Summary: Constantin and Fefferman's regularity result

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1 Introduction

We consider the incompressible Navier–Stokes equations in $\mathbb{R}^3$, assuming suitable decay of the solution at infinity. Our goal is to provide the *essential* arguments underlying the celebrated conditional regularity result of Constantin and Fefferman [1] from 1993. The main theorem they prove can be stated as follows.

Theorem 1 (Constantin and Fefferman, 1993) *Suppose there exists constants Ω and ρ such that*

$$|\sin\phi| \leq \frac{|\boldsymbol{y}|}{\rho},$$

holds whenever $|\boldsymbol{\omega}(\boldsymbol{x},t)| > \Omega$ and $|\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y},t)| > \Omega$, for $0 \leqslant t \leqslant T$ for any $T > 0$. Here $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{x},t)$ is the vorticity field and ϕ is the angle between the vorticity vectors $\boldsymbol{\omega}(\boldsymbol{x},t)$ and $\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y},t)$. Then the solution to the initial value problem of the Navier–Stokes equation is strong and hence smooth on the time interval $[0,T]$.

Our proof is brief. We will list the caveats thus induced at the end.

2 Proof

2.1 Enstrophy evolution

We start by writing the incompressible Navier–Stokes equations in the form

$$\begin{aligned} \partial_t \boldsymbol{u} + \boldsymbol{\omega} \wedge \boldsymbol{u} &= \nu\, \Delta \boldsymbol{u} - \nabla\big(p + \tfrac{1}{2}|\boldsymbol{u}|^2\big), \\ \nabla \cdot \boldsymbol{u} &= 0. \end{aligned}$$

By taking the curl of the Navier–Stokes equation we arrive at the equation for the evolution of the vorticity $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u}$ as follows:

$$\partial_t \boldsymbol{\omega} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \nu\, \Delta \boldsymbol{\omega} + D\boldsymbol{\omega},$$

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where D is the deformation matrix (defined as the symmetric part of $\nabla \boldsymbol{u}$). Considering the L^2-inner product of this evolution equation for the vorticity with the vorticity itself, we generate the equation for the evolution of the enstrophy

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\omega}\|_{L^2}^2 + \nu\|\nabla\boldsymbol{\omega}\|_{L^2}^2 = \int \boldsymbol{\omega}\cdot(D\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{x}.$$

We implicity assumed suitable decay for the vorticity at infinity when integrating by parts to derive the dissipative term. The main idea in Constantin and Fefferman's paper is to try to be more subtle about estimating the vorticity stretching term. Lastly, note that since $\boldsymbol{u}$ is divergence-free the following quantities are equivalent:

$$\|\nabla\boldsymbol{u}\|_{L^2}^2 = \|\boldsymbol{\omega}\|_{L^2}^2 = 2\int_{\Omega}\operatorname{tr}\bigl(D^2\bigr)\,\mathrm{d}\boldsymbol{x}.$$

2.2 **Biot–Savart Law**

Note that since $\nabla\cdot\boldsymbol{u}=0$, there exists a vector potential $\boldsymbol{\psi}$ such that $\boldsymbol{u}=\nabla\wedge\boldsymbol{\psi}$. Hence we see that $\boldsymbol{\omega}=-\nabla\wedge\boldsymbol{\psi}$ and thus $\boldsymbol{u}(\boldsymbol{x})=-\bigl(\nabla\wedge(\Delta^{-1}\boldsymbol{\omega})\bigr)(\boldsymbol{x})$. This is the Biot–Savart Law from potential theory. More explicitly, we have

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{1}{4\pi}\int \boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})\wedge\nabla\left(\frac{1}{|\boldsymbol{y}|}\right)\mathrm{d}\boldsymbol{y}.$$

This is a convolution and in the integrand we can freely swap $\boldsymbol{x}+\boldsymbol{y}$ and $\boldsymbol{y}$.

2.3 **Deformation matrix**

Taking the gradient with respect to $\boldsymbol{x}$ of the Biot–Savart Law we see that

$$\begin{aligned}\nabla\boldsymbol{u}(\boldsymbol{x}) &= \frac{1}{4\pi}\int \boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})\wedge\nabla\nabla\left(\frac{1}{|\boldsymbol{y}|}\right)\mathrm{d}\boldsymbol{y}\\ &= \frac{1}{4\pi}\int \boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})\wedge\bigl(3\,\hat{\boldsymbol{y}}\otimes\hat{\boldsymbol{y}}-I\bigr)\frac{\mathrm{d}\boldsymbol{y}}{|\boldsymbol{y}|^3},\end{aligned}$$

where we have used that (by direct computation)

$$\nabla\nabla\left(\frac{1}{|\boldsymbol{y}|}\right) = \frac{1}{|\boldsymbol{y}|^3}\bigl(3\,\hat{\boldsymbol{y}}\otimes\hat{\boldsymbol{y}}-I\bigr).$$

Here I is the 3×3 identity matrix and $\hat{\boldsymbol{y}} \coloneqq \boldsymbol{y}/|\boldsymbol{y}|$ is the corresponding unit vector. Note that if $V=\boldsymbol{\omega}\wedge(\boldsymbol{v}\otimes\boldsymbol{v})$, which is a 3×3 matrix, then

$$\tfrac{1}{2}\bigl(V+V^{\mathrm{T}}\bigr) = \tfrac{1}{2}\bigl((\boldsymbol{\omega}\wedge\boldsymbol{v})\otimes\boldsymbol{v}+\boldsymbol{v}\otimes(\boldsymbol{\omega}\wedge\boldsymbol{v})\bigr).$$

Hence for example, note that $I=\hat{\boldsymbol{e}}\otimes\hat{\boldsymbol{e}}$, where $\hat{\boldsymbol{e}}_i$ for $i=1,2,3$ represent the unit direction vectors. Thus set $V=\boldsymbol{\omega}\wedge(\hat{\boldsymbol{e}}\otimes\hat{\boldsymbol{e}})=(\boldsymbol{\omega}\wedge\hat{\boldsymbol{e}})\otimes\hat{\boldsymbol{e}}$, and let R be the antisymmetric part of $\nabla \boldsymbol{u}$ defined by $R\boldsymbol{v} = \frac{1}{2}\boldsymbol{\omega} \wedge \boldsymbol{v}$ for any $\boldsymbol{v} \in \mathbb{R}^3$. Then, by the antisymmetry of the cross product, we see that

$$\begin{aligned} V + V^{\mathrm{T}} &= (\boldsymbol{\omega} \wedge \boldsymbol{e}) \otimes \boldsymbol{e} + \boldsymbol{e} \otimes (\boldsymbol{\omega} \wedge \boldsymbol{e}) \\ &= 2(R\,\boldsymbol{e}) \otimes \boldsymbol{e} + 2\boldsymbol{e} \otimes (R\,\boldsymbol{e}) \\ &= 2R(\boldsymbol{e} \otimes \boldsymbol{e}) + 2\boldsymbol{e} \otimes (R\,\boldsymbol{e}) \\ &= 2(R + R^{\mathrm{T}}) \\ &= O. \end{aligned}$$

Hence we deduce that

$$D(\boldsymbol{x}) = \frac{3}{8\pi} \int \big(\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y}) \wedge \hat{\boldsymbol{y}} \otimes \hat{\boldsymbol{y}} + \hat{\boldsymbol{y}} \otimes \boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y}) \wedge \hat{\boldsymbol{y}}\big) \frac{\mathrm{d}\boldsymbol{y}}{|\boldsymbol{y}|^3}.$$

2.4 **Vorticity stretching**

Now consider the vorticity stretching term. If we set $\hat{\boldsymbol{\omega}} := \boldsymbol{\omega}/|\boldsymbol{\omega}|$, then we see that

$$\big(\hat{\boldsymbol{\omega}} \cdot (D\hat{\boldsymbol{\omega}})\big)(\boldsymbol{x}) = \frac{3}{4\pi} \int \hat{\boldsymbol{\omega}}(\boldsymbol{x}) \cdot \big(\hat{\boldsymbol{\omega}}(\boldsymbol{x}+\boldsymbol{y}) \wedge \hat{\boldsymbol{y}}\big)\big(\hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{\omega}}(\boldsymbol{x})\big)\big|\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})\big| \frac{\mathrm{d}\boldsymbol{y}}{|\boldsymbol{y}|^3}.$$

Note that the integrand contains the triple scalar product given by

$$\hat{\boldsymbol{\omega}}(\boldsymbol{x}) \cdot \big(\hat{\boldsymbol{\omega}}(\boldsymbol{x}+\boldsymbol{y}) \wedge \hat{\boldsymbol{y}}\big).$$

This is invariant to a cyclic rotation of the vectors therein and so it is equivalent to

$$\hat{\boldsymbol{y}} \cdot \big(\hat{\boldsymbol{\omega}}(\boldsymbol{x}) \wedge \hat{\boldsymbol{\omega}}(\boldsymbol{x}+\boldsymbol{y})\big).$$

It can also be thought of as $\det\big(\hat{\boldsymbol{y}}, \hat{\boldsymbol{\omega}}(\boldsymbol{x}), \hat{\boldsymbol{\omega}}(\boldsymbol{x}+\boldsymbol{y})\big)$. Importantly though, in magnitude it has an upper bound given by

$$\Big|\hat{\boldsymbol{y}} \cdot \big(\hat{\boldsymbol{\omega}}(\boldsymbol{x}) \wedge \hat{\boldsymbol{\omega}}(\boldsymbol{x}+\boldsymbol{y})\big)\Big| \leqslant |\sin\phi|,$$

where ϕ is the angle between $\hat{\boldsymbol{\omega}}(\boldsymbol{x})$ and $\hat{\boldsymbol{\omega}}(\boldsymbol{x}+\boldsymbol{y})$.

2.5 **Estimating vorticity stretching**

Using the assummption on $|\sin\phi|$ stated in the theorem, we see that the vorticity stretching term can be bounded as follows

$$\begin{aligned} \int \hat{\boldsymbol{\omega}} \cdot (D\hat{\boldsymbol{\omega}})(\boldsymbol{x})\,\big|\boldsymbol{\omega}(\boldsymbol{x})\big|^2\,\mathrm{d}\boldsymbol{x} &\leqslant \frac{3}{4\pi} \int \big|\boldsymbol{\omega}(\boldsymbol{x})\big|^2 \int |\sin\phi|\,\big|\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})\big| \frac{\mathrm{d}\boldsymbol{y}}{|\boldsymbol{y}|^3}\,\mathrm{d}\boldsymbol{x} \\ &\leqslant \frac{c}{\rho}\|\boldsymbol{\omega}\|_{L^4}^2 \cdot \left(\int \left(\int \big|\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})\big| \frac{\mathrm{d}\boldsymbol{y}}{|\boldsymbol{y}|^2}\right)^2 \mathrm{d}\boldsymbol{x}\right)^{1/2} \\ &\leqslant \frac{c}{\rho}\|\nabla\boldsymbol{\omega}\|_{L^2}^{3/2}\|\boldsymbol{\omega}\|_{L^2}^{1/2} \cdot \|\boldsymbol{\omega}\|_{L^1}^{2/3}\|\boldsymbol{\omega}\|_{L^2}^{1/3} \\ &\leqslant \frac{\nu}{8}\|\nabla\boldsymbol{\omega}\|_{L^2}^2 + \frac{c}{\nu^3\rho^4}\big(\|\boldsymbol{\omega}\|_{L^1}^{8/3}\|\boldsymbol{\omega}\|_{L^2}^{4/3}\big)\|\boldsymbol{\omega}\|_{L^2}^2, \end{aligned}$$

where we have used the Hölder, Gagliardo–Nirenberg–Sobolev and Young inequalities.

2.6 **Bounded enstrophy**

If we now include this estimate for the vortex stretching term in the evolution equation for the enstrophy in Section 2.1 we can deduce that for any $T > 0$, the enstrophy is bounded. This is sufficient to establish regular smooth solutions to the Navier–Stokes equations in $\mathbb{R}^3$ for all time.

3 Caveats

Here is the list of results that we glossed over in our proof of the Constantin and Fefferman conditional regularity result above:

1. Constantin and Fefferman establish, in a very succint proof on page 782, that $\|\boldsymbol{\omega}\|_{L_1}$ is bounded on any finite time interval. Combining this with the fact that $\|\boldsymbol{\omega}\|_{L_2}$ is square-integrable in time allows us to draw the conclusion about the enstrophy being bounded for all time in Section 2.6. (Of course we are implicitly assuming the condition on $|\sin\phi|$ stated in the theorem.);
2. If we know that the enstrophy is bounded in time, then we know that solutions to the Navier–Stokes equations are in fact strong and thus smooth;
3. More rigorously we should actually consider an approximate system to the Navier–Stokes equations for which we know global regularity—say the approximate system is a perturbation by a parameter ϵ away. We would carry through analogous estimates for the approximate system to those above, proving bounds uniform in ϵ. Then passing to a subsequence if necessary, we would take the limit as $\epsilon \to 0$. In the case of Constantin and Fefferman, they constructed their approximate system by mollifying the advecting velocity, i.e. by replacing the term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ by $\boldsymbol{u}_\epsilon\cdot\nabla\boldsymbol{u}$, where $\boldsymbol{u}_\epsilon$ is a smoother (mollified) velocity field.

References

1. Constantin, P. and Fefferman, C. 1993 *Direction of vorticity and the problem of global regularity for the Navier–Stokes equations*, Indiana University Mathematics Journal **42**(3), pp. 775–789.