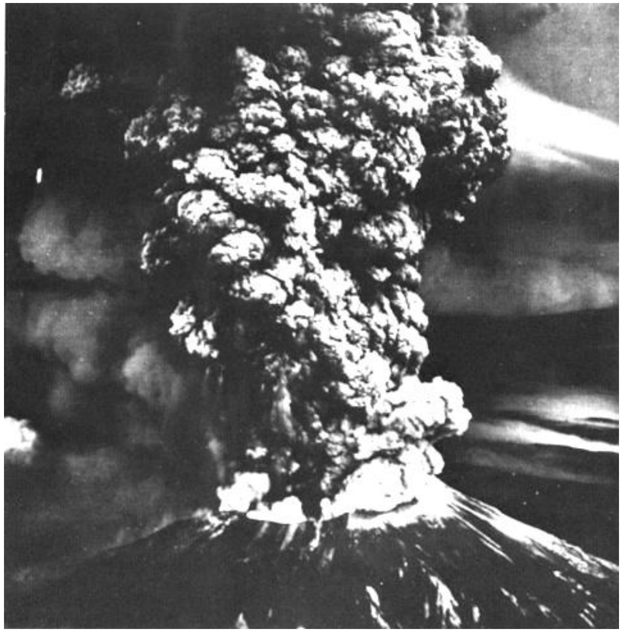
Turbulence as a problem of a (statistical) fluid mechanics

Emmanuel Lévêque
Laurent Chevillard

Laboratoire de physique de l'Ens de Lyon
France

Lecture 2: Turbulence as a problem of statistical physics



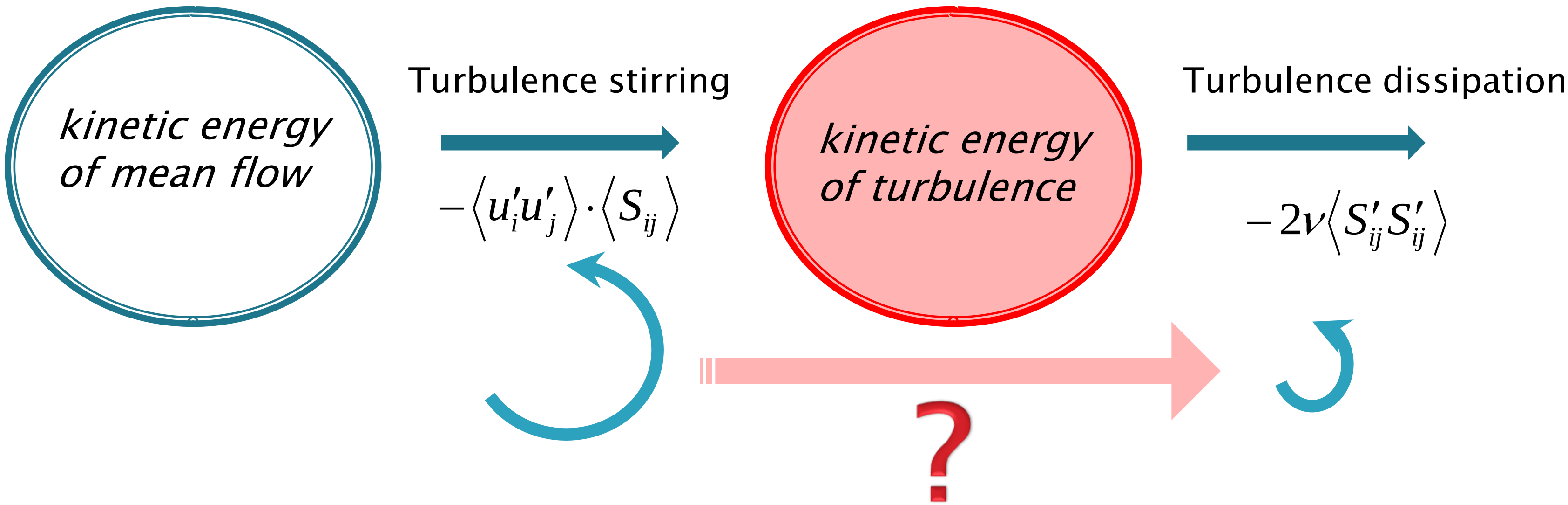
Mount St Helens, 1980

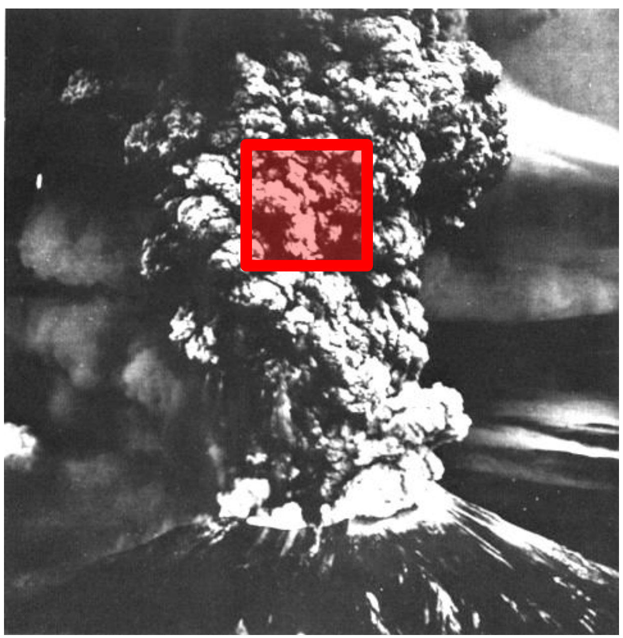
The need for a statistical description of turbulence arises both from the **complexity** of individual flow realizations and from the strong **instability of realizations to small perturbations** in initial and boundary conditions .

→ **examine ensemble of realizations rather than individual realization.**

There is the hope that **statistically distinct properties can be identified and profitably examined.**

Stationary Homogeneous and Isotropic Turbulence





Solutions of the **forced Navier-Stokes equations in a periodic cubic box**

$$\partial_t u_i' + u_k' \partial_k u_i' = -\frac{1}{\rho} \partial_i p' + \nu \partial_k^2 u_i' + f_i$$

$$\partial_i u_i' = 0$$

the external force mimics the large-scale stirring of turbulence

$$\langle f_i u_i' \rangle \approx -\langle u_i' u_j' \rangle \cdot \langle S_{ij} \rangle$$

Eulerian velocity correlation functions

The quantities of theoretical interest in the statistical description of turbulence are **velocity correlation function in space and time**

$$R_{ij\ldots}(\vec{x},t;\vec{x}',t';\ldots)=\left\langle u_i(\vec{x},t)u_j(\vec{x}',t')\ldots\right\rangle$$

G.I. Taylor 1886-1975

Taylor's development of the two-point velocity correlations laid the ground work for the modern statistical approach of turbulence

the simplest and probably most important is the **two-point correlation function in space**

$$R_{ij}(\vec{x},\vec{r})=\left\langle u_i(\vec{x},t)u_j(\vec{x}+\vec{r},t)\right\rangle$$

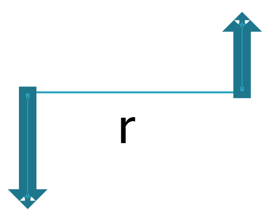
$$R_{ij}(r)=\frac{1}{3}\left\langle u_k^2\right\rangle\left(\frac{f(r)-g(r)}{r^2}r_ir_j+g(r)\delta_{ij}+\cancel{h(r)\varepsilon_{ijk}r_k}\right)$$

under hypothesis of **stationary, homogeneous and isotropic (mirror symmetric) turbulence**

$$R_{ll}(r)=\frac{1}{3}\left\langle u_k^2\right\rangle f(r)$$

longitudinal correlation

$$R_{\perp}(r)=\frac{1}{3}\left\langle u_k^2\right\rangle g(r)$$



transverse correlation

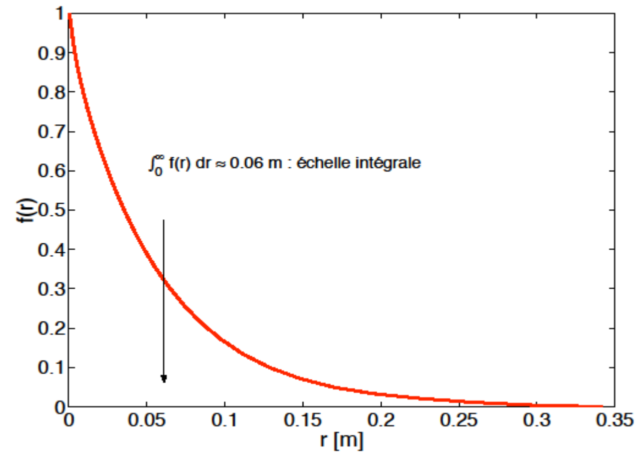
incompressibility implies $g(r)=f(r)+\frac{r}{2}f'(r)$

Micro and macro-scales of turbulence

the focus is on the **longitudinal auto-correlation function f(r):**

→ $\ell_0 \equiv \int_0^\infty f(r)dr$ **integral scale**

decorrelation length scale



→ $f(r) = 1 - \frac{1}{2}\left(\frac{r}{\lambda}\right)^2 + O(r^4)$

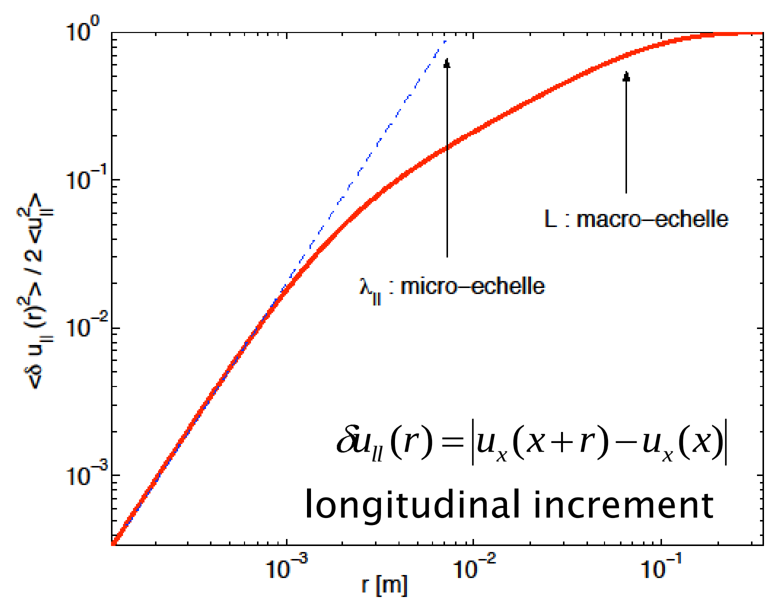
Taylor's microscale

$$\frac{1}{\lambda^2} = -f''(0) > 0 \qquad \frac{1}{\lambda^2} = \frac{1}{u^2}\left\langle (\partial_x u_x)^2 \right\rangle > 0$$

$$u^2 = \left\langle u_x^2 \right\rangle$$

λ may roughly be regarded as a ***measure of the diameter of the smallest eddies which are responsible for the dissipation of energy***, according to G. Taylor

$$\varepsilon = 2\nu \left\langle |S|^2 \right\rangle = 15\nu \frac{u^2}{\lambda^2}$$



→ the Taylor's micro-scale over-estimates the smallest scale of turbulent fluctuations

From Taylor (1935) to Kolmogorov (1941)

large eddies – turbulence stirring

small eddies – energy dissipation

$$\varepsilon = 2\nu\,\overline{|S'|^2}$$

A.N. Kolmogorov 1903–1987

if turbulence is stationary $2\nu\,\overline{|S'|^2} = -\overline{u_i'u_j'}\cdot\bar{S}_{ij} \Longrightarrow u'^2|\bar{S}| \sim \nu\left(\frac{u'}{\lambda}\right)^2$

$$\mathrm{Re} \sim \frac{-\overline{u_i'u_j'}\cdot\bar{S}_{ij}}{2\nu|\bar{S}|^2} \sim \left(\frac{u'\lambda}{\nu}\right)^2 \sim \frac{\overline{|S'|^2}}{|\bar{S}|^2} >> 1$$ locally

$$\Downarrow$$

$$R_\lambda = \frac{u\cdot\lambda}{\nu}$$ **turbulent Reynolds number**

or Reynolds number based on the Taylor's micro-scale

wrong

According to G. Taylor $\left\langle(\partial_x u_x)^2\right\rangle \propto \left(\frac{u}{\lambda}\right)^2 \Longrightarrow \cancel{\left\langle u_x(x+\lambda)u_x(x)\right\rangle \approx 0}$

$$\left\langle(\partial_x u_x)^2\right\rangle \approx \frac{\left\langle \delta u_{ll}(\eta)^2\right\rangle}{\eta^2}$$ **Kolmogorov's elementary scale**

The energy cascade in the Fourier space

see tutorial for energy cascade in physical space

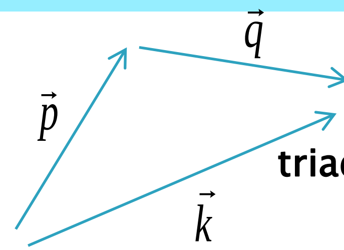
$$\vec{u}(\vec{x},t)=\sum_{\vec{k}}\hat{\vec{u}}(\vec{k},t)\exp(i\vec{k}\cdot\vec{x}) \quad \text{with} \quad \vec{k}=\frac{2\pi}{L}Z^3$$

$$\partial_t u_\alpha - \partial_\gamma(u_\alpha u_\gamma) = -\partial_\alpha \Delta^{-1}(-\partial_\beta\partial_\gamma(u_\beta u_\gamma)) + \nu\Delta u_\alpha$$

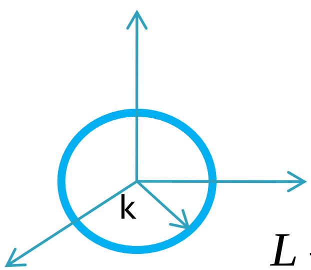
$$\left(\frac{\partial}{\partial t}+\nu k^2\right)\hat{u}_\alpha(\vec{k},t) = -k_\gamma\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right)\sum_{\vec{p}+\vec{q}=\vec{k}}\hat{u}_\beta(\vec{p},t)\hat{u}_\gamma(\vec{q},t)+f_\alpha(\vec{k},t)$$

dissipation

external force: turbulence stirring



triad interaction



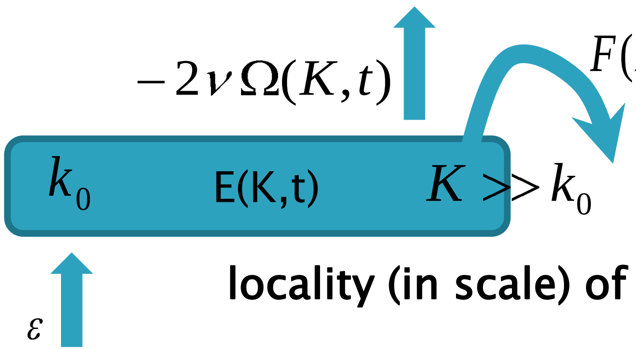
$$L\to\infty : dk\to 0$$

$$(\partial_t + 2\nu k^2)E(k,t) = T(k,t) + \varepsilon_{inj}(k,t)$$ **wavenumber-by-wavenumber energy budget**

density of kinetic energy at wavenumber k: ***energy spectrum***

$$\partial_t \int_0^K E(k,t)dk = \int_0^K T(k,t)dk + \int_0^K \varepsilon_{inj}(k,t)dk - 2\nu \int_0^K k^2 E(k,t)dk$$

$$\varepsilon \quad if \quad K >> k_0$$



$$F(K,t) = -\int_0^K T(k,t)dk$$ **energy flux**

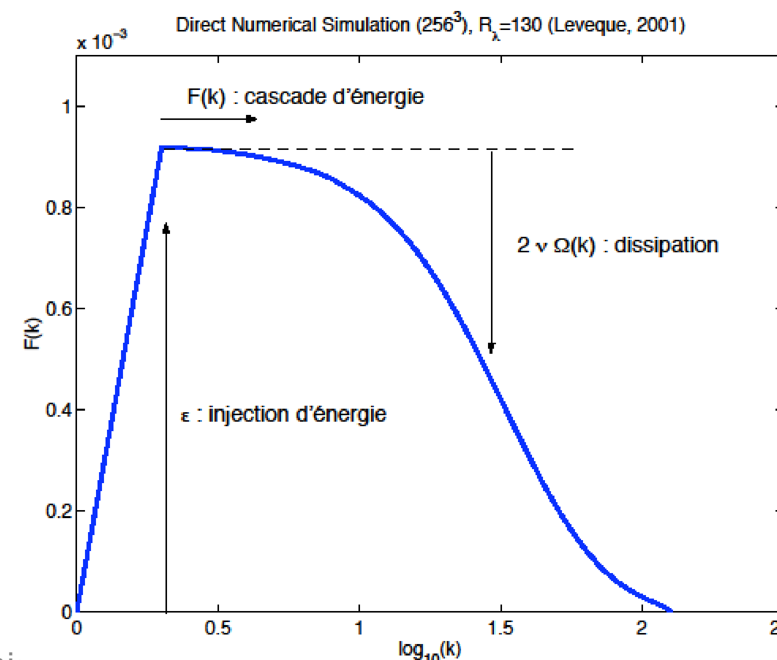
locality (in scale) of the energy transfer ~ cascade

the energy cascade characterizes the out-of-equilibrium state of turbulence

$$F(K) = \varepsilon - 2\nu\Omega(K)$$ in the stationary state

$$if \;\; K << k_{diss} \qquad F(K) = \varepsilon$$

The cascade of energy is rooted in the idea that ***a eddy of a given scale mainly interacts with eddies of similar scale****. Indeed, it is plausible that motions on much large scales should act to transport this eddy without distorting it. On the opposite, the shears associated with excitations at much smaller scales should cancel out over the extend of the eddy.*



Kolmogorov's theory of turbulence (1941)

Kolmogorov's theory focuses on $\delta u_i(\vec{x},\vec{r},t) = u_i(\vec{x}+\vec{r},t) - u_i(\vec{x},t)$

$\delta u_i(r)$ **stationary homogeneous isotropic turbulence**

Kolmogorov's similarity hypotheses:

→ for $r \ll \ell_0$ the distributions of $\delta u_i(r)$ are universal and fixed by the kinematic viscosity of the fluid and the mean energy-dissipation rate (per unit mass)

$$B_{ll}(r) = \left\langle \delta u_{ll}(r)^2 \right\rangle = \sqrt{\nu\varepsilon}\, \Phi\left(\frac{r}{\eta}\right)$$

Φ : universal function

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$ **elementary scale of turbulent motions**

→ for $r \gg \eta$ the distributions of $\delta u_i(r)$ do not depend on ν

$$B_{ll}(r) = B_2\,(\varepsilon\, r)^{2/3} \qquad \ell_0 \gg r \gg \eta$$ **inertial range**

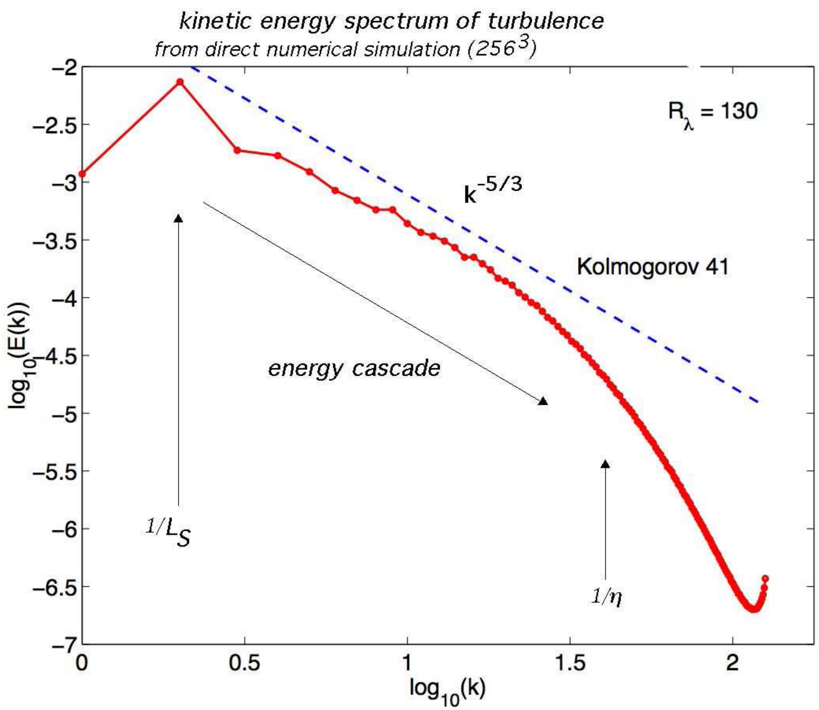
$$\frac{\ell_0}{\eta} \sim \frac{\ell_0\,\varepsilon^{1/4}}{\nu^{3/4}} \sim \frac{\ell_0 \left(\frac{u^3}{\ell_0}\right)^{1/4}}{\nu^{3/4}} \sim \mathrm{Re}^{3/4} \sim R_\lambda^{3/2}$$

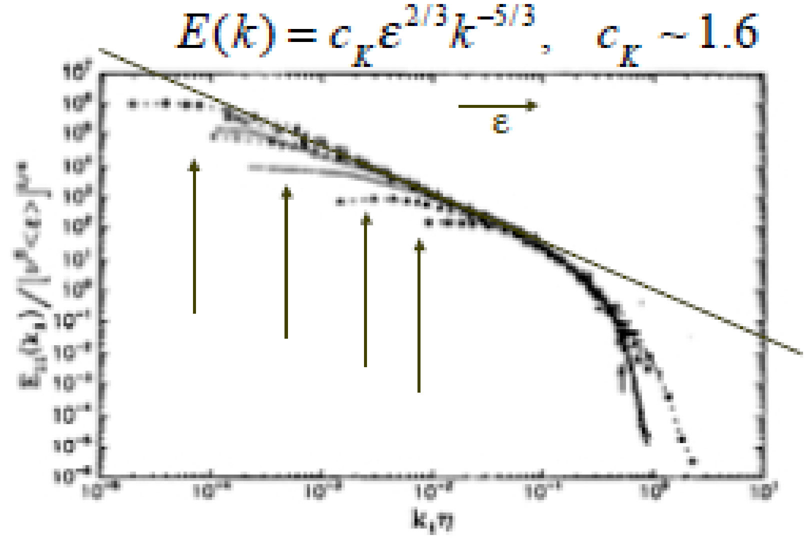
for the energy spectrum $E(k) = C_K \, \varepsilon^{2/3} \, k^{-5/3}$

Kolmogorov's constant

$$k_d \sim \frac{1}{\eta} >> \; k \; >> k_0 \sim \frac{1}{\ell_0}$$

more generally $E(k) = C_K \, \varepsilon^{2/3} \, k^{-5/3} \; f\left(\frac{k}{k_d}\right)$

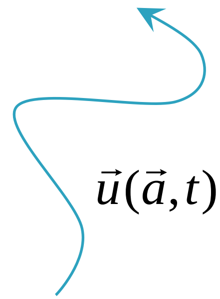




Energy spectrum in frequency

Eulerian $E(\omega) \sim (u\,\varepsilon)^{2/3}\,\omega^{-5/3}$ by substituting $k \sim \dfrac{\omega}{u}$

Lagrangian $E(\omega) \sim \varepsilon\,\omega^{-2}$



material fluid particle trajectory

Further readings:

Turbulence: The Legacy of Kolmogorov
U. Frisch
Cambridge University Press 1995