Turbulence as a problem of a (statistical) fluid mechanics

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Lecture 5: dealing with turbulence in practice for mathematicians, physicists and engineers

toys models of turbulence:

While recognizing that eventually the Navier-Stokes turbulence needs to be fully understood, it is worthwhile to examine carefully various other dynamical systems, usually **much easier to handle analytically and numerically than the N-S equations** but exhibiting the essential features of developed turbulent dynamics:

- cascade
- irreversibility in time ←→ out-of-equilibrium statistics
- intermittency and anomalous scaling laws

Studies on various turbulent systems allow us to differentiate the NS system from others and **identify the role of the essential ingredients**:

- the conservation laws
- the degree of non-linearity, etc.

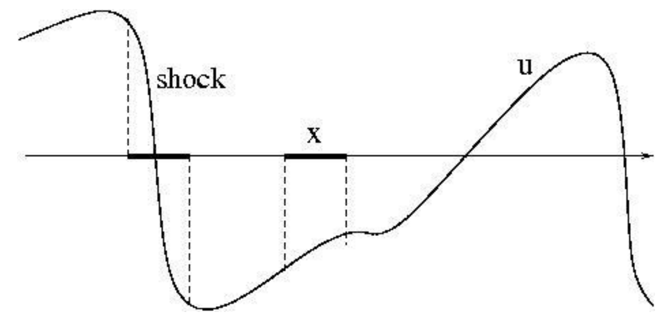
It should also provide a **test bench for analytical techniques and a convenient ground to test new ideas**

The 1d Burgers equation – Burgulence

$$\partial_t u + u \cdot \partial_x u = \nu \Delta u + f$$

simplest case of turbulence, that is a non-linear system out of statistical equilibrium

The 1D Burgers equation gives a concrete example of how energy dissipation can have a finite non-vanishing limit when the viscosity tends to zero in spite of the fact that the inviscid equation formally conserves energy.



→ cascade dynamics, coherent structures and intermittency, anomalous scaling laws

Burgulence, Firsch U. and J. Bec (2001), Les Houches 2000: New trends in turbulence, Springer EDP Science

Burgers turbulence, J. Bec and K. Khanin
Phys. Rep. 447, 1-66, 2007

Intermittency in non-linear dynamics and singularities at complex times
U. Frisch and R. Morf
Phys. Rev. A, vol. 23 (5), 1981, 2673-2705
→ investigation of singularities by nonperturbative methods

The shell-model of turbulence

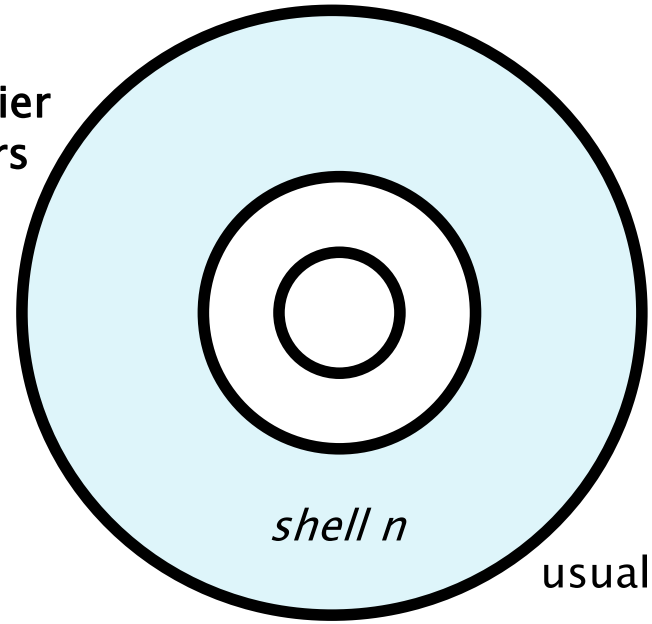
$$\left(\frac{d}{dt}+\nu k_n^2\right)u_n=(ik_n\,u_{n+1}^*u_{n+2}^*-i\varepsilon\,k_{n-1}\,u_{n+1}^*u_{n-1}^*-i(1-\varepsilon)\,k_{n-2}\,u_{n-1}^*u_{n-2}^*)+f_n$$

a caricature of the Navier-Stokes equations in the Fourier space

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)\hat{u}_\alpha(\vec{k},t)=-k_\gamma\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right)\sum_{\vec{p}+\vec{q}=\vec{k}}\hat{u}_\beta(\vec{p},t)\hat{u}_\gamma(\vec{q},t)+f_\alpha(\vec{k},t)$$

$(u_n)_{0\le n\le N}$: complex variables which model the **Fourier space excitations in shells of wavenumbers**

$$k_n=k_0\lambda^n\le k<k_{n+1}$$



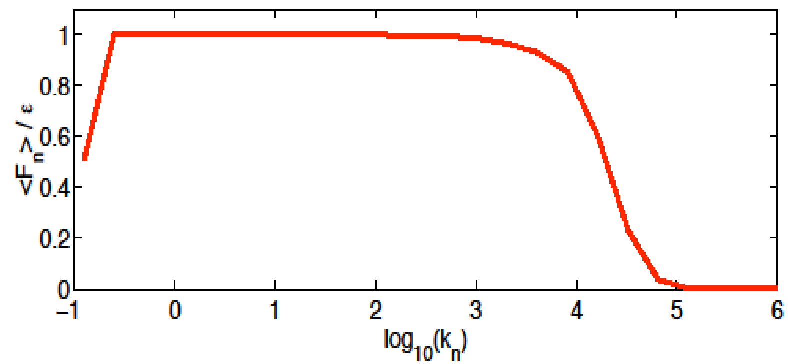
usually $\lambda=2$

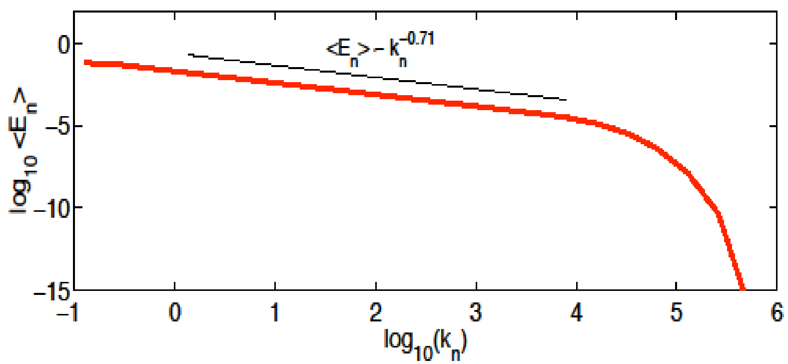
The coefficients of the non-linear term follow from the requirements that the total energy is conserved and the phase volume is conserved by the inviscid dynamics.

$$E=\frac{1}{2}\sum|u_n|^2$$

$H=\sum(-1)^n k_n\,E_n$ is conserved if $\lambda=\dfrac{1}{1-\varepsilon}$

→ dynamical system which is much easier to handle analytically and numerically… allows us to reach the *limit of infinite Reynolds numbers*





After a transitory regime, the system displays **very rich chaotic dynamics** and achieves a **cascade of energy between the forcing and the dissipative shells**

$$\left\langle \left| u_n \right|^p \right\rangle \sim k_n^{-\xi_n}$$

anomalous scaling laws

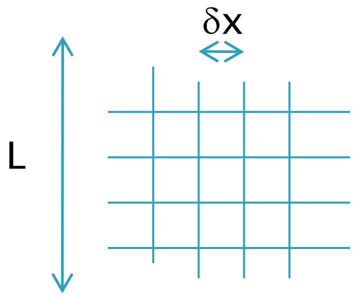
Order p	ζ_p/ζ_3 (shell model)	SL model: $0.125p+1.49(1-0.58^{p/3})$
1	0.375 ± 0.005	0.372
2	0.705 ± 0.003	0.703
3	1.000	1.000
4	1.268 ± 0.006	1.268
5	1.512 ± 0.014	1.513
6	1.738 ± 0.026	1.737
7	1.946 ± 0.040	1.946
8	2.141 ± 0.058	2.140
9	2.323 ± 0.078	2.323
10	2.50 ± 0.10	2.50

Shell models of energy cascade in turbulence
Annual Review of Fluid Mechanics vol 35, 441-468 (2003)

Analytic study of the shell model of turbulence
P. Constantin, B. Levant, E.S. Titi
Phys. Rev. E 75(1) Physica D 219 (2006)

A note on the regularity of inviscid shell models of turbulence
Phys. Rev. E 75(1) (2007)
→ **existence and uniqueness of solutions**

Treating turbulence numerically – for engineers
the large-eddy simulation (LES)



numerical grid

size of the flow

$$\frac{L}{\delta x} \sim \mathrm{Re}^{3/4}$$

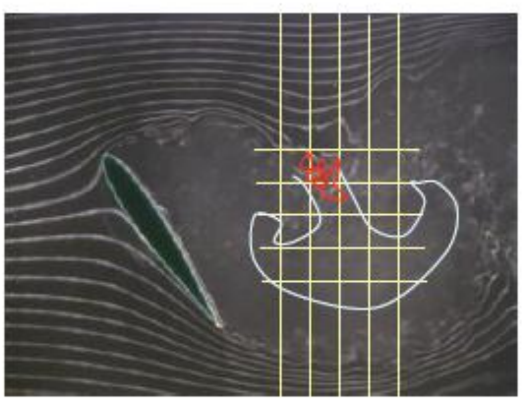
elementary size of turbulent motions

<u>Numerical application</u>: L~1m, U~10 m/s, air

→10^{14} grid points
→10^{18} floating-point operations (~Re^3)

Rank 1 supercomputer (2011) :8.10^{15} Flops (548352 cores)… 10 mins

Excessive computing costs call for development of “reduced models of turbulent flows”

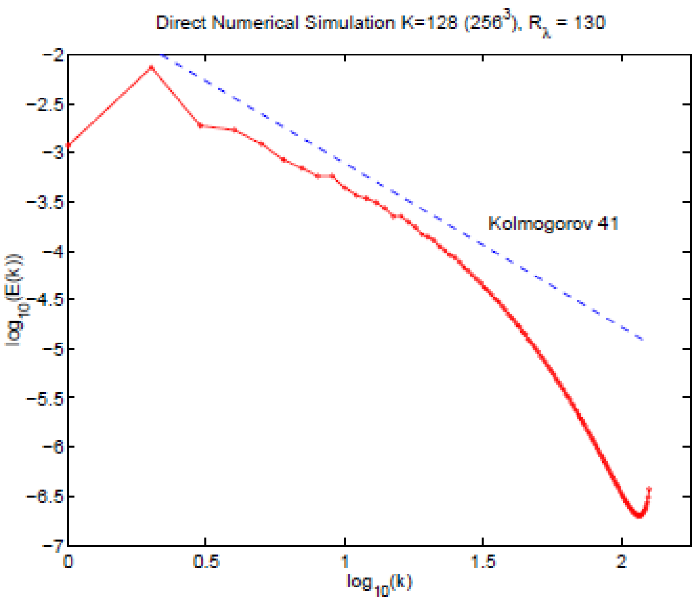


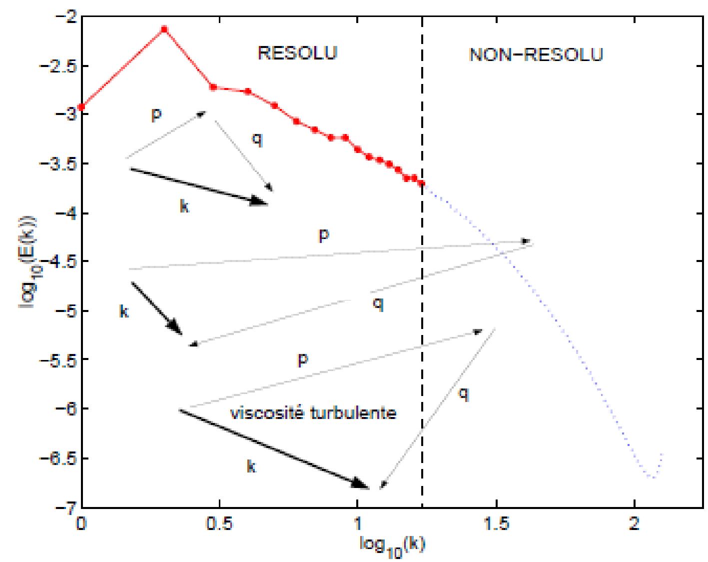
*In Large-Eddy Simulation, **the mesh resolution is deliberately reduced so that only the large-scale motions are resolved**. This is physically justifiable since large-sized eddies contain most of the kinetic energy, and their strength make them the efficient carriers of momentum, heat, mass, etc. On the contrary, small-sized eddies are mainly responsible for dissipation and contribute little to transport. Avoiding the numerical integration of small-scale motions is therefore desirable in most situations.*

How to deal with missing interactions between scales of motions ?

in Fourier space:

$$(\frac{\partial}{\partial t} + \nu k^2)\hat{u}_\alpha(\vec{k},t) = -k_\gamma\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right)\sum_{\vec{p}+\vec{q}=\vec{k}} \hat{u}_\beta(\vec{p},t)\hat{u}_\gamma(\vec{q},t) + f_\alpha(\vec{k},t)$$





$$(\frac{\partial}{\partial t} + \left(\nu + \nu_{\mathrm{turb.}}(\vec{k},t)\right) k^2)\hat{u}_\alpha(\vec{k},t) = -k_\gamma\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right)\sum_{\vec{p}+\vec{q}=\vec{k}} \hat{u}_\beta(\vec{p},t)\hat{u}_\gamma(\vec{q},t) + f_\alpha(\vec{k},t)$$

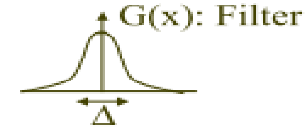
eddy-viscosity for interactions with unresolved modes

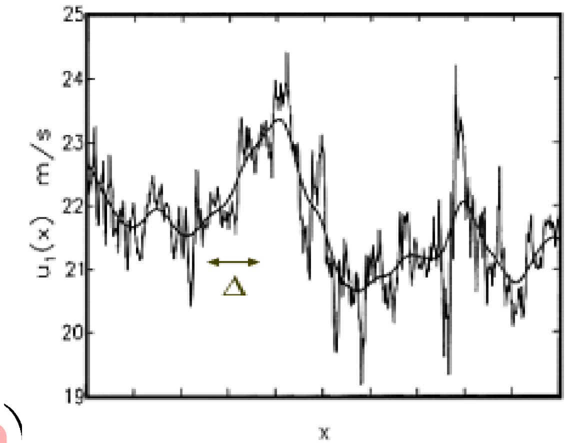
interactions between resolved modes

Equations of large-scale motions in physical space ?

the solution of a LES is expected to represent the real flow variables filtered over a "filter window" whose characteristic width Δ corresponds to the grid resolution

$$\bar{u} = G \otimes u$$





$$\partial_t u_i + u_k \partial_k u_i = -\frac{1}{\rho_0} \partial_i p + 2\nu \, \partial_k S_{ik}$$

$$\partial_t \bar{u}_i + \overline{u_k \partial_k u_i} = -\frac{1}{\rho_0} \partial_i \bar{p} + 2\nu \, \partial_k \bar{S}_{ik}$$

$$\partial_t \bar{u}_i + \bar{u}_k \partial_k \bar{u}_i = -\frac{1}{\rho_0} \partial_i \bar{p} + 2\nu \, \partial_k \bar{S}_{ik} - \partial_k \left(\overline{u_i u_k} - \bar{u}_k \bar{u}_i \right)$$

τ_{ij}: subgrid-scale tensor

$$\tau_{ij} - \frac{1}{3} \delta_{ij} \, \tau_{kk} = -2 \nu_{sgs} \, \bar{S}_{ij}$$

subgrid-scale viscosity

Modeling of the SGS viscosity

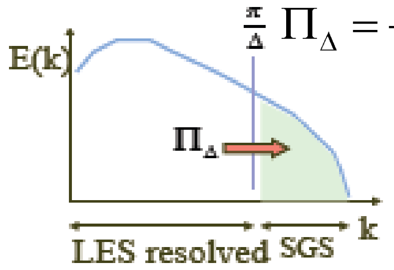
$$\partial_t \bar{k} + \bar{u}_k \partial_k \bar{k} = -\partial_k(\ldots) - 2\nu \left|\bar{S}_{ik}\right|^2 - (-\tau_{ik} \cdot \bar{S}_{ik})$$

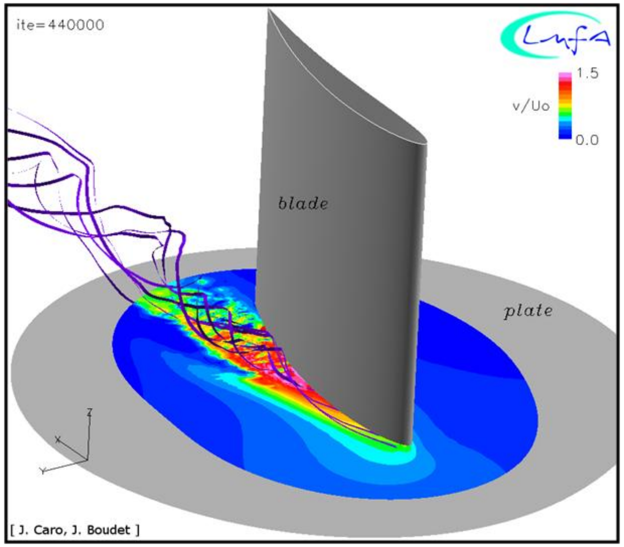
$$\bar{k} = \frac{1}{2}\bar{u}_k \bar{u}_k$$

SGS energy flux

Inertial-range flux

$$\Pi_\Delta = -\left\langle \tau_{ik} \cdot \bar{S}_{ik} \right\rangle$$





$$\left|\bar{S}\right| \equiv \sqrt{2\bar{S}_{ik}\bar{S}_{ik}}$$

Smagorinsky model (1963): $\nu_{sgs} = (C_s\Delta)^2 \left|\bar{S}\right|$

if Δ in the inertial range: $\varepsilon = (C_s\Delta)^2 \left\langle \left|\bar{S}\right|^3 \right\rangle$

which is fully consistent with $\varepsilon \sim \dfrac{\delta u(\Delta)^3}{\Delta}$

for homogeneous and isotropic turbulence

$$\varepsilon \approx (C_s\Delta)^2 \left\langle \left|\bar{S}\right|^2 \right\rangle^{3/2} \quad \left\langle \left|\bar{S}\right|^2 \right\rangle = C_K \varepsilon^{2/3} \int_0^{\pi/\Delta} k^{-5/3+2}\, dk = \ldots \qquad C_s = \left(\frac{3C_K}{2}\right)^{-3/4} \cdot \frac{1}{\pi}$$

$$C_K \approx 1.6 \quad \Rightarrow \quad C_S \approx 0.16$$

Further readings:

C. Meneveau and J. Katz
Annu. Rev. Fluid Mech. 2000

Large-eddy simulation of incompressible flows
P. Sagaut
Springer 2006

Shear-improved Smagorinsky model for the large-eddy simulation of wall-bounded turbulent flows
E. Lévêque, F. Toschi, L. Shao and J.-P. Bertoglio
J. Fluid Mech. (2007) vol. 570, pp. 491–502