Turbulence as a problem of a (statistical) fluid mechanics

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Objectives of the lectures:

- Provide physical insight into turbulence in fluids

- Introduce (briefly) some issues in turbulence research
  - Theoretical description of turbulence statistics
  - Lagrangian dynamics of velocity gradients
  - Dealing with turbulence in practice

Prerequisites:

- Basic Fluid Mechanics, Statistical Physics
- Tensor and Index Notation

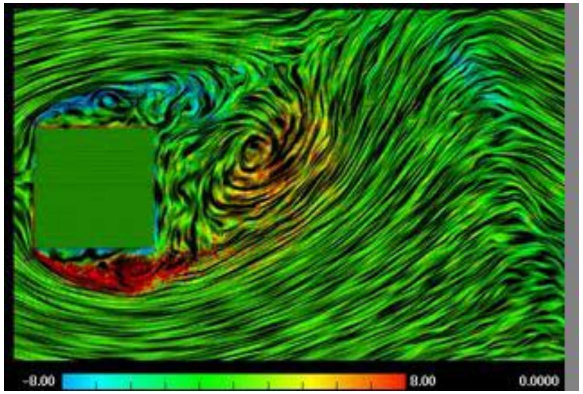
5 Lectures :

- **Physical nature of turbulence in fluids**
- **Statistical description of turbulent fluctuations**
  - Characteristic scales
  - The energy cascade and the Kolmogorov's theory
- **Vortex stretching and intermittency**
  - Anomalous scaling laws
- **Lagrangian dynamics of the velocity gradient tensor**
- **Dealing with turbulence in practice**
  - Toys models: Burgers equation, shell-models
  - Large-eddy simulations

2 Tutorials:

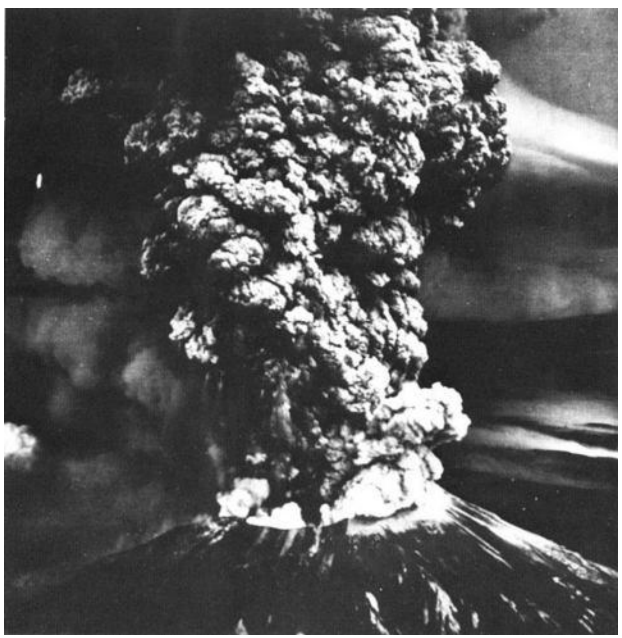
- Statistical equation for velocity correlation tensor :
  the scale-by-scale energy budget of turbulence
- The multifractral formalism for scaling properties of turbulence

Lecture 1: The physical nature of turbulence in fluids



Turbulent Flow past a square cylinder (Re=22000)

Turbulence has to do with
Chaotic… *impossible to predict accurately the details of the dynamics*
Multiscale… *widely differing scales of motions*
Vortical Fluid Motions



Mount St Helens, 1980

Turbulence appears as the net result of the interactions between eddies.

an eddy may be seen as a "glob" of fluid of a given size, or spatial scale, that has a certain structure and life history of its own.

Large-scale eddies usually originate from the instability of high-shear layers, e.g. in the vicinity of a solid boundary.

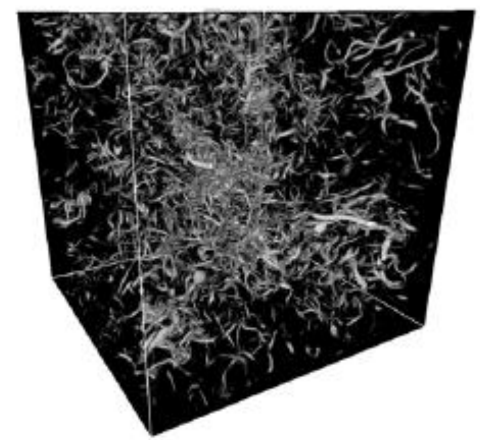
Some general remarks on Turbulence

Turbulence has enormous scientific fascination for physicists, engineers and mathematicians alike:

(hard) problems in turbulence have to do with the ***essential*** **non-linearity** of the dynamics

There is no hint of the non-linear solutions in the linearized approximations

Interplay of order and disorder : well-defined and characteristic vortex structures arise in the midst of disordered fluid motions



and with **strong departure from absolute statistical equilibrium**

Turbulence is usually approached as a stochastic problem, yet usual simplifications that can be used in statistical physics at, or near, absolute equilibrium are not applicable… no temperature

Turbulence is of the greatest practical importance: turbulent fluid motions result in **enhanced mixing and dissipation**

Turbulent transport of heat, mass and momentum is usually some orders of magnitude greater than molecular transport

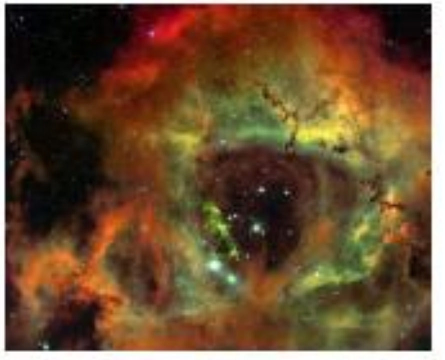
Typically, $$\frac{\nu_{turbulence}}{\nu_{molecular}} \sim \quad Re \equiv \frac{U \cdot L}{\nu_{molecular}} \quad >> \quad 1$$

Re : Reynolds number
U, L : characteristic velocity and length of the turbulent flow
ν : molecular kinematic viscosity of the fluid (ν =10^{-6} m^2s^{-1} for water)

Turbulence is an highly-dissipative process, which is responsible for the majority of human energy consumption in automobile and aircraft fuel, in pipeline pumping charges, etc.

In the atmosphere and oceans, turbulent is responsible for the transport of gases and nutrients and for the uniformity of temperature that makes life on earth possible

In astrophysics, interstellar turbulence is argued to be one of the key ingredients of modern theories of star formation



NASA pictures

We know the equations of fluid motions…

for incompressible flow: $\rho(\vec{x},t) = \rho_0$ *constant mass density*

→ $\partial_t \rho + \partial_k(\rho u_k) = 0$ *mass conservation*

therefore $\partial_k u_k = 0$

→ $\rho(\partial_t u_i + u_k \partial_k u_i) = -\partial_i p + 2\mu \cdot \partial_k \left(S_{ik} - \frac{1}{3}(\nabla \cdot \vec{u})\delta_{ik} \right)$ *momentum conservation*

for **(constant-property) Newtonian fluid**
μ is the **shear viscosity**

$$S_{ik} \equiv \frac{1}{2}(\partial_k u_i + \partial_i u_k)$$ *is the* ***rate-of-strain tensor***

$$\partial_t u_i + u_k \partial_k u_i = -\frac{1}{\rho_0}\partial_i p + 2\nu\,\partial_k S_{ik}$$
$$\partial_k u_k = 0$$

➕ *initial and boundary conditions*

$$\Delta p = -\rho_0 \partial_i u_k \cdot \partial_k u_i = -\rho_0 \partial_i \partial_k (u_i u_k)$$ the pressure can be eliminated

velocity gradients ~ source term

elliptic equation… the solution is sensitive to perturbations in the whole domain!

…the equations alone fix all the solutions but do not choose the *relevant* one among them

in a cubic box with periodic boundary conditions, all solutions verify the **global conservation laws:**

$$\frac{d}{dt}\langle \vec{u} \rangle = \vec{0}$$

brackets denote volume average

$$\frac{d}{dt}\left\langle \frac{1}{2}\vec{u}^2 \right\rangle = -\nu \langle \vec{\omega}^2 \rangle \qquad \frac{dE}{dt} = -2\nu\,\Omega$$

E: *kinetic energy*
Ω: *enstrophy*

$$\frac{d}{dt}\left\langle \frac{1}{2}\vec{u}\cdot\vec{\omega} \right\rangle = -\nu \langle \vec{\omega}\cdot \mathrm{curl}\,\vec{\omega} \rangle \qquad \frac{dH}{dt} = -2\nu\, H_\omega$$

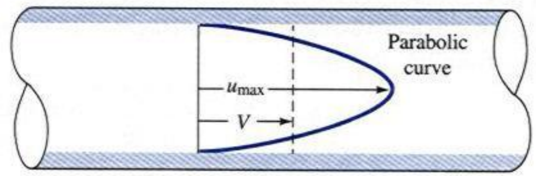
H: ***Helicity***

in two dimensions : $\frac{d}{dt}\langle \omega^n \rangle = 0 \quad (\nu = 0)$

Not sufficient… need to know about the details of the solutions?

An historical introduction: flow in a pipe

Jean-Louis Poiseuille
(1797-1869)



For low flow rate, a **laminar regime** is achieved:

$$u(r) = \frac{G}{4\mu}(R^2 - r^2) \quad \text{parabolic profile}$$

R is the radius
G is the *generalized* pressure drop by unit length

includes potential energy

flow rate: $$Q = \int_0^R 2\pi r\, u(r)\, dr = \frac{\pi G R^4}{8\mu} \quad \text{Poiseuille law}$$

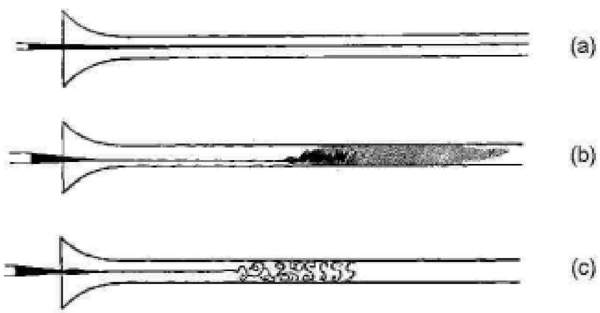
$Q = \pi R^2 U$ U : characteristic velocity

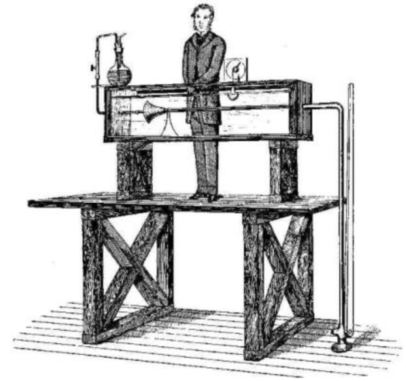
$$\varepsilon = \frac{GU}{\rho} = 8\nu \frac{U^2}{R^2}$$ **dissipation rate of kinetic energy per unit mass**

numerical application: G=500 Pa/m, water, R=10 cm

→ compute the value of U… why is it wrong?

flow in a pipe: laminar to turbulent



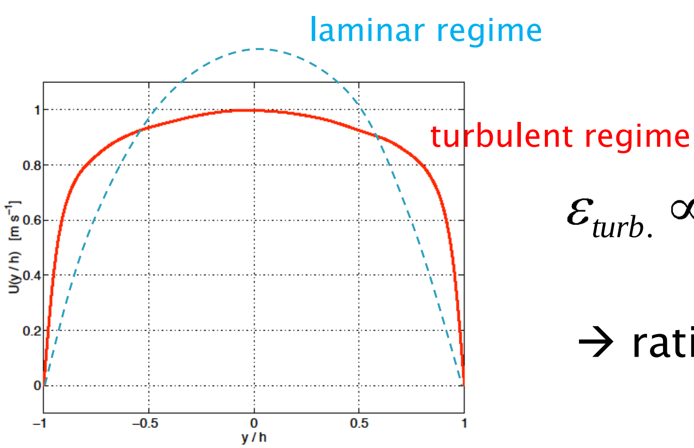


Osborne Reynolds
(1842–1912)

O. Reynolds , Phil. Trans. Roy. Soc. London Ser. A, vol 174, 1883, p. 935–982

(b) “the colour band would all at once mix up with surrounding water, and fill the rest of the tube with a mass of coloured water”

(c) “on the viewing the tube by the light of an electric spark, the mass of colour resolved itself into a mass of more or less distinct curls, showing eddies”



$$\varepsilon_{turb.} \propto \frac{U^3}{R} >> \nu \frac{U^2}{R^2}$$ dissipation rate (per unit mass)

→ ratio proportional to $\mathrm{Re} = \frac{U \cdot R}{\nu}$: Reynolds number

Turbulence strongly modifies the kinetics of the flow

Mathematical representation of a turbulent flow

$$u_i(\vec{x},t) = \bar{u}_i(\vec{x},t) + u_i'(\vec{x},t)$$ **Reynolds decomposition**

instantaneous velocity field = mean flow + rapidly fluctuating (turbulent) component

$$\overline{\alpha\lambda + \beta\mu} = \alpha\bar{\lambda} + \beta\bar{\mu}$$

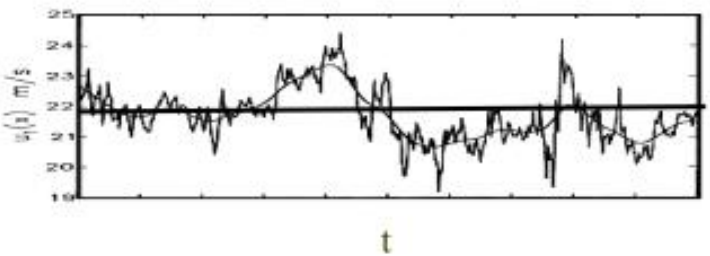
$$\frac{\overline{\partial\lambda}}{\partial s} = \frac{\partial\bar{\lambda}}{\partial s}$$

$$\overline{\bar{\lambda}\mu} = \bar{\lambda}\cdot\bar{\mu}$$

in practice: spatio-temporal average ~ statistical average

$$\bar{f}(\vec{x},t) = \int_{R^3\times R} f(\vec{x}-\vec{\xi}),t-\tau)\cdot\Phi(\vec{\xi},\tau)\,\mathrm{d}^3\xi\,\mathrm{d}\tau = f * \Phi$$

$$\int_{R^3\times R}\Phi(\vec{\xi},\tau)\,\mathrm{d}^3\xi\,\mathrm{d}\tau = 1$$ *weighted average in time and space*



"Observe the motion of the surface of water, which resembles that of hair, which has two motions, of which one is caused by the weight of hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the principal current, the other to the random and reverse motion."

A clear prefiguring of the Reynolds decomposition by Leonardo da Vinci (dated 1510)

Kinetics of the mean flow

$\overline{u_i'} = 0$ but $\mathrm{E} = \frac{1}{2}\overline{u_i}^{\,2} + \frac{1}{2}\overline{u_i'^2} = \overline{e_c} + \overline{e_c'}$ kinetic energy per unit mass

$$\rightarrow \quad \rho\left(\partial_t \overline{u}_i + \overline{u}_k \partial_k \overline{u}_i\right) = -\partial_i \overline{p} + \partial_k \left(2\eta\, \overline{S}_{ik} - \rho\, \overline{u_i' u_k'} \right)$$

$$\partial_k \overline{u}_k = 0$$

viscous stress **Reynolds (turbulent) stress**

$$\rightarrow \quad \partial_t \overline{e_c} + \overline{u}_k \partial_k \overline{e_c} = \partial_k \left(-\frac{1}{\rho}\overline{u}_k \cdot \overline{p} + \left(2\nu\, \overline{S}_{ik} - \overline{u_i' u_k'} \right) \cdot \overline{u}_i \right) - \left(2\nu\, \overline{S}_{ik} - \overline{u_i' u_k'} \right) \cdot \overline{S}_{ik}$$

viscous stress VERSUS **turbulent stress,** introduction of the **Reynolds number:**

$$\frac{1}{V}\int_V \overline{u_i' u_k'} \cdot \overline{S}_{ik} \sim \frac{U^3}{L}$$

$$\frac{1}{V}\int_V 2\nu\, \overline{S}_{ik} \cdot \overline{S}_{ik} \sim \nu \frac{U^2}{L^2}$$

$$\frac{\int_V \overline{u_i' u_k'} \cdot \overline{S}_{ik}}{\int_V 2\nu\, \overline{S}_{ik} \cdot \overline{S}_{ik}} \sim \mathrm{Re} \equiv \frac{U \cdot L}{\nu}$$

global order parameter

→ Re<<1 : viscous >> turbulent

laminar regime*: the mean flow is solution of the Navier-Stokes equations*

→ Re>>1 : viscous << turbulent

turbulent regime

the turbulent regime corresponds to high Reynolds numbers

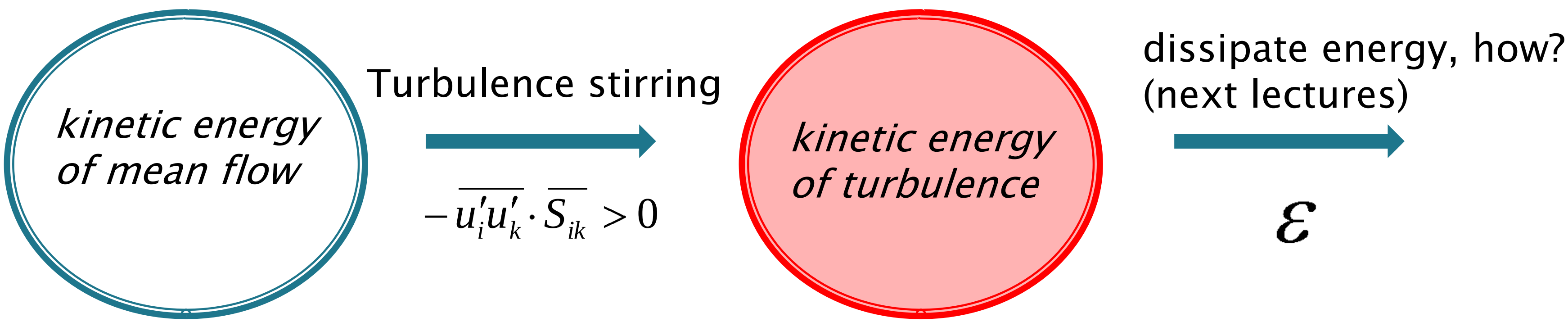
Kinetics of the mean flow (in the turbulent regime)

$$\rho\,(\partial_t \bar{u}_i + \bar{u}_k \partial_k \bar{u}_i) = -\partial_i \bar{p} + \partial_k \left(-\rho\, \overline{u_i' u_k'} \right)$$

$$\partial_t \bar{e}_c + \bar{u}_k \partial_k \bar{e}_c = \partial_k \left(-\frac{1}{\rho} \bar{u}_k \cdot \bar{p} - \overline{u_i' u_k'} \cdot \bar{u}_i \right) + \overline{u_i' u_k'} \cdot \bar{S}_{ik}$$

transport within the mean flow

energy exchange with turbulent flow



The mean dissipation rate $\varepsilon = -\frac{1}{V} \int_V \overline{u_i' u_k'} \cdot \bar{S}_{ik} \sim \frac{U^3}{L}$ *does not depend on* ν

The *mean flow* does not dissipate energy but transmits energy to the *turbulent flow*

turbulent disturbances grow by extracting energy from the *mean flow*

Closure problem

$$\partial_t \left(\overline{u_i' u_k'} \right) = \overline{u_i' \partial_t u_k'} + \overline{u_k' \partial_t u_i'} = \ldots$$ **Reynolds stress equation**

the Reynolds stress equation involves the third-order tensor $\overline{u_i' u_k' u_l'}$
Similarly, the equation for $u_i' u_k' u_l'$ involves the 4th-order tensor $\overline{u_i' u_k' u_l' u_m'}$

What does it mean ?

The Navier-Stokes equation is equivalent to an infinite hierarchy of statistical equations coupling all moments. Any finite subset of this hierarchy is not closed and possesses more unknowns than are set by the subset... a closure condition must be posed to provide a description with a reduced set of moments.

→ *The Navier-Stokes equation in itself does not provide a complete definition of the (statistical) problem of turbulence.* ***Additional physical principles must be invoked to fix the relevant solution.***

→ ***need to investigate the nature of turbulent motions in order to pose the correct principles***

A common thread is to assume (arbitrarily) that

$$-\rho\,\overline{u_i' u_k'}\,(\vec{x},t) \approx 2\,\mu_{turb.}\,(\vec{x},t)\,\bar{S}_{ik}\,(\vec{x},t) - \frac{1}{3}\,\rho\,\overline{u_l' u_l'}\,(\vec{x},t)\,\delta_{ik}$$

turbulent viscosity **turbulent pressure**

The concept of turbulent viscosity provides a valuable simple contact with the dynamics of turbulence

→ Unlike the molecular viscosity, **the turbulent viscosity is a property of the flow**, not of the fluid, which **needs to be modelled**

mixing-length model
(k–ε) model
etc.

→ $-\overline{u_i' u_k'}\cdot\bar{S}_{ik} \sim \nu_{turb.}\,\bar{S}_{ik}\cdot\bar{S}_{ik}$

$$\mathrm{Re} \sim \frac{\int_V \overline{u_i' u_k'}\cdot\bar{S}_{ik}}{\int_V 2\nu\,\bar{S}_{ik}\cdot\bar{S}_{ik}} \sim \frac{\nu_{turb.}}{\nu_{mol.}}$$

energy dissipation is greatly enhanced in the turbulent regime

Further readings:

Turbulent flows
a textbook of a course taught at Cornell University
by S. Pope
ed. Cambridge University Press, Cambridge, UK (2000)

A first course in turbulence
a reference book on turbulence
by H. Tennekes & J. L. Lumley
ed. MIT Press, Cambridge, USA (1972)