

Numerical evaluation of the Evans function

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Outline

- 1 Evans function: definition and properties
- 2 Numerical computation
- 3 Exterior product spaces
- 4 Numerical methods
- 5 Examples

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Take home message:

For non-selfadjoint stiff problems, the Evans function method, which is a shooting & matching technique, is the most accurate or even the only approach.

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$$y' = A(x, \lambda)y, \quad \text{where } A(x, \lambda) = \begin{pmatrix} 0 & 1 \\ q(x) - \lambda & 0 \end{pmatrix}, \quad (*)$$

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with boundary conditions $y_1(a) = y_1(b) = 0$.

Denote by $y^-(x)$ the solution of $(*)$ with $y^-(a) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The **miss-distance function** is

$$D(\lambda) = y_1^+(b).$$

Eigenvalues correspond to zeros of the miss-distance function.

The matching point ξ

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The miss-distance function, evaluated at $\xi \in [a, b]$ is

$$D(\lambda) = \det \begin{pmatrix} y_1^-(\xi) & y_1^+(\xi) \\ y_2^-(\xi) & y_2^+(\xi) \end{pmatrix}.$$

For $\xi = b$, we get $D(\lambda) = y_1^-(b)$, as before.

General spectral problems

Consider $y' = A(x, \lambda) y$ with $y(x) \in \mathbb{C}^n$ for $x \in \mathbb{R}$.

We assume there is a region $\Omega \subset \mathbb{C}$ such that for all $\lambda \in \Omega$:

- ▶ $A^\pm(\lambda) = \lim_{x \rightarrow \pm\infty} A(x, \lambda)$ exist and are hyperbolic;
- ▶ $A^-(\lambda)$ has k unstable eigenvalues μ_1^-, \dots, μ_k^- ;
- ▶ $A^+(\lambda)$ has $n - k$ unstable eigenvalues $\mu_1^+, \dots, \mu_{n-k}^+$.

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Let y_i^- be the solution $y_i^-(x) \sim e^{\mu_i^- x} v_i^-$ as $x \rightarrow -\infty$.

Let y_i^+ be the solution $y_i^+(x) \sim e^{\mu_i^+ x} v_i^+$ as $x \rightarrow +\infty$.

The **Evans function** is defined by

$$D(\lambda) = \det(y_1^-(\xi) \cdots y_k^-(\xi) y_1^+(\xi) \cdots y_{n-k}^+(\xi)).$$

It is analytic in Ω and its zeros correspond to eigenvalues.

(Evans '75; Alexander, Gardner & Jones '90; Sandstede '02)

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Computing the Evans function numerically

$$D(\lambda) = \det(y_1^-(\xi) \cdots y_k^-(\xi) y_1^+(\xi) \cdots y_{n-k}^+(\xi)).$$

Basic **numerical** computation:

- ▶ Compute the unstable eigenvectors v_1^-, \dots, v_k^- of A^- .
- ▶ For $i = 1, \dots, k$, solve $y' = A(x, \lambda)y$ with initial condition $y(-L) = v_i^-$ (where L is large) to get $y_i^-(\xi)$.
- ▶ Compute $y_i^+(\xi)$ similarly, and calculate the determinant.

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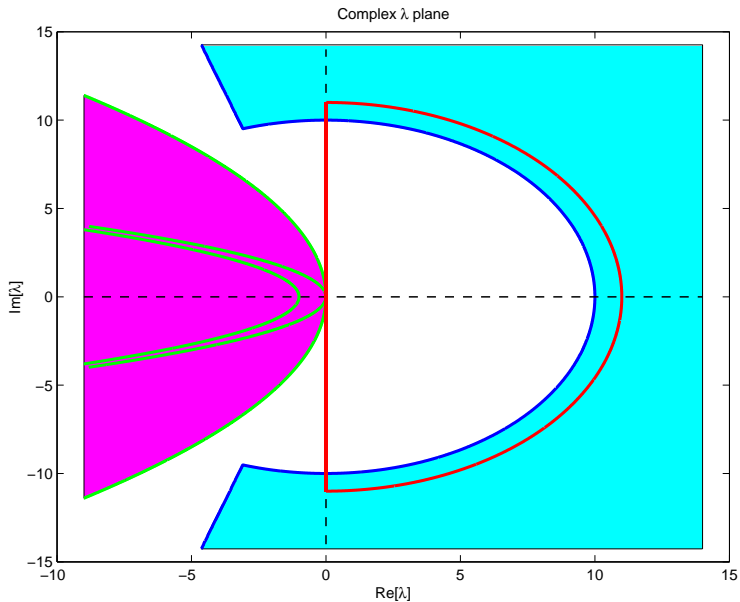
The Evans function is analytic, so we can use the **argument principle** to count the number of eigenvalues in a given region.

(Evans & Faroe '77)

Alternatively, we can use **Newton's method** to solve $D(\lambda) = 0$ and locate eigenvalues.

(Pego, Smereka & Weinstein '93)

Spectrum structure



Problems

- ▶ Unstable space \Rightarrow for example y_i^- grow with rate μ_i^- .

Solution: Rescale $y_i^- \rightarrow e^{\mu_i^- x} y_i^-$ and solving

$$y' = (A(x, \lambda) - \mu_i^- I) y.$$

- ▶ **Eigenvectors** v_i^\pm must be analytic functions of λ .

Solution: Kato's algorithm.

- ▶ If $\operatorname{Re} \mu_1^- > \operatorname{Re} \mu_2^-$, then y_1^- grows faster than y_2^- , and so any errors in the y_1^- direction will dominate the y_2^- solution.

Solution: Do not look at the Y_i^- individually, but look at the subspace $S = \operatorname{span}\{y_1^-, \dots, y_k^-\}$ and **lift** the equation $y' = A(x, \lambda) y$ to $S' = \ell(A(x, \lambda)) S$.

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Evans function

For $v_j \in \mathbb{C}^n$:

$$\det(v_1 \cdots v_n) \equiv v_1 \wedge \cdots \wedge v_n.$$

Hence Evans function

$$D(\lambda) \equiv e^{-\int_0^\xi \text{Tr} A(x, \lambda) dx} \underbrace{(y_1^- \wedge \cdots \wedge y_k^-)}_{\mathbf{w}^-(\xi, \lambda)} \wedge \underbrace{(y_1^+ \wedge \cdots \wedge y_{n-k}^+)}_{\mathbf{w}^+(\xi, \lambda)}.$$

Prefactor ensures ξ -independence, from Abel's theorem.

Exterior product of a vector space

Let V be a vector space with basis e_1, \dots, e_n .

The **exterior product space** $\Lambda^k(V)$ is a vector space with basis

$$\{e_{i_1} \wedge \dots \wedge e_{i_k} : 1 \leq i_1 \leq \dots \leq i_k \leq n\}.$$

For example, $\Lambda^2(\mathbb{C}^4)$ is six dimensional with basis

$$e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4.$$

The **Grassmannian manifold** $G_k(V)$ is the set of k -dimensional subspaces of V . We consider the identification

$$\text{span}\{v_1, \dots, v_k\} \in G_k(V) \leftrightarrow v_1 \wedge \dots \wedge v_k \in \Lambda^k(V).$$

The Grassmannian manifold

We can embed $G_k(V)$ in $\Lambda^k(V)$, or to be precise, $\mathbb{P}(\Lambda^k(V))$.

A form $w \in \Lambda^k(V)$ is **decomposable** if it can be written as $w = v_1 \wedge \dots \wedge v_k$ with $v_i \in V$.

Only decomposable forms correspond to subspaces.

Consider for example $\Lambda^2(\mathbb{C}^4)$. The form

$$S_1 e_1 \wedge e_2 + S_2 e_1 \wedge e_3 + S_3 e_1 \wedge e_4 + S_4 e_2 \wedge e_3 + S_5 e_2 \wedge e_4 + S_6 e_3 \wedge e_4$$

is decomposable iff

$$S_1 S_6 - S_2 S_5 + S_3 S_4 = 0.$$

Lifting the differential equation

A **linear** differential equation on V

$$y' = A(x)y \quad (*)$$

induces an equation on $\Lambda^k(V)$:

$$w' = \ell(A(x))w \quad (\dagger).$$

For example with $k = 2$ and $V = \mathbb{C}^n$.

If y_1 and y_2 solve $(*)$ then $w = y_1 \wedge y_2$ solves (\dagger) .

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Computing the Evans function II

Now basic **numerical** computation:

- ▶ Compute the unstable eigenvectors v_1^-, \dots, v_k^- of A^- .
- ▶ Lift the equation to $\Lambda^k(\mathbb{C}^n)$:

$$w' = \ell(A(x, \lambda)) w,$$

with initial condition $w(-L) = v_1^- \wedge \dots \wedge v_k^-$ (where L is large) to get $w^-(\xi)$.

- ▶ Compute $w^+(\xi)$ similarly, and evaluate

$$D(\lambda) = w^- \wedge w^+.$$

(Bridges '99; Brin '00; Afendikov & Bridges '01)

Same as **Compound matrix method**.

(Davey '79; Ng & Reid '79)

Magnus series

We need to solve the **linear** differential equation

$$y' = A(x) y .$$

Solution

$$y(x) = \exp(\sigma(x)) y(0) ,$$

where

$$\sigma(x) = \int_0^x A(\xi) d\xi + \frac{1}{2} \int_0^x \int_0^{\xi_1} [A(\xi_1), A(\xi_2)] d\xi_2 d\xi_1 + \dots .$$

Converges if $\int_0^x \|A(\xi)\| d\xi < \pi$.

(Moan & Niesen '06)

Magnus numerical method

Truncate Magnus series and replace $A(x)$ by interpolant at Gauss–Legendre points $x_{1,2} = \left(\frac{1}{2} \pm \frac{\sqrt{3}}{6}\right)x$:

$$y(x) \approx \exp\left(\frac{1}{2}x(A(x_1) + A(x_2)) - \frac{\sqrt{3}}{12}[A(x_1), A(x_2)]\right) y(0).$$

This is a method of **order four**.

(Iserles & Norsett '99)

Preserves Grassmannian invariants (decomposability conditions).

(Niesen '05)

Other numerical methods

- ▶ In the case $k = 2$ and $n = 4$, the Grassmannian is **attractive**, if we replace $w' = \ell(A(x, \lambda)) w$ by $w' = \ell(A(x, \lambda - \sigma I)) w$, where σ is the largest eigenvalue, provided the spectrum of A changes not too much as x varies.
(Bridges, Derks & Gottwald '02)
- ▶ If $\text{tr} A = 0$, the Grassmannian is a **strong quadratic invariant**. Gauss–Legendre methods (eg. implicit midpoint rule) conserve these.
(Allen & Bridges '02)
- ▶ In **practice**, with proof when $n = 2$, fourth order Gauss–Legendre is globally most accurate.
(Aparicio, Malham & Oliver '05; Malham & Niesen '06)

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Autocatalytic model

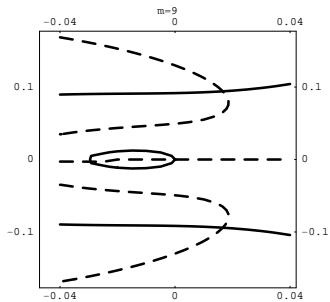
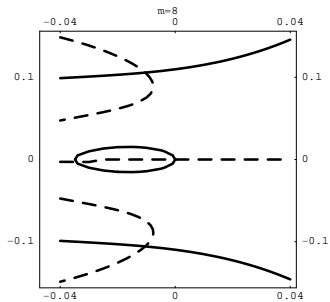
The autocatalytic reaction $U + mV \rightarrow (m + 1)V$ is modeled by

$$\begin{aligned}u_t &= \delta u_{xx} - uv^m, \\v_t &= v_{xx} + uv^m.\end{aligned}$$

There is a unique travelling wave solution with $(u, v) \rightarrow (0, 1)$ as $x \rightarrow -\infty$ and $(u, v) \rightarrow (1, 0)$ as $x \rightarrow +\infty$ for any speed $c \in [c_*, \infty)$.

Evans function evaluated with the Magnus and Gauss–Legendre methods to assess the stability of travelling wave with $c = c_*$.

Autocatalytic model II



Stability of rotating Ekman layer

Linearization of the 3d Navier–Stokes equation in a rotating frame about the Ekman layer coupled to a compliant surface leads to

$$\begin{aligned}\phi'''' - b(x)\phi'' - a(x)\phi + 2\psi' &= 0, \\ \psi'' + (\gamma^2 - b(x))\psi - i\gamma RV'(x)\phi - 2\phi' &= 0,\end{aligned}$$

for $0 \leq x < \infty$ with compliant surface BCs at $x = 0$.

Lifting yields an ODE on $\Lambda^3(\mathbb{C}^6)$, which has dimension 20.

However the dimension of the Grassmannian is only 9.

(Allen & Bridges '03)

Stability of viscous shock waves

A viscous shock wave is a travelling wave solution to the
conservation law

$$u_t + (f(u))_x = (B(u) u_x)_x.$$

As an example consider the cubic model in MHD

$$u_t + (|u|^2 u)_x = u_{xx}.$$

(Brin '00; Brin & Zumbrun '03)

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- ▶ (-) $\dim(\Lambda^k(\mathbb{C}^n)) = \binom{n}{k}$.

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- ▶ (+) Non-selfadjoint problems natural.
- ▶ (-) $\dim(\Lambda^k(\mathbb{C}^n)) = \binom{n}{k}$.
- ▶ (+) Continuous orthogonalization

(Humphreys & Zumbun '05)