Turbulence as a problem of a (statistical) fluid mechanics

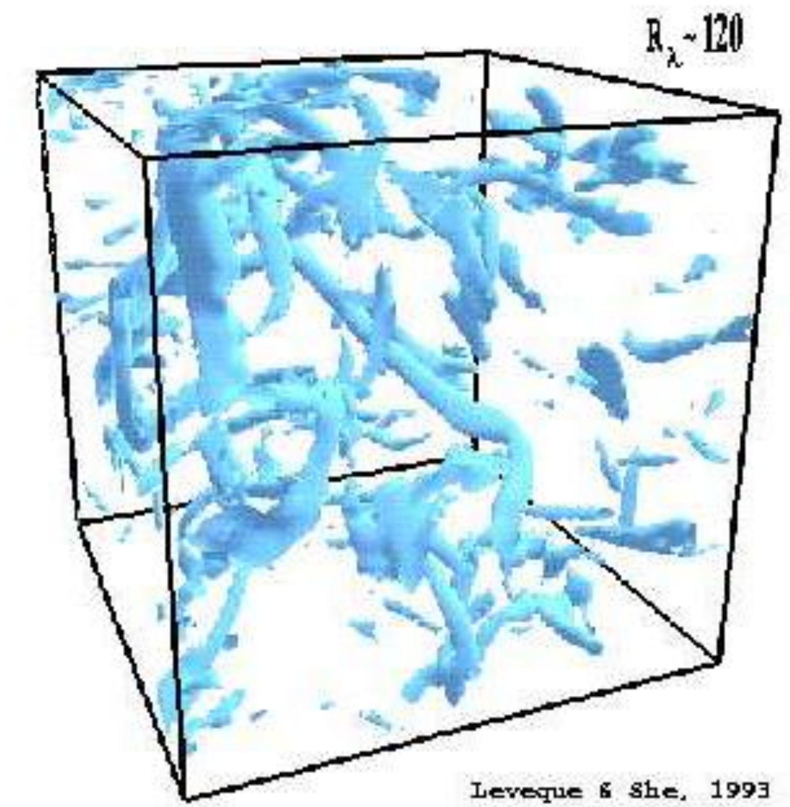
Emmanuel Lévêque
Laurent Chevillard

Laboratoire de physique de l'Ens de Lyon
France

cnrs dépasser les frontières

ENS DE LYON

Lecture 3: the dynamical mechanism of the energy cascade : vortex structures, intermittency and anomalous scaling laws



Intense vortex filaments

Well-defined characteristic vortex structures arise in the interior of homogeneous and isotropic turbulence.

Different scenario:

(1) the structures do not affect statistics. It is possible to ignore structures and construct successful approximations, e.g. Kolmogorov's theory.

(2) the structures do affect statistics but it is not necessary to capture their detailed form. Approximations can be constructed that crudely, but sufficiently, account for structures and their effects

(3) there is no statistical mechanics of turbulence!

Vortex stretching : a kinematic feature of turbulence (in 3d)

diffusion

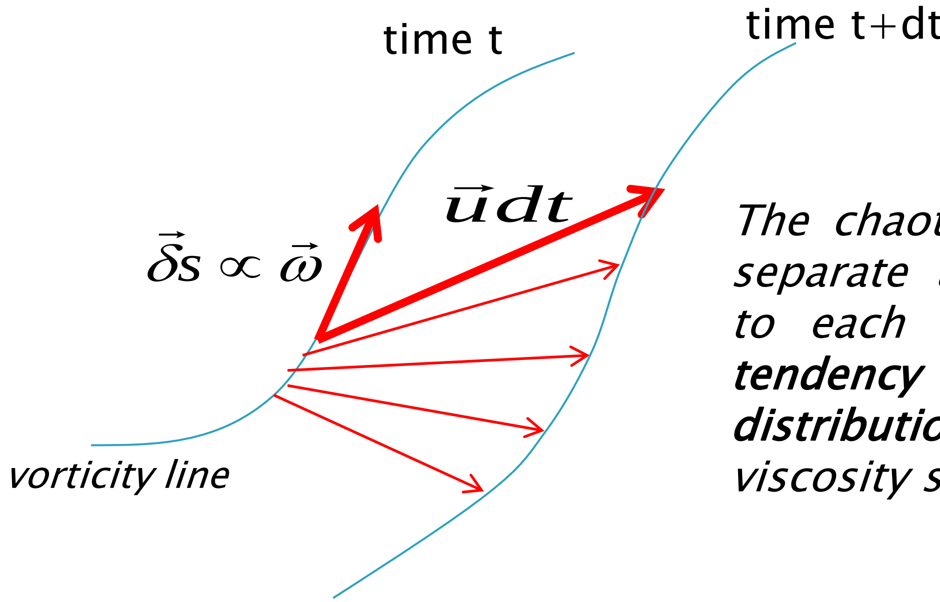
$$\partial_t \vec{\omega} + (\vec{u}\cdot\vec{\nabla})\vec{\omega} = (\vec{\omega}\cdot\vec{\nabla})\vec{u} + \nu\Delta\vec{\omega}$$

advection

vorticity stretching (in 3d only)

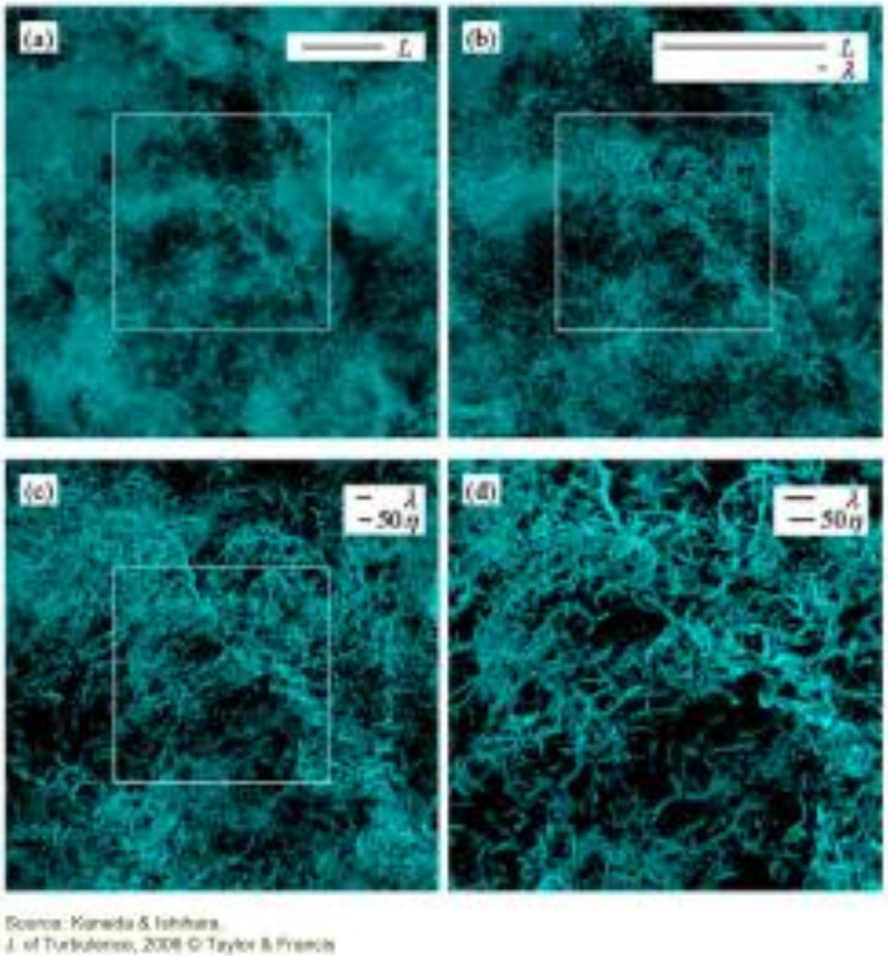
$$d_t \vec{\omega} = (\vec{\omega}\cdot\vec{\nabla})\vec{u} \quad \Longleftrightarrow \quad d_t \vec{\delta s} = (\vec{\delta s}\cdot\vec{\nabla})\vec{u}$$

purely kinematic



The chaotic nature of turbulence tends to separate two-fluid elements initially close to each other. Consequently there is a ***tendency to stretch initial vorticity distribution into elongated structures*** *until viscosity stops the thinning.*

Intensification of vorticity



The vorticity becomes concentrated in a sparse collection of intense thin filaments

Furthermore,

$\frac{\delta s}{\omega}$ is conserved during the stretching process. This means that **the stretching of vorticity line is accompanied by an intensification of the vorticity: The fluid spins harder.**

Eventually, an initial distribution of vorticity tends to stretch and concentrate on thin and elongated fluid structures.

The limitations of Kolmogorov's theory →failure (to some degree) of scenario (1)←

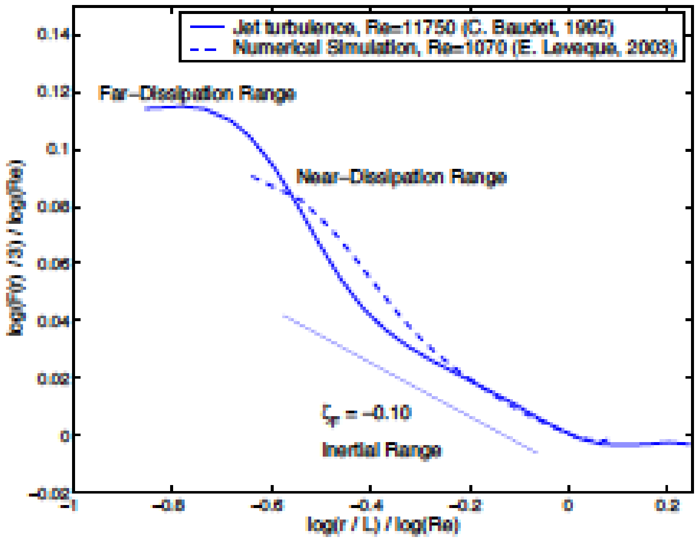
Kolmogorov's hypotheses yield

$$\left\langle |\delta u(r)|^p \right\rangle = B_p (\varepsilon r)^{p/3} \quad \text{for} \quad \eta << r << \ell_0$$ inertial range

velocity structure functions

→ the normalized moments are universal independent of the mean dissipation rate ε and the scale r (according to Kolmogorov's 1941-theory):

$$\frac{\left\langle |\delta u(r)|^p \right\rangle}{\left\langle |\delta u(r)|^2 \right\rangle^{p/2}}$$



$$F(r) = \frac{\left\langle |\delta u(r)|^4 \right\rangle}{\left\langle |\delta u(r)|^2 \right\rangle^2}$$ *flatness of velocity increment distributions*

Instead, the values of the normalized moments increase dramatically with both p and 1/r

$$\left\langle |\delta u(r)|^p \right\rangle \sim r^{\xi_p} \qquad \xi_p \neq \frac{p}{3} \quad \text{for } p \neq 3$$

anomalous scaling exponents

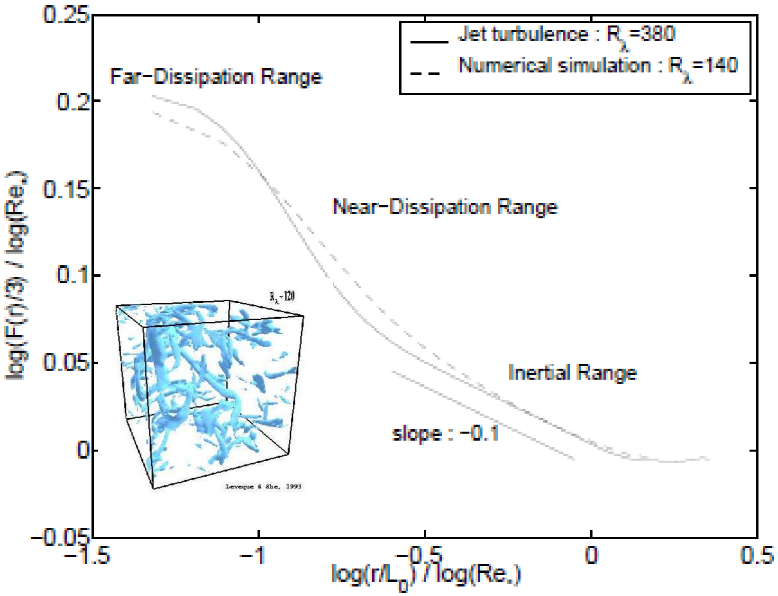
→ **Note that the scale-by-scale energy budget of the energy cascade yields as an exact result from the Navier-Stokes equations** (see tutorial):

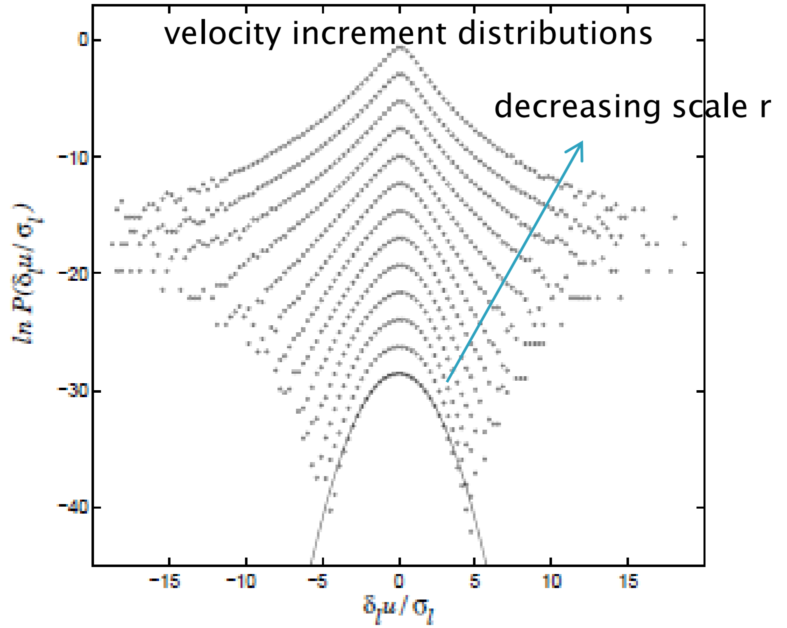
$$\left\langle \delta u_{ll}(r)^3 \right\rangle = -\frac{4}{5}\varepsilon \cdot r \qquad \xi_3 = 1$$

$$F(r) = \frac{\left\langle \left| \delta u(r) \right|^4 \right\rangle}{\left\langle \left| \delta u(r) \right|^2 \right\rangle^2}$$ flatness of velocity increment distributions

F=3 for a Gaussian distribution and increases as long tails develop

→ $\mathrm{Log}\, F(r)/3$ may be viewed as a **measure of intermittency**, i.e. the ratio of intense to quiescent fluid motions at scale r





Kolmogorov's law for the energy spectrum is well supported. However, high-order statistics are not universal in the sense of Kolmogorov's hypotheses.

The refined theory of Kolmogorov and Obouhkov →scenario (2)←

a refinement of K41 theory, in which the spatial fluctuations of the energy dissipation rate are taken into account

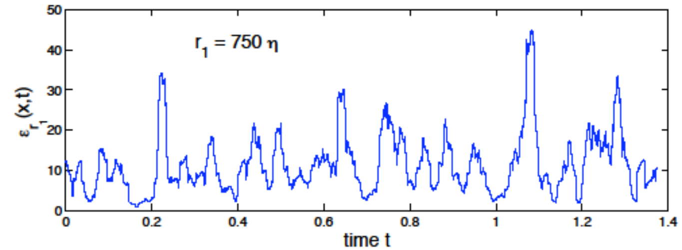
$$\varepsilon(\vec{x},t) = 2\nu \left| S(\vec{x},t) \right|^2$$

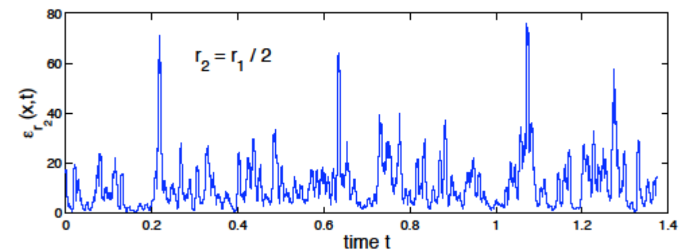
$$\varepsilon_r(\vec{x},t) = \frac{1}{V(B_r)} \int_{B_r} \varepsilon(\vec{x}+\vec{y},t)\, d^3 y$$ at scale r

Kolmogorov's similarity hypotheses are refined by considering that **turbulence is locally conditioned at scale r by ε_r**

$$B(\vec{x},t,\vec{r} \mid \varepsilon_r(\vec{x},t)) = B(\vec{x},t)\, r^{2/3}\, \varepsilon_r(\vec{x},t)^{2/3}$$

$$B(\vec{x},r,t) = B(\vec{x},t)\, r^{2/3} \left\langle \varepsilon_r^{\ 2/3} \right\rangle$$





$$\varepsilon_r \neq \varepsilon$$

$$B(r) = B\left(\varepsilon\, r\right)^{2/3} \left(\frac{r}{\ell_0} \right)^{-\mu}$$ with $$\left\langle \varepsilon_r^{\ 2/3} \right\rangle = \varepsilon^{2/3} \left(\frac{\ell_0}{r} \right)^{-\mu}$$

More generally $\left\langle \varepsilon_r^{\,p/3} \right\rangle \propto \varepsilon^{p/3} \left(\frac{r}{\ell_0} \right)^{\tau_{p/3}}$

$$\left\langle \left| \delta u(r) \right|^p \right\rangle = B_p \left(\varepsilon\, r \right)^{p/3} \left(\frac{r}{\ell_0} \right)^{\tau_{p/3}} \quad \text{with} \quad \xi_p = \frac{p}{3} + \tau_{p/3}$$

***introduction of the integral scale** is related to the idea that the energy cascade results from the iteration of the same elementary process. The number of **cascade steps required for an excitation to propagate from the integral scale l_0 to the small scale r is measured by –log(r/l_0)**. It is assumed that each step of the cascade is stochastic in nature and statistically independent from the previous steps. The result is a build up of intermittency at each cascade step, which may be viewed as a **multiplicative process**.*

$$P(\log(\varepsilon_r)) = \frac{1}{\sqrt{2\pi\sigma_r^{\,2}}} \exp\left(\frac{-\left(\log \varepsilon_r - m_r\right)^2}{2\sigma_r^{\,2}} \right)$$

log-normal model

$$\left\langle \varepsilon_r \right\rangle = \varepsilon$$

$$\left\langle \varepsilon_r^{\,2/3} \right\rangle = \varepsilon^{2/3} \left(\frac{\ell_0}{r} \right)^{-\mu}$$

$\Rightarrow$

$$\xi_p = \frac{p}{3} - \frac{1}{2} \mu\, p\,(p-3)$$

intermittency parameter…
to be adjusted ($\mu \approx 0.2$)

Attempt to connect Log-normality and vortex stretching

A (very) crude model assumes that the vorticity is stochastically independent of the local velocity shear and obeys

$$\frac{d\omega(t)}{dt} = b(t)\cdot\omega(t)$$

b(t) is an effective velocity shear: random function independent of ω(t)

$$\log\frac{\omega(t)}{\omega(0)} = \int_0^t b(s)\,ds$$

For times t very large compared to the correlation time of b(t), the statistics of ω(t) becomes log-normal (according to the central limit theorem)

$\varepsilon \sim \nu\,\omega^2$ gives support to the log-normal hypothesis

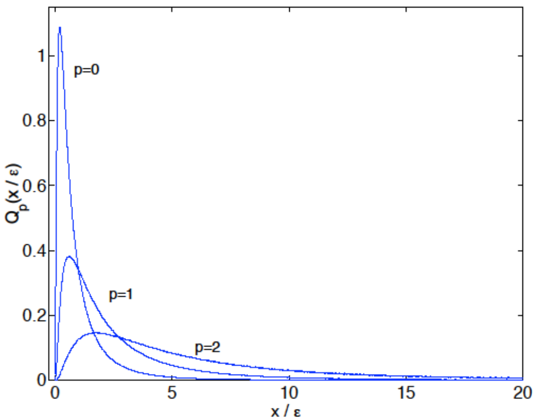
However, ***a realistic model should take into account the correlation between b(t) and*** $\boldsymbol{\omega}$***(t)****…*

The Log–Poisson model by She & Lévêque other scenario (2)

Consider the **hierarchy of fluctuation levels**

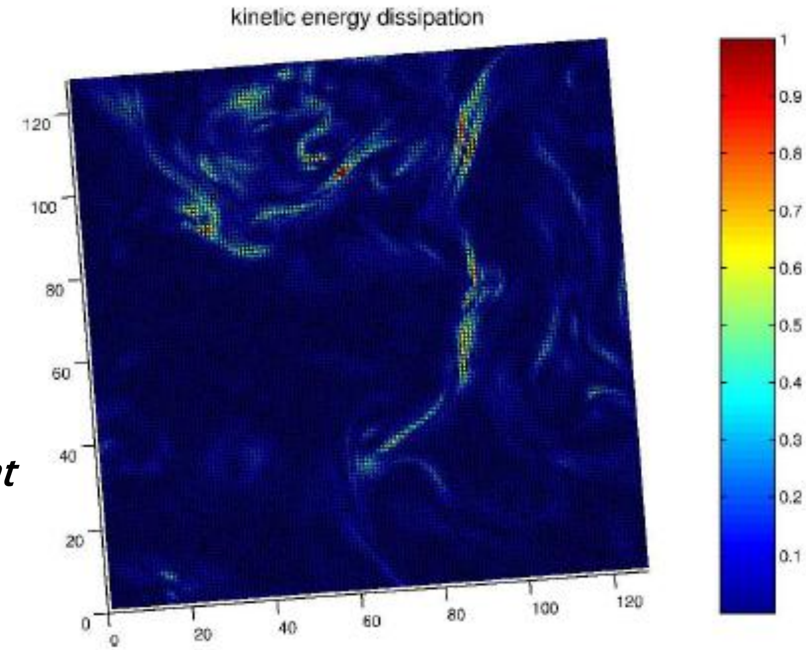
$$\varepsilon_r^{(p)} \equiv \frac{\langle \varepsilon_r^{\,p+1} \rangle}{\langle \varepsilon_r^{\,p} \rangle}$$ instead of moments

$\varepsilon_r^{(p)}$ is the mean dissipation rate weighted by



$$Q_p(\varepsilon_r) \equiv \frac{\varepsilon_r^{\,p} P(\varepsilon_r)}{\langle \varepsilon_r^{\,p} \rangle}$$

as p increases more weight on large amplitude events



Slice of dissipation field

$$\lambda_p = \tau_{p+1} - \tau_p \quad \text{with} \langle \varepsilon_r^{\,p} \rangle \sim r^{\tau_p}$$

$$\varepsilon_r^{(p)} \equiv \frac{\langle \varepsilon_r^{\,p+1} \rangle}{\langle \varepsilon_r^{\,p} \rangle} \sim r^{\lambda_p}$$ with $\lambda_0 = 0 \geq \lambda_1 \geq \ldots \geq \lambda_\infty = \lim_{p\to\infty} \lambda_p$

→ $\lambda_\infty = \lambda_0$ Kolmogorov's theory (1941) is recovered

→ $\lambda_\infty < \lambda_0$ anomalous scaling exponents

$\rightarrow \ \varepsilon_r^{(p)} \approx \varepsilon_r^{(\infty)} = a_p \cdot \varepsilon \left(\frac{r}{\ell_0} \right)^{\lambda_\infty}$ in the limit $p \to \infty$

$\rightarrow \ \frac{\varepsilon_r^{(p+1)}}{\varepsilon_r^{(\infty)}} = A_p \left(\frac{\varepsilon_r^{(p)}}{\varepsilon_r^{(\infty)}} \right)^{\beta}$ defines a relation between successive levels

Therefore $\tau_p = \lambda_\infty p + C\left(1 - \beta^p\right)$

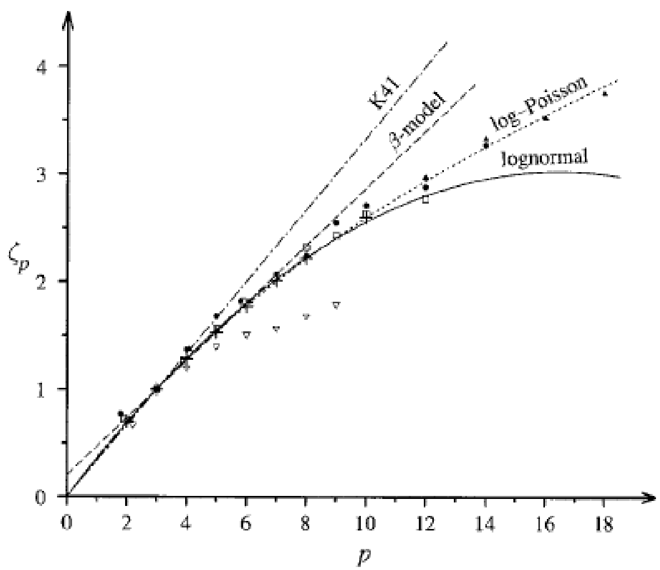
$\left\langle \varepsilon_r^{\ p} \right\rangle \approx \left(\frac{r}{\ell_0} \right)^{\lambda_\infty p + C}$ as $p \to \infty$

$\tau_1 = 0$ ➡ $\lambda_\infty + C(1 - \beta) = 0$

vortex filament

$C = 2$

$\lambda_\infty = -\frac{2}{3}$



$\left\langle \varepsilon_r^{\ p} \right\rangle \sim \left(\varepsilon_r^{(\infty)} \right)^p \times \left(\frac{r}{\ell_0} \right)^{C}$

extreme dissipation x probability

$$\tau_p = -\frac{2}{3} p + 2 \left(1 - \left(\frac{2}{3} \right)^p \right)$$

$$\xi_p = \frac{p}{9} + 2 \left(1 - \left(\frac{2}{3} \right)^{p/3} \right)$$

no empirical parameter

The importance of fluid dynamics in the cascade process naturally calls for ***a Lagrangian representation of turbulence****. The Lagrangian coordinate system moves with the fluid and therefore gets ride of advection effects by the large-sized eddies (sweeping effects) but* ***focuses on distortion effects responsible for the local energy transfer mechanism****.*

→ See next lecture on *Lagrangian dynamics of velocity gradients*

Further readings:

On Kolmogorov's inertial range theories
R.H. Kraichnan
J. Fluid Mech. (1974) vol. 62 (2), pp. 305-330