C2

31·08·11

▷ Notation mainly

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u} + \vec{\nabla} p = \nu \Delta \vec{u} \quad , \vec{u}(x,t) = \hat{i}u + \hat{j}v + \hat{k}w = \hat{e}_i u_i$$

$$\vec{\nabla}\cdot\vec{u} = 0$$

$$\vec{x} = \hat{i}x + \hat{j}y + \hat{k}z \in T^3 = [0,L]^3 \text{ periodic}$$

Initial data $\vec{u}_0(x)$ — WLOG mean zero.

Pressure $p(x,t) = -\Delta^{-1}(\vec{\nabla}\cdot(\vec{u}\cdot\vec{\nabla}\vec{u})) = -\Delta^{-1}[(\partial_i u_j)(\partial_j u_i)]$

Fourier representation: $\vec{u}(\vec{x},t) = \sum_{k\neq 0} \hat{u}(\vec{k},t)e^{ikx}$

$$\hat{u}(k,t) = \frac{1}{L^3}\int_{T^3} e^{-ikx}\,\vec{u}(\vec{x},t)\,d^3x \qquad \text{where } \vec{k} = \frac{2\pi}{L}\hat{e}_i n_i$$

$$\vec{\nabla}\cdot\vec{u} = 0 \Longleftrightarrow \vec{k}\cdot\vec{\hat{u}}(\vec{k},t) = 0 \qquad n_i = \dots,-2,-1,0,\dots$$

$$\|\vec{u}(\cdot,t)\|_2^2 = \int_{T^3} |\vec{u}(\vec{x},t)|^2 d^3x = L^3 \sum_k |\vec{\hat{u}}(\vec{k},t)|^2$$

$$\|\vec{\nabla}\vec{u}\|_2^2 = \int |\vec{\nabla}\vec{u}|^2 d^3x = \int (\partial_i u_j)(\partial_j u_i)\,d^3x$$

$$= L^3 \sum_k |k|^2 |\hat{u}(\vec{k},t)|.$$

▷ Galerkin $\quad \mathbb{P}^N(f)(\vec{x},t) = \sum_{|k|\leq N} e^{i\vec{k}\cdot\vec{x}} f(\vec{k},t).$

Then the Galerkin method to the NSE is

$$\frac{d\vec{u}^N}{dt} + \mathbb{P}^N(\vec{u}^N\cdot\vec{\nabla}\vec{u}^N) + \vec{\nabla}p^N = \nu\vec{\nabla}^2 u^N, \text{ where } \mathbb{P}^N(\vec{u}^N) = \vec{u}^N$$

$$\vec{\nabla}\cdot\vec{u}^N = 0$$

$(\mathbb{P}^N)^2 = \mathbb{P}^N \qquad [\mathbb{P}^N, \vec{\nabla}] = 0 \qquad$ this is a significant simplification.

Also $\mathbb{P}^N = (\mathbb{P}^N)^+$

If we now dot u^N with the Galerkin approximation to the NSE, then we have

$$u^N \cdot \text{(GANSE)}$$

$$\frac{1}{2}\frac{d}{dt}\|\vec{u}^N\|_2^2 + \int \underbrace{\vec{u}^N \cdot \mathbb{P}^N(\vec{u}^N \cdot \vec{\nabla}\vec{u}^N)}\, d^3x$$

$$\int \vec{u}^N \cdot (\vec{u}^N \cdot \vec{\nabla}\vec{u}^N)$$

$$\int \vec{\nabla} \cdot \left(\vec{u}^N \tfrac{1}{2}|\vec{u}^N|^2\right) \leftarrow \text{dies.}$$

and therefore

$$\frac{1}{2}\frac{d}{dt}\|\vec{u}^N\|_2^2 + \int \vec{u}^N \cdot \mathbb{P}^N(\vec{u}^N \cdot \vec{\nabla}\vec{u}^N)\, d^3x = -\nu\|\vec{\nabla}\vec{u}^N\|_2^2$$

we have global in time smooth solutions.

▷ Fact A: Given $\vec{u}_0(x)$, $\|\vec{\nabla}u_0\|_2^2 = E_0 < \infty$

This says that given a velocity, if the enstrophy is finite then there...

$\exists\, T > 0$ and $E(T) < \infty$ so that for $t \in (0,T)$ $\|\vec{\nabla}\vec{u}^N(\cdot,t)\|_2^2 < E(T)$ uniformly in N.

▷ Fact B: Given u_0 with finite enstrophy ($\|\vec{\nabla}u_0\|_2^2 = E_0 < \infty$) and any $\alpha > 0$, then $\exists\, T(\alpha) > 0$ and $g(T(\alpha)) < \infty$ so that for $t \in (0, T(\alpha))$

$$\|e^{\alpha|\vec{\nabla}|t}\vec{\nabla}\vec{u}^N(\cdot,t)\|_2^2 < G(T(\alpha)) \text{ uniformly in } N.$$

$$(|\vec{\nabla}|\vec{u})(\vec{x}) = \sum_{\vec{k}} |\vec{k}|\,\hat{\vec{u}}(\vec{k})\,e^{i\vec{k}x}$$

then

$$e^{\alpha|\vec{\nabla}|t}(|\vec{\nabla}|\vec{u})(\vec{x}) = \sum_{\vec{k}} |\vec{k}|\,\hat{\vec{u}}(\vec{k})\,\underbrace{e^{i\vec{k}x}\,e^{\alpha|\vec{\nabla}|t}}_{\text{this becomes}}$$

$$\|e^{\alpha|\vec{\nabla}|t}\vec{u}(\cdot,t)\|_2^2 = \sum_{\vec{k}} |\vec{k}|^2 e^{2\alpha|\vec{k}|t}\,|\hat{\vec{u}}(\vec{k},t)|^2$$

This kind of regularity on a function is called Gevrey regularity.

We would like to prove fact A and fact B.

▷ Fact C : (follows from fact A) multiplied by.
$\exists$ $c>0$ s.t given $\|\vec{u}_0\|_2 \times \|\vec{\nabla} u_0\|_2 < C\nu^2$ then $\vec{u}^k_k(\cdot,t) \in C^\infty$ with uniform in N norms $\forall t>0$.

▷ Fact C' : If $\||\vec{\nabla}|^{1/2} u_0\|_2 = \|\vec{u}\|_{\dot{H}^{1/2}} < c'\nu$, then the enstrophy and all other norms are bounded uniformly in N for all time.

▷ Fact D : Analysis leads to fact A "cannot be improved". i.e the estimates that go into proving fact A are saturated.

Here we will define $K^N(t) = \frac{1}{2}\|\vec{u}^N(\cdot,t)\|_2^2$ as the natural kinetic energy and the enstrophy is defined as $E(t) = \|\vec{\nabla}\vec{u}^N\|_2^2 = \|\vec{\omega}^N\|_2^2$.

Now we will go ahead with proving fact A.

~~▷ Fact A proof:~~

Note: Poincaré : $E^N(t) = L^3 \sum_{k\neq 0} |\vec{k}|^2 |\hat{\vec{u}}(\vec{k},t)|^2 \geq \frac{4\pi^2}{L^2} K^N(t)$

and Energy eq^n: $\frac{d}{dt} K^N = -\nu E^N$

Enstrophy eq^n: $\frac{d}{dt} \epsilon^N = -2\nu\|\vec{\nabla}\vec{\omega}^N\|_2^2 + 2\int \vec{\omega}^N \cdot (\vec{\nabla}\vec{u}^N)\cdot\vec{\omega}^N$

$$= -2\nu\|\Delta\vec{u}^N\|_2^2 + 2\int \vec{u}^N (\vec{\nabla}\vec{u}^N)\cdot\Delta\vec{u}^N$$

Now we will curl the Navier Stokes on the Galerkin appox.

this is the vorticity equation generated by Galerkin approximation

We will get

$$\underbrace{\frac{d\vec{w}^N}{dt} + \mathbb{P}^N(\vec{u}^N \cdot \vec{\nabla}\vec{w}^N) = \nu \vec{\nabla}^2 \vec{w}^N \cdot \vec{\nabla}\vec{u}^N}$$

~~If we dot this equation with $\vec{w}^N$, we will get~~

Also if we take NS and $(NS)\cdot(-\vec{\nabla}^2\vec{u}^N)$ (the Laplacian).
These operations give us terms in the enstrophy eq^n given previously.
Now we will work with the enstrophy eq^n but drop the N.

$$\frac{d}{dt}E = -2\nu \|\vec{\nabla}\vec{u}\|_2^2 + 2\int \vec{u}\cdot\vec{\nabla}\vec{u} \cdot \vec{\nabla}^2\vec{u}$$

Now we want to consider the non-linear term; i.e,
$\left|\int \vec{u}\cdot\vec{\nabla}\vec{u}\cdot\vec{\nabla}^2\vec{u}\right|$

Note: the term $\vec{w}^N\cdot\vec{\nabla}\vec{u}^N\cdot\vec{w}^N$. If we consider the symmetric part of this then we know that the trace is zero as $\vec{\nabla}\cdot\vec{u}=0$ (divergence free). Thus the real part of eigenvalues ~~some~~ will be zero. Some $\lambda<0$, some $\lambda>0$ and that will tell us if the vortex is stretching or contracting.

$$\left|\int \vec{u}\cdot\vec{\nabla}\vec{u}\cdot\vec{\nabla}\vec{u}^2\right| \le \|\vec{u}\|^\infty \underbrace{\|\vec{\nabla}\vec{u}\|_2}_{(\sim E^{1/2})} \|\vec{\nabla}^2\vec{u}\|_2 \quad \text{(due to Hölder).}$$

Fact: $\|\vec{u}\|_\infty \le \|\vec{\nabla}\vec{u}\|_2^{1/2} \|\vec{\nabla}^2\vec{u}\|_2^{1/2}$ in three dimensions (3D)

▷ Proof: $|u(x)| = \left|\sum_{\vec{k}\neq 0} e^{i\vec{k}\vec{x}}\hat{\vec{u}}(\vec{k})\right| \le \sum_{k\neq 0} |\hat{\vec{u}}(\vec{k})|$

$$= \sum_{0\neq \vec{k}\le\Lambda} \frac{|\vec{k}|}{|\vec{k}|}|\hat{u}(\vec{k})| + \sum_{k>\Lambda} \frac{|\vec{k}|^2}{|\vec{k}|^2}\hat{\vec{u}}(\vec{k})$$

$$\leq \left(\sum_{|\vec{k}|\leq\Lambda} \frac{1}{|\vec{k}|^2}\right)^{1/2} \left(\sum_{|\vec{k}|\leq\Lambda} |\vec{k}|^2 |\hat{\vec{u}}(\vec{k})|^2\right)^{1/2} + \left(\sum_{|\vec{k}|>\Lambda} \frac{1}{|\vec{k}|^2}\right)^{1/2} \left(\sum_{|\vec{k}|>\Lambda} |\vec{k}|^4 |\hat{\vec{u}}(\vec{k})|^2\right)^{1/2}$$

$$\sum_{0<|\vec{k}|\leq\Lambda} \frac{1}{|\vec{k}|^2}\left(\frac{2\pi}{L}\right)^3\left(\frac{L}{2\pi}\right)^3 = \left(\frac{L}{2\pi}\right)^3 \int_{2\pi/L}^{\Lambda} \frac{4\pi k^2\,dk}{k^2} \leq \left(\frac{L}{2\pi}\right)^3 4\pi\Lambda$$

$$\sum_{|\vec{k}|>\Lambda} \frac{1}{|\vec{k}|^4}\left(\frac{2\pi}{L}\right)^3\left(\frac{L}{2\pi}\right)^3 = \left(\frac{L}{2\pi}\right)^3 4\pi \int_{\Lambda}^{\infty} \frac{k^2\,dk}{k^4} \leq \left(\frac{L}{2\pi}\right)^3 4\pi \frac{1}{\Lambda}$$

this calculation tells us that ~~eac~~ at each point:

$$|\vec{u}(\vec{x})| \leq \left(\frac{L}{2\pi}\right)^{3/2} \cdot (4\pi)^{1/2} \Lambda^{1/2} \|\vec{\nabla}\vec{u}\|_2 + \left(\frac{L}{2\pi}\right)^{3/2} (4\pi)^{1/2} \frac{1}{\Lambda^{1/2}} \|\vec{\nabla}^2\vec{u}\|_2$$

Now choose $\Lambda^{1/2} = \dfrac{\|\vec{\nabla}^2\vec{u}\|_2^{1/2}}{\|\vec{\nabla}\vec{u}\|_2^{1/2}}$

$$\leq 2\left(\frac{4\pi}{(2\pi)^3}\right)^{1/2} \|\vec{\nabla}\vec{u}\|_2^{1/2} \|\vec{\nabla}^2\vec{u}\|_2^{1/2}$$

$$= \left(\frac{2}{\pi^2}\right)^{1/2} \|\vec{\nabla}\vec{u}\|_2^{1/2} \|\vec{\nabla}^2\vec{u}\|_2^{1/2}$$

01 · 09 · 2011

Trying to do something with Fact C':

$$\left(\sum_{\vec{k}\neq 0} |\vec{k}|\,|\hat{u}(\vec{k})|^2\right)^{1/2} \leq E^{1/4}(2K)^{1/4}$$

$(|\vec{\nabla}|\vec{u}) \cdot NSE$:

$$\frac{d}{dt}\frac{1}{2}\||\vec{\nabla}|^{1/2}u_0\|_2^2 + \int (|\vec{\nabla}|\vec{u})\cdot(\vec{u}\cdot\vec{\nabla}\vec{u}) = -\nu\||\vec{\nabla}|^{3/2}u\|_2^2$$

$$\int (\vec{u}\cdot\vec{\nabla}\vec{u})\cdot(|\vec{\nabla}|\vec{u}) \leq \|u\|_{L^3} \|\vec{\nabla}\vec{u}\|_{L^3}^2$$

Sobolev 3D says 3-d : $\|f\|_{L^3} \leq c\||\vec{\nabla}|^{1/2}f\|_{L^2}$

then we can use this to obtain:

$$\int (\vec{u}\cdot\vec{\nabla}\vec{u})\cdot(|\vec{\nabla}|\vec{u}) \leq \| |\vec{\nabla}|^{1/2}\vec{u}\|_2 \, \| |\vec{\nabla}|^{3/2}\vec{u}\|_2^2$$

then

$$\frac{d}{dt}\frac{1}{2}\| |\vec{\nabla}|^{1/2}\vec{u}\|_2^2 + \left(\nu - c\| |\vec{\nabla}|^{1/2}\vec{u}\|_2\right)\| |\vec{\nabla}|^{3/2}\vec{u}\|_2^2 \leq 0$$

Again, we will write out the enstrophy equation

$$\frac{d}{dt}\frac{1}{2}\|\vec{\nabla}\vec{u}\|_2^2 = -\nu\|\vec{\nabla}^2\vec{u}\|_2^2 + \int \underbrace{\vec{u}}_{L^3}\cdot\underbrace{(\vec{\nabla}\vec{u})}_{L^6}\cdot\underbrace{\vec{\nabla}^2\vec{u}}_{L^2}$$

use these norms for the terms above.

$$\leq -\left(\nu - \tilde{\tilde{c}}\,\| |\vec{\nabla}|^{1/2}\vec{u}\|_2\right)\|\Delta\vec{u}\|_2^2 .$$

And I think that was supposed to show something:
This is said to be the 'cutting-edge' of knowledge.

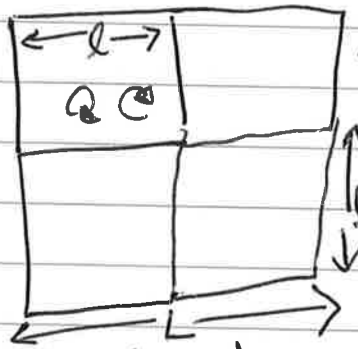
C2

02.09.2011

~~Th~~ we have the Navier Stokes:

$$\frac{d\vec{u}}{dt} + \vec{u}\cdot\vec{\nabla}\vec{u} + \vec{\nabla}p = \nu\vec{\nabla}^2\vec{u} + \vec{f}(x) \quad , \quad \vec{\nabla}\cdot\vec{u} = 0$$

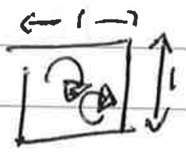
where $\vec{f}$ is some forcing term, in any terms of space.



periodic cells

we should consider velocity length scales that are not necessarily periodic.

$$\frac{L}{l} = \alpha \text{ (integer)}$$



let $\vec{f}(\vec{x}) = F\,\vec{\Phi}(l^{-1}x)$

Φ lives on the unit taurus.

e.g $\|\Phi\|_{L_2^2}([0,1]^3) = 1$

if $f(x)$ is square integrable then the ~~fun~~ kinetic energy must remain bounded.

we take the average of NS we obtain

$$\frac{d}{dt}\frac{1}{2}\|\vec{u}\|_2^2 = -\nu\|\vec{\nabla}\vec{u}\|_2^2 + \int \vec{u}\cdot\vec{f} \qquad *$$

Suppose f is in L_2. $(\vec{f} \in L_2)$

by Poincare (periodic function)

$$\Rightarrow \int_{(0,L)^3} \vec{f} = 0, \quad \int_{(0,L)^3} u_0 = 0, \quad \int_{(0,L)^2} \vec{u} = 0$$

then

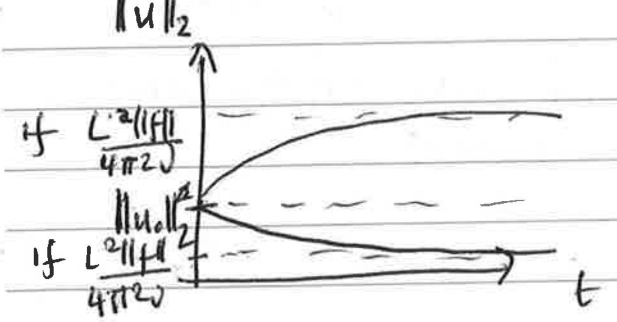
$$\frac{d}{dt}\frac{1}{2}\|\vec{u}\|_2^2 \leq -\frac{4\pi^2}{L^2}\nu\|\vec{u}\|_2^2 + \|u\|_2\|f\|_2$$

↑ by Cauchy-Schwartz

$$\cancel{\|u\|_2}\frac{d}{dt}\|u\|_2 \leq -\frac{4\pi^2}{L^2}\nu\|\vec{u}\|_2^{\cancel{2}} + \cancel{\|u\|_2}\|f\|_2$$

by Gronwalls inequality:

$$\|u(\cdot,t)\|_2 \leq \|u_0\|_2 e^{-\frac{4\pi^2\nu}{L^2}t} + \|f\|_2 \frac{\left(1-e^{-\frac{4\pi^2}{L^2}\nu t}\right)}{\frac{4\pi\nu^2}{L^2}}$$



going back to (*) we can now take time-averaging. we get

$$\frac{1}{T}\int_0^T dt\left(\frac{d}{dt}\frac{1}{2}\|\vec{u}\|_2^2\right) = -\nu\frac{1}{T}\int_0^T dt\,\|\vec{\nabla}u(\cdot,t)\|_2^2 + \frac{1}{T}\int_0^T dt\, f(x)\cdot\vec{u}(x,t)$$

$$\frac{1}{T}\left(\frac{1}{2}\|u(\cdot,T)\|_2^2 - \frac{1}{2}\|u_0\|_2^2\right)$$

by fundamental theorem of calculus.

But as $T \to \infty$, that term $\to 0$.
This is conservation of energy.

$$\langle F \rangle_T = \frac{1}{T}\int_0^T dt \, \frac{1}{L^3}\int_{(0,L)^3} d^3x \, F(x,t).$$

This equation says that

$$\langle \nu |\vec{\nabla} u|^2 \rangle_T = \langle \vec{f} \cdot \vec{u} \rangle_T + O\left(\frac{1}{T}\right)$$

$$\langle F \rangle = \lim_{T \to \infty} \langle F \rangle_T$$ (we could always say it goes to the supremum limit).

Now we define:
$$\boxed{\varepsilon = \langle \nu |\vec{\nabla} \vec{u}|^2 \rangle}$$
the rate power is converted to heat

force x velocity ~~is the~~
is the force going out

these equal.

Given $\varepsilon = \langle \nu |\vec{\nabla} \vec{u}|^2 \rangle = \langle \vec{f} \cdot \vec{u} \rangle \leq \langle |\vec{f}|^2 \rangle^{1/2} \langle |u|^2 \rangle^{1/2}$

and also let $U = \langle |u|^2 \rangle^{1/2}$

$\Rightarrow$

$$= F \langle |\Phi|^2 \rangle^{1/2} \, U$$

but we want to get rid of the 'F'. If we go back to NS:

$$\frac{d\vec{u}}{dt} + \underset{(FU^2)}{\vec{u} \cdot \vec{\nabla} u} + \vec{\nabla} p = \underset{FU}{\nu \vec{\nabla}^2 \vec{u}} + \underset{F^2}{\vec{f}}$$ (looking at powers)

we want to dot this now with $\vec{f}$, where $\vec{\nabla} \cdot \vec{f} = 0$.
and we get

The time average of the time-derivative vanishes.

~~Now we~~ But we should multiply NS by $(-\Delta^{-2}\vec{f})$ where

$$-\Delta^{-2} f = \sum_{\vec{k}\neq 0} e^{i\vec{k}\cdot\vec{x}} \frac{\hat{f}(\vec{k})}{|k|^4}$$

But we need to do a precalculation

$$f(x) = F\,\vec{\Phi}(\ell^{-1}x)$$

$$\Rightarrow \langle |\vec{f}|^2 \rangle = F^2 \langle |\Phi|^2 \rangle$$

Δ^{-2} divergence free

$$\cancel{\vec{\nabla}^a \vec{f} = } \quad \vec{\nabla}^{a-1}\vec{f} = \frac{F^2}{\ell^{2a}} \langle |\vec{\nabla}^a \Phi|^2 \rangle \leftarrow ☺$$

(for scaling purposes)

$$\langle \Delta^{-2}\vec{f} \cdot (NS) \rangle \to 0$$

$$\overset{①}{\cancel{\left\langle (\Delta^{-2}\vec{f}) \cdot \frac{d\vec{u}}{dt} \right\rangle}} + \overset{②}{\left\langle (\Delta^{-2}\vec{f}) \cdot (\vec{u} \cdot \vec{\nabla}\vec{u}) \right\rangle} = -\nu \overset{③}{\langle (\Delta^{-2}\vec{f}) \cdot (\Delta\vec{u}) \rangle} + \overset{④}{\langle (\Delta^{-2}\vec{f}) \cdot \vec{f} \rangle}$$

$\vec{\nabla}\cdot\vec{u}\vec{u}$

integrate by parts

from integration by parts

$$\overset{④}{\langle |\Delta^{-1}\vec{f}|^2 \rangle} = \overset{③}{\nu \langle (\Delta^{-1}\vec{f}) \cdot \vec{u} \rangle} - \overset{②}{\langle (\vec{\nabla}\Delta^{-2}\vec{f}) : (\vec{u}\vec{u}) \rangle}$$

Now from ☺ and by Cauchy-Schwartz:

$$F^2 \ell^4 \langle |\Delta^{-1}\Phi|^2 \rangle \leq \nu \langle |\Delta^{-1}\vec{f}|^2 \rangle^{1/2} + \sup_x |\vec{\nabla}\Delta^{-2}\vec{f}|\, U^2$$

$$\leq \nu F \ell^2 \langle |\tilde{\Delta}^{-1}\vec{\Phi}|^2 \rangle^{1/2} U + \|\Delta^{-1}\vec{f}\|^{1/2}_{L^2(0,\ell)^3} \|\vec{\nabla}^{-1}\vec{f}\|^{1/2}_{L^2(0,\ell)^3} U$$

$$= \nu F \ell^2 \langle |\tilde{\Delta}^{-1}\Phi|^2 \rangle^{1/2} U + \ell^3 F \langle |\tilde{\Delta}^{-1}\Phi|^2 \rangle^{1/4} \langle |\hat{\nabla}^{-1}\Phi|^2 \rangle^{1/4} U^2$$

Exercise 2: use Poincaré inequality and discover lower bound on ε, i.e,

$$\frac{\varepsilon \ell}{U^3} \geq \frac{\cancel{4}\pi^2}{Re} \cdot \frac{1}{\alpha}$$ where α is aspect ratio.

Note that the amplitude of the force is bounded from below, i.e.

$$F \leq \frac{\nu}{\ell^2} \frac{\langle |\Delta^{-1} \Phi|^2 \rangle^{1/2} U}{\langle |\Delta^{-1} \Phi|^2 \rangle} + \frac{U^2}{\ell} \frac{\langle |\Delta^{-1} \Phi|^2 \rangle^{1/4} \langle |\vec{\nabla}^{-1} \Phi|^2 \rangle^{1/4}}{\langle |\Delta^{-1} \Phi|^2 \rangle}$$

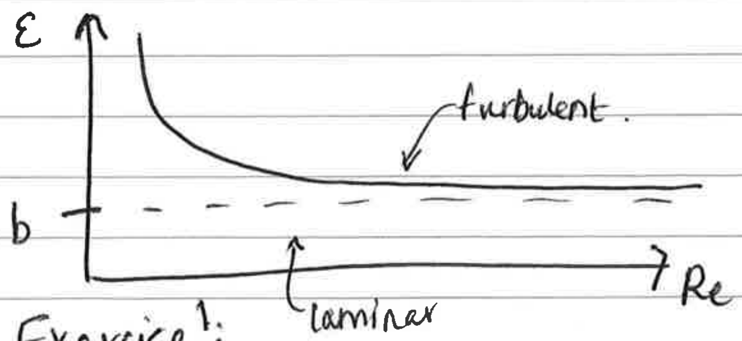
Remember that
$$\varepsilon \leq F U \langle |\Phi|^2 \rangle^{1/2} \leq a \frac{\nu U^2}{\ell} + b \frac{U^3}{\ell}$$

$$\frac{\varepsilon \ell}{U^3} \leq \frac{a \nu}{U \ell} + b = \frac{a}{Re} + b$$

where coefficients a and b are shape ~~con~~ functions in the form of

$$a = \frac{\langle |\Phi|^2 \rangle^{1/2}}{\langle |\Delta^{-1} \Phi|^2 \rangle^{1/2}}$$

$$b = \frac{\langle |\vec{\nabla}^{-1} \Phi|^2 \rangle^{1/4} \langle |\phi|^2 \rangle^{1/2}}{\langle |\Delta^{-1} \Phi|^2 \rangle^{3/4}}$$



this only depends on the shape of the forcing function.

Exercise 1:

$$\lambda^2 = \frac{U^2 \nu}{\varepsilon}, \quad R_\lambda = \frac{U \lambda}{\nu}$$

$$\frac{\varepsilon \ell}{U^3} \leq \frac{a}{Re} + b \Rightarrow \frac{\varepsilon \ell}{U^3} \leq \frac{b}{2}\left(1 + \sqrt{1 + \frac{4a^2}{b^2 R_\lambda^2}}\right)$$

Doering and Foias 2002.
periodic.