Introductory fluid mechanics

Simon J.A. Malham

Heriot–Watt University, Edinburgh, UK

LMS-EPSRC Short Course on Theoretical Fluid Dynamics
August 29th to September 2nd, 2011

Introduction

- *Ideal fluid*: Euler (1755);
- *Viscous fluid*: Navier (1822) and Stokes (1845).

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nu \nabla^2 \boldsymbol{u} - \nabla p + \boldsymbol{f},$$
$$\nabla \cdot \boldsymbol{u} = 0.$$

- $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ fluid velocity;
- $p = p(\boldsymbol{x}, t)$ pressure;
- $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}, t)$ external force;
- ν viscosity;

Flow

Material exhibits flow if shear forces, however small, lead to a deformation which is unbounded; i.e. a fluid.

- Liquids: incompressible, eg. brakes!;
- Gases: compressible, eg. aerosols and air canisters.

Subcatergorization:

- *Ideal/inviscid*: *only* internal force is pressure;
- *Viscous*: internal frictional forces also present;
- *Non-Newtonian/complex*: reaction to deformation depend on:
  - past history, eg. paints;
  - temperature, eg. some polymers or glass;
  - deformation size, eg. plastics or silly putty.

Continuum hypothesis

Natural length scales:

$$L_{\text{molecular}} \ll L_{\text{fluid}} \ll L_{\text{macro}}.$$

Continuum assumption:

> *properties of the fluid at scale L_{fluid} propagate all the way down and through the molecular scale $L_{\text{molecular}}$.*

Everyday fluid mechanics: this is extremely accurate (Chorin and Marsden).

Conservation principles

1. *Conservation of mass*;
2. *Newton's 2nd law/balance of momentum*;
3. *Conservation of energy*.

These principles generate:

1. *Continuity equation*;
2. *Navier–Stokes equations*;
3. *Equation of state*.

Trajectories

Small fluid particle or a speck of dust:

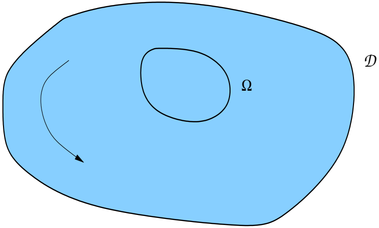
- Velocity flow field $\boldsymbol{u}(\boldsymbol{x}, t) = (u, v, w)$;
- Position recorded by $\big(x(t), y(t), z(t)\big)$.

$$\dot{x}(t) = u\big(x(t), y(t), z(t), t\big),$$
$$\dot{y}(t) = v\big(x(t), y(t), z(t), t\big),$$
$$\dot{z}(t) = w\big(x(t), y(t), z(t), t\big).$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t).$$

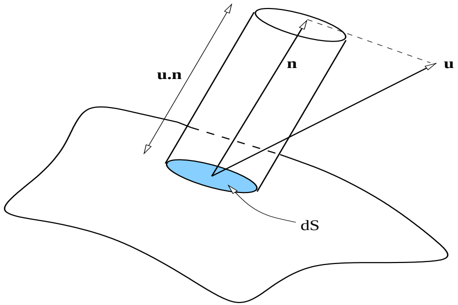
Continuity equation



Mass density $\rho(\boldsymbol{x}, t)$:

$$\text{Mass}(\Omega, t) := \int_{\Omega} \rho(\boldsymbol{x}, t)\, \mathrm{d}V.$$

Continuity equation II



$$\text{mass density} \times \text{vol leaving per unit time} = \rho(\boldsymbol{x}, t)\, \boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x})\, \mathrm{d}S$$

Continuity equation III

Conservation of mass:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho(\boldsymbol{x},t)\,\mathrm{d}V = -\int_{\partial\Omega}\rho\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S.$$

$$\Leftrightarrow \qquad \int_{\Omega}\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u})\,\mathrm{d}V = 0.$$

Ω arbitrary $\Longrightarrow$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u}) = 0.$$

Incompressible flow

Definition (Incompressibility)

$\nabla \cdot \boldsymbol{u} = 0$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{u} = 0$$

$$\Leftrightarrow \qquad \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho = 0$$

Homogeneous when ρ constant in space:

Incompressible $\Leftrightarrow$ ρ is constant in time.

Differentiation following the fluid

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t) \\
\Longrightarrow \quad & \frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}\big(\boldsymbol{x}(t), t\big) \\
& = \frac{\partial \boldsymbol{u}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \boldsymbol{u}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \boldsymbol{u}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial \boldsymbol{u}}{\partial t} \\
& = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial}{\partial z}\right)\boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial t} \\
& = \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial t}
\end{aligned}
$$

Material derivative:

$$
\frac{\mathrm{d}}{\mathrm{d}t}F\big(x(t), y(t), z(t), t\big) = \frac{\partial F}{\partial t} + \boldsymbol{u}\cdot\nabla F = \frac{\mathrm{D}F}{\mathrm{D}t}
$$

Rate of strain tensor

$$\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{h}) = \boldsymbol{u}(\boldsymbol{x}) + \big(\nabla \boldsymbol{u}(\boldsymbol{x})\big)\cdot \boldsymbol{h} + \mathcal{O}(h^2)$$

Rate of strain tensor:

$$\nabla \boldsymbol{u} = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{pmatrix}$$

$$\nabla \boldsymbol{u} = \underbrace{\tfrac{1}{2}\Big((\nabla \boldsymbol{u}) + (\nabla \boldsymbol{u})^{\mathrm{T}}\Big)}_{D} + \underbrace{\tfrac{1}{2}\Big((\nabla \boldsymbol{u}) - (\nabla \boldsymbol{u})^{\mathrm{T}}\Big)}_{R}$$

Rate of strain tensor II

$$R = \begin{pmatrix} 0 & \partial u/\partial y - \partial v/\partial x & \partial u/\partial z - \partial w/\partial x \\ \partial v/\partial x - \partial u/\partial y & 0 & \partial v/\partial z - \partial w/\partial y \\ \partial w/\partial x - \partial u/\partial z & \partial w/\partial y - \partial v/\partial z & 0 \end{pmatrix}$$

$$\omega_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \qquad \omega_2 = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \qquad \omega_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$R = \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad \Leftrightarrow \qquad R\,\boldsymbol{h} = \tfrac{1}{2}\boldsymbol{\omega} \times \boldsymbol{h}$$

Rate of strain tensor III

Theorem

$$\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{h}) = \boldsymbol{u}(\boldsymbol{x}) + D(\boldsymbol{x})\cdot\boldsymbol{h} + \tfrac{1}{2}\boldsymbol{\omega}(\boldsymbol{x})\times\boldsymbol{h} + \mathcal{O}(h^2).$$

D is the symmetric *deformation tensor*.

$$X^{-1}DX = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}.$$

$\boldsymbol{\omega}$ is the *vorticity field*, indeed: $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$.

Rate of strain tensor IV

$\boldsymbol{x}$ is fixed and $\boldsymbol{h}$ is small:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x}+\boldsymbol{h}) = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{h})$$

$$\Leftrightarrow \qquad \frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{h})$$

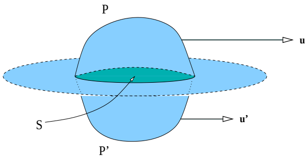
$$\Leftrightarrow \qquad \frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} \approx \boldsymbol{u}(\boldsymbol{x}) + D(\boldsymbol{x})\cdot\boldsymbol{h} + \tfrac{1}{2}\omega(\boldsymbol{x})\times\boldsymbol{h}.$$

$\boldsymbol{h} = X\hat{\boldsymbol{h}} \implies$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\hat{h}_1\\ \hat{h}_2\\ \hat{h}_3\end{pmatrix} = \begin{pmatrix} d_1 & 0 & 0\\ 0 & d_2 & 0\\ 0 & 0 & d_3\end{pmatrix}\begin{pmatrix}\hat{h}_1\\ \hat{h}_2\\ \hat{h}_3\end{pmatrix}.$$

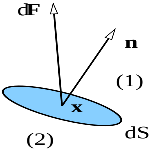
$$\frac{\mathrm{d}}{\mathrm{d}t}(\hat{h}_1\hat{h}_2\hat{h}_3) = \underbrace{(d_1+d_2+d_3)}_{\mathrm{Tr}(D)=\nabla\cdot\boldsymbol{u}}(\hat{h}_1\hat{h}_2\hat{h}_3)$$

Internal fluid forces



1. *external or body forces* per unit volume;
2. *surface or stress forces*, molecular in origin.

Internal fluid forces II



Force $\mathrm{d}\boldsymbol{F}$ on side (2) by side (1) of $\mathrm{d}S$:

$$\mathrm{d}\boldsymbol{F} = \boldsymbol{\Sigma}(\boldsymbol{n})\,\mathrm{d}S$$

$$\boldsymbol{\Sigma}(\boldsymbol{n}) = \sigma(\boldsymbol{x})\,\boldsymbol{n}$$

where $\sigma = [\sigma_{ij}]$ = stress tensor (3×3 matrix)

Write:

$$\sigma = -p\,I + \hat{\sigma}$$

where

- $p := -\frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) > 0$ represents *pressure*;
- $\hat{\sigma}$ is the *deviatoric stress tensor*.

Note $-p\,I$ generates the normal stresses:

$$\sigma = -p\,I \qquad \Rightarrow \qquad \boldsymbol{\Sigma}(\boldsymbol{n}) = -p\,\boldsymbol{n}$$

while $\hat{\sigma}$ generates the shear stresses.

Internal fluid forces IV

Assumptions on $\hat{\sigma}$:

1. Linear homogeneous function of $\nabla \boldsymbol{u}$ (*Newtonian fluid*);
2. Invariant under rigid body rotations:

$$\hat{\sigma}\left(U \cdot \nabla \boldsymbol{u} \cdot U^{-1}\right) \equiv U \cdot \hat{\sigma}(\nabla \boldsymbol{u}) \cdot U^{-1};$$

3. Symmetric.

$$\Rightarrow \qquad \hat{\sigma}_{ij} = \sum_{k,l} A_{ijkl} D_{kl}$$

$$\text{isotropy} \Rightarrow \qquad A_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

Internal fluid forces V

Properties imply $\hat{\sigma} D = D\hat{\sigma}$

$$D\hat{\mathbf{e}}_i = d_i\hat{\mathbf{e}}_i \qquad \Longrightarrow \qquad \hat{\sigma} D\,\hat{\mathbf{e}}_i = D(\hat{\sigma}\,\hat{\mathbf{e}}_i) = d_i\,(\hat{\sigma}\,\hat{\mathbf{e}}_i)$$

$\Longrightarrow$ $\hat{\sigma}\hat{\mathbf{e}}_i \propto \hat{\mathbf{e}}_i$ $\Longrightarrow$ $\hat{\mathbf{e}}_i$ evec of $\hat{\sigma}$ also

$\Longrightarrow$ evals of $\hat{\sigma}$ are homogeneous linear functions of d_i

Prop 2 $\Longrightarrow$ symmetric functions $\Longrightarrow$ $\hat{\sigma}_i = \lambda(d_1 + d_2 + d_3) + 2\mu d_i$

$$\hat{\sigma} = \lambda(\nabla \cdot \boldsymbol{u})I + 2\mu D$$

$$\zeta = \lambda + \tfrac{2}{3}\mu \quad \Rightarrow \quad \hat{\sigma} = 2\mu\Big(D - \tfrac{1}{3}(\nabla \cdot \boldsymbol{u})I\Big) + \zeta(\nabla \cdot \boldsymbol{u})I$$

Navier–Stokes equations

$$\int_{\partial\Omega} (-pI + \hat{\sigma})\, \boldsymbol{n}\, \mathrm{d}S \equiv \int_{\Omega} (-\nabla p + \nabla \cdot \hat{\sigma})\, \mathrm{d}V$$

$$\begin{aligned}
[\nabla \cdot \hat{\sigma}]_i &= \sum_{j=1}^{3} \frac{\partial \hat{\sigma}_{ij}}{\partial x_j} \\
&= \lambda [\nabla(\nabla \cdot \boldsymbol{u})]_i + 2\mu \sum_{j=1}^{3} \frac{\partial D_{ij}}{\partial x_j} \\
&= \lambda [\nabla(\nabla \cdot \boldsymbol{u})]_i + \mu \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\
&= (\lambda + \mu) [\nabla(\nabla \cdot \boldsymbol{u})]_i + \mu \nabla^2 u_i.
\end{aligned}$$

Navier–Stokes equations II

Balance of momentum $\Longrightarrow$

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \nabla\cdot\hat{\sigma} + \rho\,\boldsymbol{f}.$$

$$\Longrightarrow \qquad \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nu\,\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f},$$

$$\nabla\cdot\boldsymbol{u} = 0,$$

where $\nu = \mu/\rho =$ kinematic viscosity. Rigid boundary conditions:

- Ideal fluid flow: $\boldsymbol{u}\cdot\boldsymbol{n} = 0$;
- Viscous flow: $\boldsymbol{u} = \boldsymbol{0}$.

Evolution of vorticity

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} \equiv \tfrac{1}{2}\nabla\left(|\boldsymbol{u}|^2\right) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})$$

$$\nabla\times(\boldsymbol{u}\times\boldsymbol{\omega}) \equiv \boldsymbol{u}\,(\nabla\cdot\boldsymbol{\omega}) - \boldsymbol{\omega}\,(\nabla\cdot\boldsymbol{u}) + (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega}$$

Two identities imply:

$$\frac{\partial\boldsymbol{u}}{\partial t} + \tfrac{1}{2}\nabla\left(|\boldsymbol{u}|^2\right) - \boldsymbol{u}\times\boldsymbol{\omega} = \nu\,\Delta\boldsymbol{u} - \nabla p$$

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} = \nu\,\Delta\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\nabla\boldsymbol{u}$$

$$\Delta\boldsymbol{u} = -\nabla\times\boldsymbol{\omega},$$

Note: $\boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = (\nabla\boldsymbol{u})\boldsymbol{\omega} = D\boldsymbol{\omega} + R\boldsymbol{\omega} = D\boldsymbol{\omega}$.

Transport theorem

Theorem (Transport theorem)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \rho\, F\, \mathrm{d}V = \int_{\Omega_t} \rho\, \frac{\mathrm{D}F}{\mathrm{D}t}\, \mathrm{d}V.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \rho\, \boldsymbol{u}\, \mathrm{d}V = \int_{\Omega_t} -\nabla p + \nabla \cdot \hat{\sigma} + \rho\, \boldsymbol{f}\, \mathrm{d}V.$$

Corollary (Equivalent statements)

1. *Fluid is incompressible;*
2. *Jacobian of flowmap equals* 1*;*
3. *Volume of* Ω_t *is constant in time.*

Simple exact flows

Lemma (**Majda and Bertozzi, p. 8**)

Given $D = D(t)$*, real symmetric matrix with* $\mathrm{Tr}(D) = 0$*. Suppose* $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ *solves ODEs*

$$\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = D(t)\,\boldsymbol{\omega}.$$

Then

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}, t) &= \tfrac{1}{2}\boldsymbol{\omega}(t) \times \boldsymbol{x} + D(t)\,\boldsymbol{x}, \\ p(\boldsymbol{x}, t) &= -\tfrac{1}{2}\left(\frac{\mathrm{d}D}{\mathrm{d}t} + D^2(t) + R^2(t)\right)\boldsymbol{x} \cdot \boldsymbol{x}, \end{aligned}$$

are exact solutions to Euler and Navier–Stokes equations.

Simple exact flows II

Proof.

$$\frac{\partial}{\partial t}(\nabla \boldsymbol{u}) + \boldsymbol{u}\cdot\nabla(\nabla \boldsymbol{u}) + (\nabla \boldsymbol{u})^2 = \nu\,\Delta(\nabla \boldsymbol{u}) - \nabla\nabla p.$$

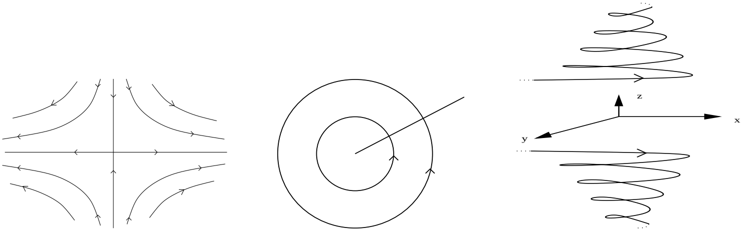
$$(\nabla \boldsymbol{u})^2 = (D+R)^2 = (D^2+R^2) + (DR+RD),$$

$$\Longrightarrow \qquad \begin{aligned} \frac{\partial D}{\partial t} + \boldsymbol{u}\cdot\nabla D + D^2 + R^2 &= \nu\,\Delta D - \nabla\nabla p, \\ \frac{\partial R}{\partial t} + \boldsymbol{u}\cdot\nabla R + DR + RD &= \nu\,\Delta R. \end{aligned}$$

$$\boldsymbol{u}(\boldsymbol{x},t) = \frac{1}{2}\omega(t)\times\boldsymbol{x} + D(t)\,\boldsymbol{x} \quad \Rightarrow \quad \frac{\mathrm{d}D}{\mathrm{d}t} + D^2 + R^2 = -\nabla\nabla p.$$

□

Example (jet flow with swirl)



$$\boldsymbol{u}(\boldsymbol{x},t) == \begin{pmatrix} d_1\, x - \frac{1}{2}\omega(t)y \\ d_2\, y + \frac{1}{2}\omega(t)x \\ d_3\, z \end{pmatrix}$$

Shear-layer flows

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \nu\, \Delta \boldsymbol{\omega} + D\boldsymbol{\omega}$$

$$\Delta \boldsymbol{u} = -\nabla \times \boldsymbol{\omega}$$

$$\boldsymbol{u}(\boldsymbol{x}, t) = \begin{pmatrix} -\gamma x \\ \gamma y \\ w(x, t) \end{pmatrix} \qquad \text{and} \qquad p(\boldsymbol{x}, t) = \frac{1}{2}\gamma\big(x^2 + y^2\big)$$

$$\frac{\partial w}{\partial t} - \gamma x \frac{\partial w}{\partial x} = \nu \frac{\partial^2 w}{\partial x^2}$$

$$\boldsymbol{\omega}(\boldsymbol{x}, t) = \big(0 \quad -\big(\partial w / \partial x\big)(x, t) \quad 0\big)^{\mathrm{T}}$$

Shear-layer flows II

$$\frac{\partial\omega}{\partial t} - \gamma x\frac{\partial\omega}{\partial x} = \gamma\omega + \nu\frac{\partial^2\omega}{\partial x^2}$$

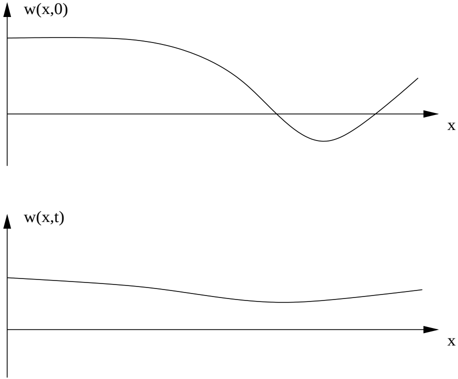
$$w(x,t) = -\int_{-\infty}^{x} \omega(\xi,t)\,\mathrm{d}\xi.$$

Viscous shear-layer solution where $\gamma = 0$:

$$\omega(x,t) = \int_{\mathbb{R}} G(x-\xi,\nu t)\,\omega_0(\xi)\,\mathrm{d}\xi,$$

$$G(\xi,t) := \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\xi^2/4t}.$$

Shear-layer flows II



Shear-layer flows II

Characteristic scales L, U and $T = L/U$:

$$\boldsymbol{x}' = \frac{\boldsymbol{x}}{L}, \qquad \boldsymbol{u}' = \frac{\boldsymbol{u}}{U} \qquad \text{and} \qquad t' = \frac{t}{T}$$

$$\Longrightarrow \qquad \frac{U}{T}\frac{\partial \boldsymbol{u}'}{\partial t'} + \frac{U^2}{L}\boldsymbol{u}' \cdot \nabla_{\boldsymbol{x}'}\boldsymbol{u}' = \frac{\nu U}{L^2}\Delta_{\boldsymbol{x}'}\boldsymbol{u}' - \frac{1}{\rho L}\nabla_{\boldsymbol{x}'}p$$

$$\Longrightarrow \qquad \frac{\partial \boldsymbol{u}'}{\partial t'} + \boldsymbol{u}' \cdot \nabla_{\boldsymbol{x}'}\boldsymbol{u}' = \frac{\nu}{UL}\Delta_{\boldsymbol{x}'}\boldsymbol{u}' - \frac{1}{\rho U^2}\nabla_{\boldsymbol{x}'}p.$$

Set $p' = p/\rho U^2$, and then $\mathrm{Re} := UL/\nu$.

Continuous differentiability

Throughout: $\Omega \subset \mathbb{R}^d$ is a bounded domain.

Generalized partial derivative operator:

$$\partial^\alpha := \frac{\partial^{\alpha_1+\cdots+\alpha_d}}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ a multi-index with $\alpha_1 + \cdots + \alpha_d = m$.

Definition (Continuously differentiable functions)

$C^m(\Omega)$ denotes the space of all functions whose partial derivatives, up to and including order m, are all continuous on Ω.

Continuous differentiability II

$C^0(\Omega) = C(\Omega) =$ space of continuous functions

$C^1(\Omega) =$ derivatives also continuous

$$C^\infty(\Omega) := \bigcap_{m=0}^{\infty} C^m(\Omega) = \text{smooth functions}$$

$C_0^m(\Omega)$ have compact support in Ω

$$||f||_{C^m(\overline{\Omega})} := \max_{0 \leqslant |\alpha| \leqslant m} \sup_{\boldsymbol{x} \in \Omega} |\partial^\alpha f(\boldsymbol{x})|.$$

$C(\Omega; V) =$ continuous functions whose image lies in V

Lebesgue integrability

Definition (**Lebesgue integrable functions**)

$L^p(\Omega)$: space of equivalence classes of p-integrable functions on Ω:

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p \, \mathrm{d}\boldsymbol{x} \right)^{1/p}.$$

Example

Suppose $\Omega = [0, 1]$ and $f = f(x)$ takes the values 1 if x is irrational in $[0, 1]$ and 0 if x is rational.

Lemma

For $1 \leqslant p < \infty$, the spaces $C_0(\Omega)$ and $C_0^\infty(\Omega)$ are both dense in $L^p(\Omega)$.

Bounded functions

Definition (Bounded functions)

f *essentially bounded* if $\exists\, K$: $|f(\boldsymbol{x})| \leqslant K$ a.e. on Ω. The greatest lower bound of such constants K is $\operatorname{ess\,sup}_{\boldsymbol{x}\in\Omega} |f(\boldsymbol{x})|$. Then $L^\infty(\Omega)$ given by

$$\|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{\boldsymbol{x}\in\Omega} |f(\boldsymbol{x})| < \infty.$$

The spaces $C(\Omega)$, $C_0(\Omega)$ and $C_0^\infty(\Omega)$ are proper subspaces of L^∞.

Definition (Inner product)

$L^2(\Omega; \mathbb{R}^n)$ is a separable *Hilbert space* with *inner product*

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{g} \,\mathrm{d}\boldsymbol{x}.$$

Sobolev spaces

Definition (**Local integrability**)

A function f is locally L^p-integrable on Ω provided $f \in L^p(\mathcal{D})$ for every measurable $\mathcal{D}$ such that $\overline{\mathcal{D}} \subseteq \Omega$ and $\overline{\mathcal{D}}$ is compact in $\mathbb{R}^d$: $f \in L^p_{\mathrm{loc}}(\Omega)$.

Definition (**Weak derivatives**)

$f, h \in L^1_{\mathrm{loc}}(\Omega)$: $h = \partial^\alpha f$ weak derivative of f if $\forall\ \varphi \in C_0^\infty(\Omega)$:

$$\int_\Omega f(\boldsymbol{x})\, \partial^\alpha \varphi(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = (-1)^{|\alpha|} \int_\Omega h(\boldsymbol{x}) \varphi(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.$$

Definition (**Sobolev space**)

$$W^{m,p}(\Omega) := \left\{ f \in L^p(\Omega) : \partial^\alpha f \in L^p(\Omega),\ \forall \alpha : 0 \leqslant |\alpha| \leqslant m \right\}$$

Sobolev spaces II

$W^{m,p}(\Omega)$ is a Banach space with norm

$$\|f\|_{W^{m,p}(\Omega)} := \left(\sum_{0 \leqslant |\alpha| \leqslant m} \|\partial^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p}$$

and the completion of $\{f \in C^m(\Omega) : \|f\|_{W^{m,p}(\Omega)} < \infty\}$ wrt $\|\cdot\|_{W^{m,p}(\Omega)}$.

Lemma (Sobolev Hilbert spaces)

$H^m(\Omega) := W^{m,2}(\Omega)$ is a separable Hilbert space with inner product

$$\langle f, g \rangle_{H^m(\Omega)} := \sum_{0 \leqslant |\alpha| \leqslant m} \langle \partial^\alpha f, \partial^\alpha g \rangle_{L^2(\Omega)}.$$

Sobolev spaces: Space and time regularity

$$f \in C\Big([0,T]; L^2(\Omega;\mathbb{R}^n)\Big) \qquad \Leftrightarrow \qquad \|f(\cdot,t)\|_{L^2(\Omega;\mathbb{R}^n)} \text{ continuous in } t$$

$$f \in L^2\Big([0,T]; H^1(\Omega;\mathbb{R}^n)\Big) \qquad \Leftrightarrow \qquad \int_0^T \|f(\cdot,t)\|^2_{H^1(\Omega;\mathbb{R}^n)}\,\mathrm{d}t < \infty$$

Embeddings

Definition (**Embedding**)

Suppose V and H are two function spaces. We say that V is embedded in H and write

$$V \hookrightarrow H$$

if the following holds:

1. V is a vector subspace of H;
2. The identity map $\mathrm{id}\colon V \to H$ given by $\mathrm{id}\colon f \mapsto f$ is *continuous*, i.e. for all $f \in V$ and some positive constant c, we have:

$$\|f\|_H \leqslant c\,\|f\|_V.$$

Examples: $L^\infty(\Omega) \hookrightarrow L^p(\Omega)$; $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ and for any $p > p'$:

$$L^p(\Omega) \hookrightarrow L^{p'}(\Omega).$$

Fundamental inequalities

Pair (p, q) are *conjugate pair* if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Lemma (**Young's inequality**)

For any $a \geqslant 0$, $b \geqslant 0$, $\epsilon > 0$ and conjugate pair (p, q) with $1 < p < \infty$:

$$ab \leqslant \frac{1}{p}(a\epsilon)^p + \frac{1}{q}\left(\frac{b}{\epsilon}\right)^q.$$

Theorem (**Hölder's inequality**)

For any conjugate pair (p, q):

$$\int_\Omega \big| f(\boldsymbol{x})\, g(\boldsymbol{x}) \big| \, \mathrm{d}\boldsymbol{x} \leqslant \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

Poincaré and Sobolev–Gagliardo–Nirenberg

Theorem (Poincaré's inequality)

Suppose Ω is connected and has a C^1 boundary $\partial\Omega$. For any $f \in W^{1,p}(\Omega)$ $\exists$ constant $c = c(d, p, \Omega)$:

$$\left\| f - \langle f \rangle \right\|_{L^p(\Omega)} \leqslant c \, \|\nabla f\|_{L^p(\Omega)}.$$

Functions with mean zero: $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$.

Theorem (Sobolev–Gagliardo–Nirenberg inequality)

For all $f \in H^1(\Omega)$ and some constant $c = c(\Omega)$:

$$\|f\|_{L^p(\Omega)} \leqslant c \, \|\nabla f\|^a_{L^2(\Omega)} \|f\|^{1-a}_{L^2(\Omega)},$$

where $a = d(p-2)/2p$ and $2 \leqslant p \leqslant 2d/(d-2)$.

$\implies W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$

Compact embeddings

Definition (Compact operator)

$A: V \to H$ *compact* if $A(U)$ is precompact in H whenever U is bounded in V.

Definition (Compact embedding)

If $V \hookrightarrow H$ and $\mathrm{id}: V \to H$ is compact then: $V \hookrightarrow\hookrightarrow H$.

Theorem (Rellich–Kondrachov theorem)

Assume $\partial\Omega$ is C^1, $d > 2$ and $1 \leqslant p < 2d/(d-2)$: $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$.

$$L^\infty\big([0,T]; L^2(\Omega)\big) \cap L^2\big([0,T]; H^1(\Omega)\big) \cap W^{1,\frac{4}{d}}\big([0,T]; H^{-1}(\Omega)\big) \hookrightarrow\hookrightarrow L^2\big([0,T]; L^2(\Omega)\big)$$