Navier–Stokes equations tutorial solutions

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We consider the incompressible Navier–Stokes equations on a bounded domain $\Omega \subset \mathbb{R}^d$ with viscous boundary conditions. Our goal is to derive some basic energy estimates which assume that the solutions are smooth, but which indicate properties we might expect weak/strong solutions to satisfy. For example weak solutions satisfy an energy inequality.

Question 1 (a) To show that the Navier–Stokes equations can be expressed in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{u} = \nu\, \Delta \boldsymbol{u} - \nabla\big(p + \tfrac{1}{2}|\boldsymbol{u}|^2\big) + \boldsymbol{f},$$

we simply substitute the identity

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \tfrac{1}{2}\nabla\big(|\boldsymbol{u}|^2\big) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u})$$

into the standard formulation of the Navier–Stokes equations.

(b) There are several ways to derive the energy inequality for the forced Navier–Stokes equations, one of which is to directly compute $(\mathrm{d}/\mathrm{d}t)\|\boldsymbol{u}\|_{L^2}^2$ as follows:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|\boldsymbol{u}\|_{L^2}^2 &= \tfrac{1}{2}\int_\Omega \partial_t(\boldsymbol{u}\cdot\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} \\
&= \tfrac{1}{2}\int_\Omega 2\boldsymbol{u}\cdot(\partial_t \boldsymbol{u})\,\mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \boldsymbol{u}\cdot(-\boldsymbol{\omega}\times\boldsymbol{u} + \nu\,\Delta\boldsymbol{u} - \nabla\big(p + \tfrac{1}{2}|\boldsymbol{u}|^2\big) + \boldsymbol{f})\,\mathrm{d}\boldsymbol{x} \\
&= -\nu\int_\Omega |\nabla \boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x} + \int_\Omega \boldsymbol{u}\cdot\boldsymbol{f}\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

In the last step we have used that $\boldsymbol{u}\cdot(\boldsymbol{\omega}\times\boldsymbol{u}) \equiv 0$, and that

$$\int_\Omega \boldsymbol{u}\cdot\nabla\big(p+\tfrac{1}{2}|\boldsymbol{u}|^2\big)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \nabla\cdot\Big(\boldsymbol{u}\,\big(p+\tfrac{1}{2}|\boldsymbol{u}|^2\big)\Big)\,\mathrm{d}\boldsymbol{x} - \int_\Omega (\nabla\cdot\boldsymbol{u})\big(p+\tfrac{1}{2}|\boldsymbol{u}|^2\big)\,\mathrm{d}\boldsymbol{x}$$

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and

$$\int_\Omega \boldsymbol{u}\cdot(\Delta\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} = \int_\Omega \nabla\cdot\big((\nabla\boldsymbol{u})\cdot\boldsymbol{u}\big)\,\mathrm{d}\boldsymbol{x} - \int_\Omega |\nabla\boldsymbol{u}|^2\,\mathrm{d}\boldsymbol{x}.$$

Note that in these last two integral identites we simply use the product formulae for the vector fields indicated and thus these are also just 'integration by parts' formulae. The divergence terms on the right, by the divergence theorem, generate surface integrals on $\partial\Omega$, the bounding surface of Ω. Since $\boldsymbol{u}=\boldsymbol{0}$ on $\partial\Omega$ due to the viscous boundary conditions assumed, these terms are zero. Of course, we are also assuming incompressibility $\nabla\cdot\boldsymbol{u}=0$ everywhere. Further note, to be clear, we define

$$\int_\Omega |\nabla\boldsymbol{u}|\,\mathrm{d}\boldsymbol{x} = \sum_{i,j=1}^{2}\int_\Omega (\partial_{x_j}u_i)(\partial_{x_j}u_i)\,\mathrm{d}\boldsymbol{x}.$$

Now using the Hölder and Young inequalities we see that

$$\langle\boldsymbol{u},\boldsymbol{f}\rangle_{L^2} \leqslant \|\boldsymbol{u}\|_{L^2}\,\|\boldsymbol{f}\|_{L^2} \leqslant \frac{\delta}{2}\|\boldsymbol{u}\|_{L^2}^2 + \frac{1}{2\delta}\|\boldsymbol{f}\|_{L^2}^2,$$

for some $\delta>0$. Combining these results gives us the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_{L^2}^2 + 2\nu\|\nabla\boldsymbol{u}\|_{L^2}^2 \leqslant \delta\|\boldsymbol{u}\|_{L^2}^2 + \frac{1}{\delta}\|\boldsymbol{f}\|_{L^2}^2.$$

(c) To derive a uniform upper bound for $\|\boldsymbol{u}\|_{L^2}^2$ in time, we can set $\delta=\nu$ and use Poincaré's inequality

$$\|\boldsymbol{u}\|_{L^2} \leqslant c\,\|\nabla\boldsymbol{u}\|_{L^2}$$

for some constant c (note that if $\boldsymbol{u}(\boldsymbol{x},0)$ has mean zero then $\boldsymbol{u}(\boldsymbol{x},t)$ also has mean zero, for $t>0$). Using this in the evolution inequality above, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_{L^2}^2 + \frac{\nu}{c^2}\|\boldsymbol{u}\|_{L^2}^2 \leqslant \frac{1}{\nu}\|\boldsymbol{f}\|_{L^2}^2.$$

Now we can integrate this time differential inequality, more rigorously we apply the Gronwall lemma, to get (using the integrating factor technique for linear differential equations, i.e. variation of constants formula):

$$\|\boldsymbol{u}(\cdot,t)\|_{L^2}^2 \leqslant \|\boldsymbol{u}(\cdot,0)\|_{L^2}^2\exp\big(-\nu t/c^2\big) + \Big(\frac{c^2}{\nu}\Big)^2\|\boldsymbol{f}(\cdot,t)\|_{L^2}^2\Big(1-\exp\big(-\nu t/c^2\big)\Big).$$

This establishes for any time $T>0$, we know

$$\boldsymbol{u}\in L^\infty\big([0,T];L^2(\Omega;\mathbb{R}^d)\big).$$

(d) Now note if we simply time integrate the time differential inequality (before we applied the Poincaré inequality) we get

$$\|\boldsymbol{u}(\cdot,t)\|_{L^2}^2 + \frac{\nu}{c^2}\int_0^T\|\nabla\boldsymbol{u}(\cdot,\tau)\|_{L^2}^2\,\mathrm{d}\tau \leqslant \nu\int_0^T\|\boldsymbol{u}(\cdot,\tau)\|_{L^2}^2\,\mathrm{d}\tau + \frac{1}{\nu}\int_0^T\|\boldsymbol{f}(\cdot,\tau)\|_{L^2}^2\,\mathrm{d}\tau.$$

From this we can deduce for any time $T>0$, we have

$$\boldsymbol{u}\in L^2\big([0,T];H^1(\Omega;\mathbb{R}^d)\big).$$

Question 2 We consider the L^2-inner product of '$-\Delta \boldsymbol{u}$' with the Navier–Stokes formulation quoted above, this generates:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}\|_{L^2}^2 + \nu\|\Delta \boldsymbol{u}\|_{L^2}^2 = \langle \Delta \boldsymbol{u}, \boldsymbol{\omega}\times\boldsymbol{u}\rangle_{L^2} - \langle \Delta \boldsymbol{u}, \boldsymbol{f}\rangle_{L^2}.$$

Note that for the time derivative term, we have used the product formula again (integration by parts) and that $\boldsymbol{u} = \boldsymbol{0}$ on the boundary $\partial\Omega$ so that $\partial_t \boldsymbol{u} = \boldsymbol{0}$ on the boundary as well. Now consider the term generated from the nonlinear term in the Navier–Stokes equations. Since $-\Delta \boldsymbol{u} \equiv \nabla\times(\nabla\times\boldsymbol{u})$ for divergence free fields $\boldsymbol{u}$ we see that

$$\begin{aligned}\langle \Delta \boldsymbol{u}, \boldsymbol{\omega}\times\boldsymbol{u}\rangle_{L^2} &= -\langle \nabla\times\boldsymbol{\omega}, \boldsymbol{\omega}\times\boldsymbol{u}\rangle_{L^2}\\ &= \langle \boldsymbol{\omega}, \nabla\times(\boldsymbol{\omega}\times\boldsymbol{u})\rangle_{L^2}\\ &= \langle \boldsymbol{\omega}, \boldsymbol{u}\cdot\nabla\boldsymbol{\omega}\rangle_{L^2} + \langle \boldsymbol{\omega}, \boldsymbol{\omega}\cdot\nabla\boldsymbol{u}\rangle_{L^2}\\ &= \langle \boldsymbol{\omega}, D\boldsymbol{\omega}\rangle_{L^2},\end{aligned}$$

where D is the deformation matrix (the symmetric part of $\nabla\boldsymbol{u}$). In this sequence of equalities, we have used the product formula

$$\langle \nabla\times\boldsymbol{\omega}, \boldsymbol{v}\rangle_{L^2} = \int_\Omega \nabla\cdot\big(\boldsymbol{\omega}\times\boldsymbol{v}\big)\,\mathrm{d}\boldsymbol{x} - \langle \boldsymbol{\omega}, \nabla\times\boldsymbol{v}\rangle_{L^2}$$

with $\boldsymbol{v} = \boldsymbol{\omega}\times\boldsymbol{u}$ together with the divergence theorem and that $\boldsymbol{u} = \boldsymbol{0}$ on the boundary. We have also used that for divergence free fields $\boldsymbol{u}$, with $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$, we have

$$\nabla\times(\boldsymbol{\omega}\times\boldsymbol{u}) = \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\nabla\boldsymbol{u},$$

and that $\boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = D\boldsymbol{\omega}$ as $R\boldsymbol{\omega} \equiv \boldsymbol{0}$ (where R is the antisymmetric part of $\nabla\boldsymbol{u}$) and that

$$\langle \boldsymbol{\omega}, \boldsymbol{u}\cdot\nabla\boldsymbol{\omega}\rangle_{L^2} = \int_\Omega \nabla\cdot\big(\tfrac{1}{2}\boldsymbol{u}|\boldsymbol{\omega}|^2\big)\,\mathrm{d}\boldsymbol{x} = 0.$$

Putting all this together, we arrive at the equality:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}\|_{L^2}^2 + \nu\|\Delta \boldsymbol{u}\|_{L^2}^2 = \langle \boldsymbol{\omega}, D\boldsymbol{\omega}\rangle_{L^2} - \langle \Delta \boldsymbol{u}, \boldsymbol{f}\rangle_{L^2}.$$

(a) Assume $d = 2$. Then direct calculation shows that $D\boldsymbol{\omega} = 0$ and thus, using the Hölder and then Young inequalities

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}\|_{L^2}^2 + \nu\|\Delta \boldsymbol{u}\|_{L^2}^2 \leqslant \frac{\delta}{2}\|\Delta \boldsymbol{u}\|_{L^2}^2 + \frac{1}{2\delta}\|\boldsymbol{f}\|_{L^2}^2,$$

for some $\delta > 0$ (exactly as we did for the energy inequality above). Now choose $\delta = \nu$. Hence we can deduce for any $T > 0$, we have

$$\boldsymbol{u} \in L^\infty\big([0,T]; H^1(\Omega;\mathbb{R}^d)\big).$$

Note that for any divergence free vector field $\boldsymbol{u}$ we have

$$\|\nabla \boldsymbol{u}\|_{L^2}^2 = \|\boldsymbol{\omega}\|_{L^2}^2 = 2\int_\Omega \mathrm{tr}\big(D^2\big)\,\mathrm{d}\boldsymbol{x}.$$

(b) Now assume $d = 3$. The vorticity stretching term $D\boldsymbol{\omega}$ is now an important mechanism of the flow. Using the Hölder inequality, followed by the Gagliardo–Sobolev–Nirenberg inequality

$$\|\nabla \boldsymbol{u}\|_{L^4} \leqslant c\,\|\Delta \boldsymbol{u}\|_{L^2}^{3/4}\|\nabla \boldsymbol{u}\|_{L^2}^{1/4},$$

and then the Young inequality, we see that

$$\begin{aligned}\langle \boldsymbol{\omega}, D\boldsymbol{\omega}\rangle_{L^2} &\leqslant c\,\|\nabla \boldsymbol{u}\|_{L^2}\|\nabla \boldsymbol{u}\|_{L^4}^2\\ &\leqslant c\,\|\nabla \boldsymbol{u}\|_{L^2}\big(\|\Delta \boldsymbol{u}\|_{L^2}^{3/4}\|\nabla \boldsymbol{u}\|_{L^2}^{1/4}\big)^2\\ &= c\,\|\Delta \boldsymbol{u}\|_{L^2}^{3/2}\|\nabla \boldsymbol{u}\|_{L^2}^{3/2}\\ &\leqslant \frac{3}{4}\nu\|\Delta \boldsymbol{u}\|_{L^2}^2 + \frac{c}{4\nu^3}\big(\|\nabla \boldsymbol{u}\|_{L^2}^2\big)^3.\end{aligned}$$

Inserting this into the equality we have above and using similar estimates for the forcing term (now use $\delta = \nu/4$) we see for some constant c, we have

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}\|_{L^2}^2 + \frac{\nu}{8}\|\Delta \boldsymbol{u}\|_{L^2}^2 \leqslant \frac{c}{\nu^3}\big(\|\nabla \boldsymbol{u}\|_{L^2}^2\big)^3 + \frac{2}{\nu}\|\boldsymbol{f}\|_{L^2}^2.$$

For simplicity assume $\boldsymbol{f} \equiv \boldsymbol{0}$. Then using the Gronwall lemma we see, as a function of time, the best we can show is $\|\nabla \boldsymbol{u}\|_{L^2}^2$ has an upper bound that blows up in finite time.