LMS-EPSRC Short Course: Theoretical Fluid Dynamics

Fundamental functional analysis

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Introduction

We outline briefly here the principal results from *functional analysis* required for the nonlinear analysis of solutions to partial differential equations. The concepts and statements provided are not meant to be exhaustive, but are intended to be a minimum set of tools with which most of the basic results concerning the existence, uniqueness and regularity of solutions can be proved. We assume the reader has a rough notion of Lebesgue measure and integrability. We state results without proof as most of them can be found in standard texts; for example Evans [2].

1 Function spaces

We introduce spaces of continuous and continuously differentiable functions as well as Lebesgue and Sobolev spaces for functions. Throughout assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain of dimension $d \geqslant 1$. The generalized partial derivative operator ∂^α of order $m = |\alpha|$ is defined as

$$\partial^\alpha := \frac{\partial^{\alpha_1+\cdots+\alpha_d}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index with $\alpha_1 + \cdots + \alpha_d = m$.

1.1 Continuous differentiability

Definition 1 (Continuously differentiable functions) For any non-negative integer m, $C^m(\Omega)$ denotes the space of all functions whose partial derivatives, up to and including order m, are all continuous on Ω.

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Example 1 The spaces $C^0(\Omega) = C(\Omega)$ and $C^1(\Omega)$ denote the spaces of continuous functions and continuously differentiable functions, respectively, on Ω. The space

$$C^\infty(\Omega) \coloneqq \bigcap_{m=0}^{\infty} C^m(\Omega),$$

denotes the space of smooth functions on Ω.

Remark 1 We shall denote by $C_0^m(\Omega)$ the subspace of $C^m(\Omega)$ of functions which have compact support in Ω.

The subset of functions of $C^m(\Omega)$ which are bounded and uniformly continuous on Ω can be uniquely extended to the closure $\overline{\Omega}$ of Ω. We shall use $C^m\big(\overline{\Omega}\big)$ to denote the space of functions whose partial derivatives, up to and including order m, are all bounded and uniformly continuous on Ω. This is a Banach space with norm

$$\|f\|_{C^m(\overline{\Omega})} \coloneqq \max_{0 \leqslant |\alpha| \leqslant m} \sup_{\boldsymbol{x} \in \Omega} |\partial^\alpha f(\boldsymbol{x})|.$$

Remark 2 If V is a vector space, we shall use $C(\Omega; V)$ to denote the space of continuous functions whose image lies in V. For example, $\boldsymbol{f} \in \mathbb{R}^n$ lies in the space of vector valued continuous functions if each component of $\boldsymbol{f}$ is in $C(\Omega; \mathbb{R})$ and we write $\boldsymbol{f} \in C(\Omega; \mathbb{R}^n)$. When the image space is clear from the context we will simply write $C(\Omega)$.

1.2 **Lebesgue integrability**

Definition 2 (Lebesgue integrable functions) For any real number $p > 0$, the *Lebesgue space* $L^p(\Omega)$ is the space of equivalence classes of p-integrable functions on Ω (in the Lebesgue sense), i.e. it is the set of all Lebesgue measurable functions f defined on Ω for which the following functional is finite:

$$\|f\|_{L^p(\Omega)} \coloneqq \left(\int_\Omega |f|^p \,\mathrm{d}\boldsymbol{x} \right)^{1/p}.$$

Remark 3 If we are interested in $\mathbb{R}^d$-valued functions then we use $|\boldsymbol{f}|^p$ in the right-hand side in the definition above, where $|\boldsymbol{f}|$ denotes the Euclidean norm/length of $\boldsymbol{f} \in \mathbb{R}^n$; and denote the space $L^p(\Omega; \mathbb{R}^n)$ unless the image space is obvious from the context and we simply write $L^p(\Omega)$.

Remark 4 Note that we identify functions that differ on a set of Lebesgue measure zero, hence this is a space of equivalence classes of Lebesgue measurable functions. For example, if one function differs from another at only a finite number of points in Ω, we will consider them to be the same function, and the same for any other function that differs from it on a set of Lebesgue measure zero on Ω. Indeed from this perspective Lebesgue integrable functions need only be defined *almost everywhere* on Ω, meaning everywhere except on a set of Lebesgue measure zero.

Example 2 Suppose $\Omega = [0, 1]$ and $f = f(x)$ takes the values 1 if x is irrational in $[0, 1]$ and 0 if x is rational. The infimum and supremum processes that are used to define the Riemann integral do not converge for this function and so it is not defined. However the Lebesgue integral is defined and takes the value 1. Here we could use $\hat{f}(x) \equiv 1$ on $[0, 1]$ as the representative of the class of functions (all differing from $\hat{f}$ on $[0, 1]$ on a set of Lebesgue measure zero) with this property.

Remark 5 In practice as we shall see, when we compute estimates we rarely need to keep tabs on this subtlety. Most of the time we can assume the functions we are computing with are smooth, and then use density arguments (see the next lemma) in the final steps to then relax this assumption.

Lemma 1 *For $1 \leqslant p < \infty$, the spaces $C_0(\Omega)$ and $C_0^\infty(\Omega)$ are both dense in $L^p(\Omega)$.*

Hence it is always possible to find a smooth function arbitrarily close to an L^p function in the L^p-norm, at least for $1 \leqslant p < \infty$. For the limit $p \to \infty$ though, we must be more careful.

Definition 3 (Bounded functions) A measurable function f on Ω is *essentially bounded* if there exists a constant K such that $|f(\boldsymbol{x})| \leqslant K$ almost everywhere on Ω. The greatest lower bound of such constants is called the essential supremum of $|f|$ on Ω which we write as $\operatorname{ess\,sup}_{\boldsymbol{x}\in\Omega} |f(\boldsymbol{x})|$. Then $L^\infty(\Omega)$ is the vector space consisting of all functions f for which the norm $\|f\|_{L^\infty(\Omega)} \coloneqq \operatorname{ess\,sup}_{\boldsymbol{x}\in\Omega} |f(\boldsymbol{x})|$ is finite.

Remark 6 The spaces $C(\Omega)$, $C_0(\Omega)$ and $C_0^\infty(\Omega)$ are proper subspaces of L^∞.

Definition 4 (Inner product) The space of square integrable functions $L^2(\Omega;\mathbb{R}^n)$ is an inner product space, indeed it is a separable *Hilbert space*, with *inner product*

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle \coloneqq \int_\Omega \boldsymbol{f} \cdot \boldsymbol{g} \,\mathrm{d}\boldsymbol{x}.$$

1.3 **Sobolev spaces**

Definition 5 (Local integrability) A function f defined almost everywhere on Ω is locally L^p-integrable on Ω provided $f \in L^p(\mathcal{D})$ for every measurable $\mathcal{D}$ such that $\overline{\mathcal{D}} \subseteq \Omega$ and $\overline{\mathcal{D}}$ is compact in $\mathbb{R}^d$. We write $f \in L^p_{\mathrm{loc}}(\Omega)$.

Definition 6 (Weak derivatives) Let f and h be two locally L^1-integrable functions on Ω. We call $h = \partial^\alpha f$ the (partial) weak derivative of f provided h satisfies, for every $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega f(\boldsymbol{x})\,\partial^\alpha \varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = (-1)^{|\alpha|} \int_\Omega h(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

We now define vector subspaces of $L^p(\Omega)$ spaces where we additionally demand that their weak derivatives are also L^p-integrable.

Definition 7 (Sobolev space) For any $1 \leqslant p \leqslant \infty$, the vector subspace of $L^p(\Omega)$ functions given by

$$W^{m,p}(\Omega) \coloneqq \big\{ f \in L^p(\Omega) \colon \partial^\alpha f \in L^p(\Omega),\ \forall \alpha \colon 0 \leqslant |\alpha| \leqslant m \big\}$$

is the *Sobolev space* of $L^p(\Omega)$ functions whose weak derivatives up to order m are also $L^p(\Omega)$ functions.

The Sobolev space $W^{m,p}(\Omega)$ is a Banach space with norm

$$\|f\|_{W^{m,p}(\Omega)} \coloneqq \Big(\sum_{0\leqslant|\alpha|\leqslant m} \|\partial^\alpha f\|^p_{L^p(\Omega)} \Big)^{1/p},$$

for $1 \leqslant p < \infty$ and

$$\|f\|_{W^{m,\infty}(\Omega)} \coloneqq \max_{0\leqslant|\alpha|\leqslant m} \|\partial^\alpha f\|_{L^\infty(\Omega)}.$$

For $1 \leqslant p < \infty$, Meyers and Serrin proved that $W^{m,p}(\Omega)$ is the completion of the set $\{f \in C^m(\Omega) \colon \|f\|_{W^{m,p}(\Omega)} < \infty\}$ with respect to the norm $\|\cdot\|_{W^{m,p}(\Omega)}$.

Lemma 2 (Sobolev Hilbert spaces) *The Sobolev space $H^m(\Omega) \coloneqq W^{m,2}(\Omega)$ is a separable Hilbert space with inner product*

$$\langle f, g\rangle_{H^m(\Omega)} \coloneqq \sum_{0\leqslant|\alpha|\leqslant m} \langle \partial^\alpha f, \partial^\alpha g\rangle_{L^2(\Omega)}.$$

Remark 7 Often in estimates for evolutionary partial differential equations we will want to separate the time regularity of solutions from spatial regularity. If V is a vector space, we have already seen the notation such as $L^2(\Omega; V)$ which denotes the space of functions whose image lies in V that are square integrable from Ω into V. For example if $V = \mathbb{R}^n$, then $L^2(\Omega;\mathbb{R}^n)$ denotes the space of vector valued functions, each of whose n components are themselves square-integrable. Typically we want to use this notation as follows. If $[0, T]$ represents a time interval of interest for some $T > 0$ and $\Omega \subseteq \mathbb{R}^d$ is the spatial domain, then we might want to show that the solution lies in

$$C\big([0, T]; L^2(\Omega;\mathbb{R}^n)\big).$$

This is the space of functions $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}, t)$ on $\Omega \times [0, T]$ that are square-integrable in space on Ω and continuous in time on $[0, T]$. In other words, for such functions the quantity

$$\|f(\cdot, t)\|_{L^2(\Omega;\mathbb{R}^n)}$$

is continuous in time. As another example consider the space of functions f lying in

$$L^2\big([0, T]; H^1(\Omega;\mathbb{R}^n)\big).$$

Such functions satisfy the condition

$$\int_0^T \|f(\cdot, t)\|^2_{H^1(\Omega;\mathbb{R}^n)}\,\mathrm{d}t < \infty.$$

2 Embeddings

The goal here is to understand the relative structure underlying all of the function spaces we saw in the last section. For example, a function that lies in $L^\infty(\Omega)$ is in $L^p(\Omega)$ for any $1 \leqslant p < \infty$. As another example, the Sobolev space of functions $W^{1,2}(\Omega)$, i.e. the space of functions which themselves as well as their derivatives are L^2-integrable, are also naturally L^p-integrable for any $2 \leqslant p \leqslant 2d/(d-2)$. These are examples of function space embeddings.

Definition 8 (Embedding) Suppose V and H are two function spaces. We say that V is embedded in H and write

$$V \hookrightarrow H$$

if the following holds:

1. V is a vector subspace of H;
2. The identity map $\mathrm{id}\colon V \to H$ given by $\mathrm{id}\colon f \mapsto f$ is *continuous*, i.e. for all $f \in V$ and some positive constant c, we have:

$$\|f\|_H \leqslant c\,\|f\|_V.$$

Example 3 Hence from our discussion above we would write $L^\infty(\Omega) \hookrightarrow L^p(\Omega)$ and $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$. We also have the natural sequence, for any $p > p'$ then

$$L^p(\Omega) \hookrightarrow L^{p'}(\Omega).$$

Remark 8 Establishing such embeddings gives us the overall picture of the relationships between the function spaces. The definition above shows the inequality $\|f\|_V \leqslant c\|f\|_H$ naturally mediates the embedding. Hence most of the work involved in establishing such embeddings revolves around proving/using such inequalities.

2.1 Fundamental inequalities

We first outline some simple inequalities as well as some more sophisticated functional and interpolation inequalities that are often useful in practice when computing estimates and establishing function space embeddings. For convenience we will call a pair (p, q) of real numbers, where $p \geqslant 1$ and $q \geqslant 1$, a *conjugate pair* if they satisfy

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Lemma 3 (Young's inequality) *For any $a \geqslant 0$, $b \geqslant 0$, $\epsilon > 0$ and conjugate pair (p, q) with $1 < p < \infty$, we have*

$$ab \leqslant \frac{1}{p}(a\epsilon)^p + \frac{1}{q}\Big(\frac{b}{\epsilon}\Big)^q.$$

Theorem 1 (Hölder's inequality) *For any conjugate pair (p, q) suppose that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then $fg \in L^1(\Omega)$ and we have*

$$\int_\Omega \big|f(\boldsymbol{x})\,g(\boldsymbol{x})\big|\,\mathrm{d}\boldsymbol{x} \leqslant \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}.$$

Recall that $H^m(\Omega) \equiv W^{m,2}(\Omega)$. Further we shall use the notation $H_0^m(\Omega)$ and $W_0^{m,p}(\Omega)$ to denote the subspaces of functions of $H^m(\Omega)$ and $W^{m,p}(\Omega)$, respectively, that have compact support. Indeed for example, we can also think of $W_0^{m,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. We have the following important theorem that is a natural consequence of the fundamental theorem of calculus; see Evans [2, p. 275]. We shall denote the average of f over Ω by

$$\langle f\rangle := \big(\mathrm{vol}(\Omega)\big)^{-1}\int_\Omega f\,\mathrm{d}\boldsymbol{x}.$$

Theorem 2 (Poincaré's inequality) *Suppose Ω is connected and has a C^1 boundary $\partial\Omega$. Assume that $1 \leqslant p \leqslant \infty$. Then for any $f \in W^{1,p}(\Omega)$ there exists a constant $c = c(d, p, \Omega)$ such that*

$$\big\| f - \langle f \rangle \big\|_{L^p(\Omega)} \leqslant c \, \|\nabla f\|_{L^p(\Omega)}.$$

Example 4 For functions with mean zero, we can deduce that $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$.

Theorem 3 (Sobolev–Gagliardo–Nirenberg inequality) *For all $f \in H^1(\Omega)$ we have for some constant $c = c(\Omega)$:*

$$\|f\|_{L^p(\Omega)} \leqslant c \, \|\nabla f\|^{a}_{L^2(\Omega)} \|f\|^{1-a}_{L^2(\Omega)},$$

where $a = d(p-2)/2p$ and $2 \leqslant p \leqslant 2d/(d-2)$.

This is an example of an interpolation inequality and a special case of the more general Gagliardo–Nirenberg inequalities. In particular, it tells us that under the conditions stated $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$.

2.2 Compact embeddings

Definition 9 (Compact operator) Let $A\colon V \to H$ be an operator from the vector space V to the vector space H. Then A is said to be *compact* if $A(U)$ is precompact (meaning that the image set $\overline{A(U)}$ is compact) in H whenever U is bounded in V. If A is continuous and compact, then it is said to be *completely continuous*.

Remark 9 Any compact *linear* operator is completely continuous.

Definition 10 (Compact embedding) If $V \hookrightarrow H$ and the identity operator $\mathrm{id}\colon V \to H$ in the embedding is compact, we say that V is *compactly embedded* in H and write

$$V \hookrightarrow\hookrightarrow H.$$

Example 5 (Rellich–Kondrachov theorem) Assume that the boundary $\partial\Omega$ is C^1. Then for any $d > 2$ and $1 \leqslant p < 2d/(d-2)$ we have (see Evans [2, p. 272])

$$W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega).$$

Remark 10 The significance of such a compact embedding is as follows. Suppose we have established that a sequence $\{f_n\}_{n\geqslant 1} \subset V$ is bounded. Then we can deduce that there is a subsequence $\{f_{n_k}\}_{k\geqslant 1} \subset \{f_n\}_{n\geqslant 1} \subset V$ and an $f \in V$ such that f_{n_k} converges weakly to f, written $f_{n_k} \rightharpoonup f$. This means that for every bounded linear functional f^* on V then $f^*(f_{n_k}) \to f^*(f)$. However, since V is compactly embedded in H, we can deduce that f_{n_k} converges strongly to f in H.

Remark 11 Compact embeddings are extremely important for proving the existence of solutions to nonlinear partial differential equations. For example for the incompressible Navier–Stokes equations, to prove the existence of Leray weak solutions, we would typically start by considering an associated system, perturbed from the incompressible Navier–Stokes equations themselves, for which we already know existence. Suppose the perturbation parameter is δ and the limit $\delta \to 0$ recovers the Navier–Stokes equations.

Typical perturbations are to: use Galerkin projection onto a finite number ($\sim 1/\delta$) of spatial modes; smooth/mollify the advecting velocity (with the smoothness parameterized by δ) or to add hyperviscosity (parameterized by δ). We would then prove Sobolev norm bounds uniform in δ, and try to establish the correct convergence in the limit $\delta \to 0$. As you might expect, the nonlinear terms involve the most work. The following compact embedding (there are several variant approaches) is sufficient to establish the appropriate convergence:

$$L^{\infty}\big([0,T];L^2(\Omega)\big)\cap L^2\big([0,T];H^1(\Omega)\big)\cap W^{1,\frac{4}{d}}\big([0,T];H^{-1}(\Omega)\big) \hookrightarrow\hookrightarrow L^2\big([0,T];L^2(\Omega)\big).$$

References

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