Navier–Stokes equations tutorial

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Consider the incompressible Navier–Stokes equations on a bounded domain Ω with viscous boundary conditions. The goal of this tutorial is to derive some energy estimates and achieve some fluency in performing these types of calculations.

Question 1 Show that the Navier–Stokes equations can be expressed in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{u} = \nu \, \Delta \boldsymbol{u} - \nabla\big(p + \tfrac{1}{2}|\boldsymbol{u}|^2\big) + \boldsymbol{f}.$$

Directly compute the quantity $(\mathrm{d}/\mathrm{d}t)\|\boldsymbol{u}\|_{L^2}^2$ and use Hölder's inequality to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_{L^2}^2 + 2\nu\|\nabla \boldsymbol{u}\|_{L^2}^2 \leqslant \delta\|\boldsymbol{u}\|_{L^2}^2 + \frac{1}{\delta}\|\boldsymbol{f}\|_{L^2}^2,$$

for some constant $\delta > 0$. Hence derive a uniform upper bound for $\|\boldsymbol{u}\|_{L^2}^2$ in time and deduce that for any time $T > 0$:

$$\boldsymbol{u} \in L^\infty\big([0,T]; L^2(\Omega;\mathbb{R}^d)\big) \cap L^2\big([0,T]; H^1(\Omega;\mathbb{R}^d)\big).$$

Question 2 By considering the L^2-inner product of $-\Delta \boldsymbol{u}$ with the Navier–Stokes formulation quoted above, using that $-\Delta \boldsymbol{u} = \nabla \times \boldsymbol{\omega}$ show that

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}\|_{L^2}^2 + \nu\|\Delta \boldsymbol{u}\|_{L^2}^2 = \int_\Omega \boldsymbol{\omega}^{\mathrm{T}}(\nabla \boldsymbol{u})\boldsymbol{\omega}\,\mathrm{d}\boldsymbol{x} - \langle \Delta \boldsymbol{u}, \boldsymbol{f}\rangle_{L^2}.$$

1. Assume $d = 2$. Show that

$$\boldsymbol{u} \in L^\infty\big([0,T]; H^1(\Omega;\mathbb{R}^d)\big) \cap L^2\big([0,T]; H^2(\Omega;\mathbb{R}^d)\big).$$

2. Assume $d = 3$. Use the Hölder and Gagliardo–Sobolev–Nirenberg inequalities to show that for some constant c:

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{u}\|_{L^2}^2 + \nu\|\Delta \boldsymbol{u}\|_{L^2}^2 \leqslant \frac{c}{\nu^3}\|\nabla \boldsymbol{u}\|_{L^2}^6 + \frac{2}{\nu}\|\boldsymbol{f}\|_{L^2}^2.$$

   What can you deduce from this inequality?