpha \circ v}) \prod_{i=1}^{n+1} J_{u_i} \right) \\
&\quad - \epsilon^2 \left(\prod_{i,j=1}^{n+1} J_{u_i} J_{v_j} \right)
\end{aligned}$$

Concluding remarks

- exponential series when diffusion vector fields commute
- optimal series when drift included