

Efficient strong integrators for linear stochastic systems

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Introduction

$$dy_t = V_0(y_t) dt + V_1(y_t) dW_t^1 + V_2(y_t) dW_t^2$$

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- 1 Stochastic Taylor series
- 2 Exponential Lie series
- 3 Castell–Gaines method
- 4 Quadrature & efficiency
- 5 Lie group stochastic integrators
- 6 Conclusions

Introduction

Take home messages:

For numerical SDE methods that are conditioned on two or more driving Wiener paths, order 1 numerical schemes provide the most accuracy for a given computational effort.

Stochastic Lie group integration methods can preserve the evolution of an SDE solution in a smooth manifold.

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$$V_i = \sum_{j=1}^n V_i^j(y) \partial_y$$

Flow map and abstract SDE

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$$y_t = y_0 + K \circ V \circ y_0 + (K \circ V)^2 \circ y_0 + (K \circ V)^3 \circ y_0 + \dots$$

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$$\varphi_t = \text{Id} + K \circ V + (K \circ V)^2 + (K \circ V)^3 + \dots$$

Integral operators

$$\begin{aligned}K \circ V \circ y_0 &= K_0 \circ V_0 \circ y_0 + K_1 \circ V_1 \circ y_0 + K_2 \circ V_2 \circ y_0 \\ &= tV_0 \circ y_0 + \underbrace{\int_0^t dW_\tau^1}_{J_1} V_1 \circ y_0 + \underbrace{\int_0^t dW_\tau^2}_{J_2} V_2 \circ y_0\end{aligned}$$

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$$K_j K_i \circ V_j V_i \circ y_0 \equiv \underbrace{\int_0^t \int_0^{\tau_1} dW_{\tau_2}^i dW_{\tau_1}^j}_{J_{ij}} V_j V_i \circ y_0$$

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$$\begin{aligned} \varphi_{3/2} = & J_{10} V_0 V_1 + J_{01} V_1 V_0 + J_{20} V_0 V_2 + J_{02} V_0 V_2 \\ & + J_{111} V_1^3 + J_{112} V_2 V_1^2 + J_{121} V_1 V_2 V_1 + J_{122} V_2^2 V_1 \\ & + J_{211} V_1^2 V_2 + J_{212} V_2 V_1 V_2 + J_{221} V_1 V_2^2 + J_{222} V_2^3 \\ & + J_{00} V_0^2 + J_{011} V_1^2 V_0 + J_{101} V_1 V_0 V_1 + J_{110} V_0 V_1^2 + J_{022} V_2^2 V_0 \\ & + J_{202} V_2 V_0 V_2 + J_{220} V_0 V_2^2 + J_{1111} V_1^4 + J_{1122} V_2^2 V_1^2 \\ & + J_{1212} V_2 V_1 V_2 V_1 + J_{2121} V_1 V_2 V_1 V_2 + J_{2211} V_1^2 V_2^2 + J_{2222} V_2^4. \end{aligned}$$

Exponential Lie series

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- Magnus 1954, Kunita 1980, Ben Arous 1989, Castell 1993, Burrage 1999.

Numerical SDE schemes

General integrators

- ▶ Euler-Maruyama and Milstein methods
- ▶ Runge–Kutta type methods (*Kloeden & Platen* 1999)
- ▶ Magnus (*Burrage* 1999; *Misawa* 2001; *P-C. Moan* 2004)
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Exponential Lie series integrators

- ▶ **Castell–Gaines method: more accurate.**

Castell–Gaines method

Truncated exponential Lie series across $[t_n, t_{n+1}]$:

$$\hat{\psi}_{t_n, t_{n+1}} = J_1 V_1 + J_2 V_2 + J_0 V_0 + \frac{1}{2}(J_{21} - J_{12})[V_1, V_2].$$

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$$u'(\tau) = \hat{\psi}_{t_n, t_{n+1}} \circ u(\tau)$$

across $\tau \in [0, 1]$ **with** $u(0) = y_{t_n}$ **gives** $u(1) \approx y_{t_{n+1}}$.

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Need approximations for iterated integrals.

Accuracy I

For order 1/2 schemes:

$$R^{\text{els}} = \frac{1}{2}(J_{21} - J_{12})[V_1, V_2] \circ y_0;$$

$$R^{\text{st}} = R^{\text{els}} + \underbrace{\frac{1}{2}(J_{21} + J_{12})}_{\hat{R}}(V_1 V_2 + V_2 V_1) \circ y_0.$$

Local error:

$$\begin{aligned}\mathbb{E}((R^{\text{st}})^T R^{\text{st}}) &= \mathbb{E}((R^{\text{els}} + \hat{R})^T (R^{\text{els}} + \hat{R})) \\ &= \mathbb{E}((R^{\text{els}})^T R^{\text{els}}) + \mathbb{E}(\hat{R}^T \hat{R}) \\ &\quad + \underbrace{\mathbb{E}(\hat{R}^T R^{\text{els}})}_{=0} + \underbrace{\mathbb{E}((R^{\text{els}})^T \hat{R})}_{=0}.\end{aligned}$$

Accuracy II

For order 1 schemes:

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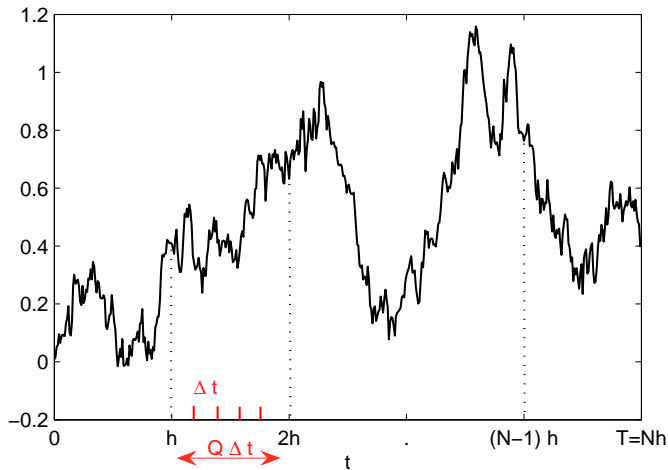
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Uniformly accurate exponential Lie integrator:

$$\begin{aligned}\hat{\psi}^* &= J_1 V_1 + J_2 V_2 + J_0 V_0 + \frac{1}{2}(J_{21} - J_{12})[V_1, V_2] \\ &\quad + \frac{h^2}{12}([V_1, [V_1, V_0]] + [V_2, [V_2, V_0]]) \\ &\quad + \frac{h^2}{12}(V_1[V_2, [V_2, V_1]] + V_2[V_1, [V_1, V_2]]).\end{aligned}$$

Quadrature approximation



Quadrature approximation

Two inherent scales:

- ▶ Wiener-discrete-path scale Δt —the *smallest scale* on which the Wiener paths W^1 and W^2 are generated;
- ▶ Time-step scale $h = Q\Delta t$ —on which the SDE is stepped forward.

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- ▶ Polygonal volumes.
- ▶ **Wiktorsson 2001; Stump & Hill 2005.**

Quadrature approximation II

$$\begin{aligned} & \mathbb{E}\left(\left|J_{12}(t_n, t_{n+1}) - \hat{J}_{12}(t_n, t_{n+1})\right|^2\right) \\ &= \sum_{q=0}^{Q-1} \mathbb{E}\left(\text{Var}[J_{12}(\tau_q, \tau_{q+1}) | \mathcal{F}_Q]\right) \\ &= \mathcal{O}((\Delta t)^2 Q) \\ &= \mathcal{O}(h^2/Q). \end{aligned}$$

Quadrature approximation III

Quadrature		\hat{J}_{12}	\hat{J}_{112}	\hat{J}_{120}	\hat{J}_{1112}
$\mathcal{E}^{\text{loc}}(n)$		$h/Q^{1/2}$	$h^{3/2}/Q^{1/2}$	$h^2/Q^{1/2}$	$h^2/Q^{1/2}$
\mathcal{U}	$\mathcal{O}(h^{3/2})$	h^{-1}
	$\mathcal{O}(h^2)$	h^{-2}	h^{-1}
	$\mathcal{O}(h^{5/2})$	h^{-3}	h^{-2}	h^{-1}	h^{-1}

Global error vs computational effort

Theorem

$$\mathcal{U} = \underbrace{(k_{\mathcal{U}} K_{\mathcal{E}}^2) \mathcal{E}^{-2}}_{\mathcal{U}^{\text{quad}}} + \underbrace{((c_M n^2 + c_E) K_{\mathcal{E}}^{1/M}) \mathcal{E}^{-1/M}}_{\mathcal{U}^{\text{eval}}}.$$

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Proof.

$$\mathcal{E} = K_Q(M) \frac{h^{\frac{1}{2}}}{\sqrt{Q}} + K_T(M) h^M.$$

$$Q = \frac{1}{h^{2M-1}} \Rightarrow \mathcal{E} = K_{\mathcal{E}} h^M.$$

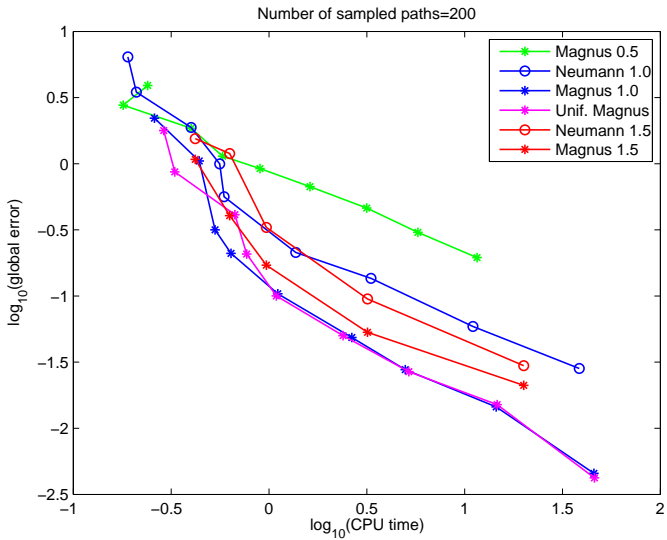
$$\mathcal{U} = (k_{\mathcal{U}} Q + c_M n^2 + c_E) N.$$

□

Linear system

$$a_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{51}{200} \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{pmatrix},$$

Linear system



Lie group action

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▶ **If** $y = \Lambda(S, y_0) \Leftrightarrow y_0 = \Lambda(S^{-1}, y)$:

$$\begin{aligned} (\partial_1 \Lambda)(S, y_0) \circ a &= \partial_\tau \Lambda(R(\tau), \Lambda(S^{-1}, y)) \Big|_{\tau=0} \\ &= \partial_\tau \Lambda(R(\tau)S^{-1}, y) \Big|_{\tau=0} \\ &= (\partial_1 \Lambda)(\text{Id}, y) \circ (aS^{-1}) . \end{aligned}$$

Pullback to the Lie group I

$$y_t = y_0 + \sum_{i=0}^d \int_0^t V_i(\tau, y_\tau) dW_\tau^i$$

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Pullback to the Lie group II

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Examples:

$\mathcal{G} = \text{Diff}(\mathcal{M})$ & $\Lambda(S, y) = S \circ y \implies y_t = S_t \circ y_0$ with $S_t \equiv \varphi_t$.

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Examples:

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$V_i(t, y) = a_i(t, y) y$ with $a_i \in \mathfrak{so}(n)$, $\mathcal{M} = \mathcal{G} = \text{SO}(n)$:

$\Lambda(S, y) = Sy \implies (\partial_1 \Lambda)(\text{Id}, y) \circ a = ay \implies y_t = S_t y_0$

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Stochastic Taylor Munthe-Kaas method

$$\sigma_t = \sum_{i=0}^d \int_0^t \text{dexp}_{\sigma_\tau}^{-1} \circ a_i(\tau, \Lambda(\exp \sigma_\tau, y_0)) \, dW_\tau^i,$$

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Algorithm:

$$\hat{\sigma}_{t_n, t_{n+1}} = (J_0 v_0 + J_1 v_1 + J_2 v_2 + \frac{1}{2} J_1^2 v_1^2 + J_{12} v_2 v_1 + J_{21} v_1 v_2 + \frac{1}{2} J_2^2 v_2^2) \circ O.$$

Stochastic Taylor Munthe-Kaas method

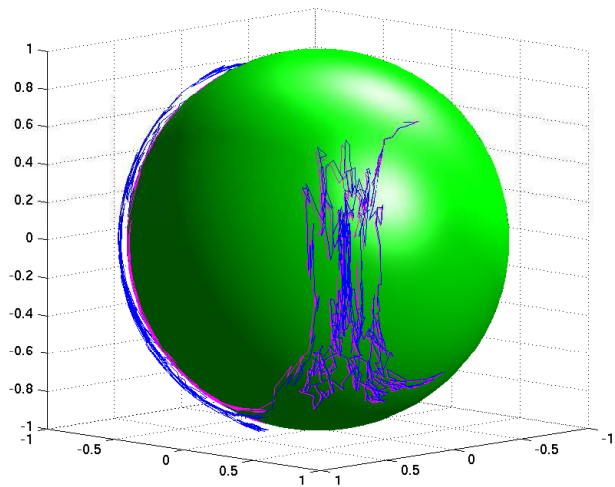
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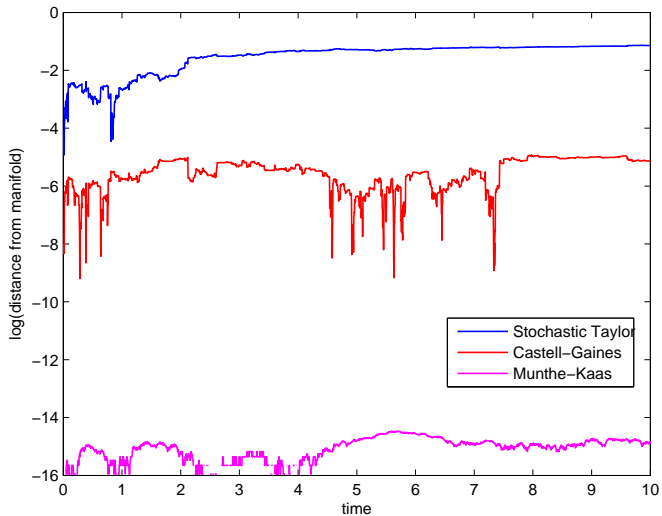
$$\hat{\sigma}_{t_n, t_{n+1}} = (J_0 v_0 + J_1 v_1 + J_2 v_2 + \frac{1}{2} J_1^2 v_1^2 + J_{12} v_2 v_1 + J_{21} v_1 v_2 + \frac{1}{2} J_2^2 v_2^2) \circ O.$$

$$y_{t_{n+1}} \approx \Lambda(\exp \hat{\sigma}_{t_n, t_{n+1}}, y_{t_n})$$

Rigid body



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Conclusions

Take home messages:

For numerical SDE methods that are conditioned on two or more driving Wiener paths, order 1 numerical schemes provide the most accuracy for a given computational effort.

Stochastic Lie group integration methods can preserve the evolution of an SDE solution in a smooth manifold.

Conclusions

Take home messages:

For numerical SDE methods that are conditioned on two or more driving Wiener paths, order 1 numerical schemes provide the most accuracy for a given computational effort.

Stochastic Lie group integration methods can preserve the evolution of an SDE solution in a smooth manifold.

Future directions:

- ▶ Variable step scheme (*Gaines & Lyons 1997*).
- ▶ Zakai equation: Markov chain filters.
- ▶ Backwards SDEs

Global error I

Global interval: $[0, T] = \cup_{n=0}^{N-1} [t_n, t_{n+1}]$, $t_n = nh$.

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Local remainder across $[t_n, t_{n+1}]$:

$$R = \exp(\psi) - \exp(\hat{\psi}) = \exp(\hat{\psi} + \rho) - \exp(\hat{\psi}) = \rho + \mathcal{O}(\psi\rho).$$

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Stochastic Taylor or exponential Lie series:

$$y_{t_n, t_{n+1}} = \hat{y}_{t_n, t_{n+1}} + R_{t_n, t_{n+1}}.$$

Global error II

$$\begin{aligned}\mathcal{E}^2 &\equiv \mathbb{E} \left\| \left(\prod_{n=N-1}^0 \varphi_{t_n, t_{n+1}} - \prod_{n=N-1}^0 \hat{\varphi}_{t_n, t_{n+1}} \right) \circ y_0 \right\|^2 \\ &= \mathbb{E} \left\| \underbrace{\left(\sum_{n=0}^{N-1} \varphi_{t_{n+1}, t_N} R_{t_n, t_{n+1}} \hat{\varphi}_{t_0, t_n} \right)}_{\mathcal{R}} \circ y_0 \right\|^2.\end{aligned}$$

Recall: exponential Lie series (order 1)

$$y_t = \exp(\psi_t) \circ y_0, \quad \psi_t = \psi_{1/2} + \psi_1 + \psi_{3/2} + \cdots$$

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$$\begin{aligned} \psi_{3/2} = & \frac{1}{2}(J_{10} - J_{01})[V_0, V_1] + \frac{1}{2}(J_{20} - J_{02})[V_0, V_2] \\ & + (J_{112} - \frac{1}{2}J_1 J_{12} + \frac{1}{12}J_1^2 J_2) [V_1, [V_1, V_2]] \\ & + (J_{221} - \frac{1}{2}J_2 J_{21} + \frac{1}{12}J_2^2 J_1) [V_2, [V_2, V_1]] \\ & + (J_{110} - \frac{1}{2}J_1 J_{10} + \frac{1}{12}J_1^2 J_0) [V_1, [V_1, V_0]] \\ & + (J_{220} - \frac{1}{2}J_2 J_{20} + \frac{1}{12}J_2^2 J_0) [V_2, [V_2, V_0]]. \end{aligned}$$

Global error III

$$\mathcal{R} = \sum_{n=0}^{N-1} \sum_{\alpha} (\varphi_{t_{n+1}, t_N} V_{\alpha} \varphi_{t_0, t_n} \circ y_0) J_{\alpha}(t_n, t_{n+1}).$$

$$\mathbb{E}(\mathcal{R}^T \mathcal{R})$$

$$\begin{aligned} &= \sum_{n=0}^{N-1} \sum_{\alpha, \beta} \mathbb{E} \left((\varphi_{t_{n+1}, t_N} V_{\alpha} \varphi_{t_0, t_n} \circ y_0)^T (\varphi_{t_{n+1}, t_N} V_{\beta} \varphi_{t_0, t_n} \circ y_0) \right) \\ &\quad \cdot \mathbb{E}(J_{\alpha}(t_n, t_{n+1}) J_{\beta}(t_n, t_{n+1})) \\ &+ \sum_{n \neq m} \sum_{\alpha, \beta} \mathbb{E} \left((\varphi_{t_{n+1}, t_N} V_{\alpha} \varphi_{t_0, t_n} \circ y_0)^T (\varphi_{t_{m+1}, t_N} V_{\beta} \varphi_{t_0, t_m} \circ y_0) \right) \\ &\quad \cdot \mathbb{E}(J_{\alpha}(t_n, t_{n+1})) \mathbb{E}(J_{\beta}(t_m, t_{m+1})). \end{aligned}$$

Stochastic Riccati system

$$S(t) = I + \sum_{i=0}^d \int_0^t f_i(\tau, S(\tau)) dW_i(\tau).$$

$$f_i(t, S) = S(t)A_i(t)S(t) + B_i(t)S(t) + S(t)C_i(t) + D_i(t).$$

- ▶ Stochastic linear-quadratic optimal control.
- ▶ eg. mean-variance hedging in finance (*Bobrovnytska & Schweizer 2004; Kohlmann & Tang 2003*).

Riccati II

If $\mathbb{A}_i(t) \equiv \begin{pmatrix} B_i(t) & D_i(t) \\ -A_i(t) & -C_i(t) \end{pmatrix}$ and $\mathbb{U} = \begin{pmatrix} U \\ V \end{pmatrix}$ satisfies

$$\mathbb{U}(t) = \mathbb{I} + \sum_{i=0}^d \int_0^t \mathbb{A}_i(\tau) \mathbb{U}(\tau) dW_i(\tau),$$

then $S = UV^{-1}$ solves the Riccati system.

$$A_0 = \begin{pmatrix} -1 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}, \quad C_0 = \begin{pmatrix} -\frac{1}{2} & 0 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad D_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix},$$

$$D_1 = a_1 \quad \text{and} \quad D_2 = a_2.$$

Riccati III

Kloeden & Platen:

$$\begin{aligned} S_{n+1} &= S_n + f(S_n)h + D_1 J_1 + D_2 J_2 \\ &+ \frac{h}{4} (f(Y_1^+) + f(Y_1^-) + f(Y_2^+) + f(Y_2^-) - 4f(S_n)) \\ &+ \frac{1}{2\sqrt{h}} ((f(Y_1^+) - f(Y_1^-))J_{10} + (f(Y_2^+) - f(Y_2^-))J_{20}) , \end{aligned}$$

$$Y_j^\pm = S_n + \frac{h}{2} f(S_n) \pm D_j \sqrt{h}$$

$$f(S) = SA_0S + B_0S + SC_0 + D_0 .$$

Riccati IV

