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An introduction to the rigorous theory of weak solutions of the Navier–Stokes equations

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Introduction

These notes outline proofs of the three key results for weak solutions of the Navier–Stokes equations. Chapter One covers the global existence result of Leray (1934) and Hopf (1951). Chapter Two gives a version of (part of) the local regularity result of Caffarelli, Kohn, & Nirenberg (1982). Chapter Three applies this to give a quick proof of Serrin's regularity criterion (1962), and bounds on the dimension of the set of space-time singularities.

I have tried to avoid the more tricky technicalities, while I hope giving a reasonable idea of the flavour of the arguments. In particular, Chapters Two and Three ignore the pressure, which forms a large part of the full, rigorous analysis.

Chapter One draws on various sources; the notes of Galdi (2000), which are available online, are very useful. Chapter Two summarises a proof-by-contradiction of the CKN result by Ladyzenskaya & Seregin (1999). The majority of Chapter Three is based on a recent paper by Robinson & Sadowski (2011); the final bound on the parabolic Hausdorff dimension of the singular set uses the original argument of CKN.

Good general Navier-Stokes books are those by Constantin & Foias (1988) and Temam (2001); Doering & Gibbon (1995) give a more gentle introduction. You can find a significantly longer version of these notes, with a different take on the CKN result following Kukavica (2009a), on my webpage, and an introductory treatment of much of the functional analysis required in my book on infinite-dimensional dynamical systems (2001).

1

Existence of weak solutions

We will study the incompressible Navier–Stokes equations

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0 \qquad \operatorname{div} u = 0 \tag{1.1}$$

in a smooth, bounded, domain $\Omega \subset \mathbb{R}^3$, with Dirichlet boundary conditions $u = 0$ on $\partial\Omega$ and a given initial condition $u(x,0) = u_0(x)$.

Generally speaking the equations include a viscosity coefficient in front of the Laplacian,

$$u_t - \nu\Delta u + (u \cdot \nabla)u + \nabla p = 0, \tag{1.2}$$

but if we are interested in questions of existence and uniqueness, we lose nothing by restricting to $\nu = 1$. Indeed, if we have a solution $u(x,t)$ of (1.2) and we consider

$$u_\nu(x,t) = \nu u(x, \nu t),$$

it is easy to see that $u_\nu(x,t)$ satisfies (1.1). So, for example, if we could prove the existence and uniqueness of solutions of (1.1) for all positive times, we would have the same result for (1.2) for any $\nu > 0$.

We begin with some simple estimates, which lie at the heart of the existence of weak solutions. Take an initial condition $u_0(x)$ with

$$\int_\Omega |u_0|^2 \, \mathrm{d}x < \infty$$

(this corresponds to finite kinetic energy) and suppose that the equations have a classical solution on $[0,T]$. Then we can take the dot product of the equations with u and integrate over Ω to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 = 0. \tag{1.3}$$

We obtain the second term after integrating by parts and using the boundary condition $u = 0$ on $\partial\Omega$. The nonlinear term vanishes – this is a particular case of the more general identity

$$\int_\Omega [(u \cdot \nabla)v] \cdot v = 0, \tag{1.4}$$

which holds provided that u has divergence zero and one of u and v vanishes on $\partial\Omega$; we will use this many times. In fact this follows from the useful antisymmetry identity

$$\int_\Omega [(u \cdot \nabla)v] \cdot w = -\int_\Omega [(u \cdot \nabla)w] \cdot v. \tag{1.5}$$

To obtain this identity, write the integral more explicitly in components and integrate by parts

$$\int_\Omega u_i(\partial_i v_j)w_j = -\int_\Omega (\partial_i u_i)v_j w_j - \int_\Omega u_i v_j(\partial_i w_j) = -\int_\Omega u_i(\partial_i w_j)v_j.$$

The pressure term has also vanished, since

$$\int_\Omega \nabla p \cdot u = -\int_\Omega p(\nabla \cdot u) = 0,$$

using the fact that u is divergence free (and the $u = 0$ boundary condition).

We can integrate (1.3) in time to give an energy equality

$$\frac{1}{2}\|u(t)\|^2 + \int_0^t \|Du(s)\|^2 \,\mathrm{d}s = \frac{1}{2}\|u_0\|^2, \tag{1.6}$$

where now we are using the notation $\|u\|$ for the $L^2(\Omega)$ norm of u.

This 'formal' calculation (we are not careful, and assume that everything is as regular as it needs to be for all our manipulations to be justified), leads us to expect that if we have an initial condition in L^2, we will obtain a solution that is bounded in L^2, and whose H^1 norm is square integrable. We write

$$u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1). \tag{1.7}$$

More generally, the notation $u \in L^p(0,T;X)$, where X is a Banach space, means that the map $t \mapsto \|u(t)\|_X$ is in L^p,

$$\int_0^T \|u(t)\|_X^p \,\mathrm{d}t < \infty.$$

If we equip the space $L^p(0,T;X)$ with the natural L^p norm it is a Banach space.

A weak solution will be a function u with the regularity in (1.7) that satisfies the Navier–Stokes equations in a weak sense, which we now make precise. We will show that there exists at least one weak solution that also satisfies an energy inequality, i.e. (1.6) with the equality replaced by $\leq$.

1.1 Weak formulation of the equation

For the rest of this chapter, the notes by Galdi (2000) are recommended for filling in the gaps (adjustment on sets of measure zero, etc.); but he has a different approach for the convergence of the approximate solutions to a solution of the original equation.

Take a test function $\psi \in \mathscr{D}(\Omega) = C^\infty(\Omega)$ functions that have divergence zero and vanish on $\partial\Omega$; multiply (1.1) by ψ, integrate over Ω, and from 0 to $t \in [0,T]$:

$$\int_0^t \int_\Omega u_t \cdot \psi + \nabla u : \nabla\psi + [(u\cdot\nabla)u]\cdot\psi + \nabla p \cdot \psi = 0. \tag{1.8}$$

We have already seen that $\int_\Omega (\nabla p, \psi) = 0$, so we obtain the following weak formulation of the problem.

Definition 1.1 *A weak solution of (1.1) on* $[0,T)$ *with initial condition* $u_0 \in H$ *is a function*

$$u \in L^\infty(0,T;H) \cap L^2(0,T;H^1)$$

such that

$$(u(t),\psi) - (u_0,\psi) + \int_0^t (\nabla u, \nabla\psi) + ((u\cdot\nabla)u,\psi) = 0 \tag{1.9}$$

for every $\psi \in \mathscr{D}(\Omega)$ *and for every* $t \in [0,T)$.

One immediate consequence of the definition in this form is the weak continuity of $u(t)$:

Lemma 1.2 *Any weak solution is weakly continuous into L^2, i.e. for any $v \in L^2$,*

$$\lim_{t \to t_0} (u(t), v) = (u(t_0), v). \tag{1.10}$$

Proof First we show that (1.10) holds if $v \in \mathscr{D}(\Omega)$. In this case we can use (1.9) to write

$$(u(t), v) - (u(t_0), v) = -\int_{t_0}^{t} (\nabla u, \nabla v) + ((u \cdot \nabla)u, v)\, \mathrm{d}s$$

Since v is smooth, v and ∇v are bounded; since $u \in L^2(0, T; H^1)$ the first of the two terms on the right-hand side is integrable; since $u \in L^\infty(0, T; H)$, so is the second. It follows that the right-hand side converges to zero as $t \to t_0$, as required.

For $v \in L^2$, we approximate v by a sequence in $\mathscr{D}(\Omega)$ and use the fact that $u \in L^\infty(0, T; H)$ to pass to the limit. □

Note that this gives a sense in which the initial condition is satisfied by the weak solution: $u(t) \rightharpoonup u_0$ as $t \to 0$.

We are going to prove the existence of (at least) one weak solution that satisfies two additional properties.

Theorem 1.3 *For any $u_0 \in H$ there exists at least one weak solution of the Navier–Stokes equations. This solution is weakly continuous into L^2, and in addition satisfies the energy inequality*

$$\frac{1}{2}\|u(t)\|^2 + \int_0^t \|Du(s)\|^2 \, \mathrm{d}s \le \frac{1}{2}\|u_0\|^2 \tag{1.11}$$

for every $t \in [0, T)$. As a consequence, $u(t) \to u_0$ as $t \to 0$.

Note that every weak solution is weakly continuous into L^2; but it is not known whether every weak solution satisfies (1.11). That the solution approaches the initial condition strongly is a consequence of the weak continuity and (1.11), so again is not known for every weak solution, only those of the form we construct here, termed 'Leray–Hopf' weak solutions (since they were first constructed by Leray (1934) and Hopf (1951)).

Proof We will use the Galerkin method: we construct a series of approximate solutions of the form

$$u_k(x,t) = \sum_{j=1}^{k} \hat{u}_{k,j}(t)\psi_j(x), \qquad u_k(0) = P_k u_0 = \sum_{j=1}^{k}(u_0,\psi_j)\psi_j, \qquad (1.12)$$

where the $\{\psi_j\}_{j=1}^{\infty}$ are an orthonormal basis for L^2 formed from the eigenfunctions of the Stokes operator,

$$-\Delta\psi_j + \nabla p_j = \lambda_j\psi_j \qquad \psi_j|_{\partial\Omega} = 0 \qquad \nabla\cdot\psi_j = 0.$$

These functions ψ_j are C^∞ in Ω, are divergence-free, and satisfy the zero boundary condition.

If we use u_k in place of u in (1.9), and test with each of the functions $\{\psi_1,\ldots,\psi_j\}$, we obtain a set of k coupled ordinary differential equations (ODEs) for the coefficients $\hat{u}_{k,j}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_{k,j} + \lambda_j\hat{u}_{k,j} + \sum_{i,l=1}^{k}((\psi_i\cdot\nabla)\psi_l,\psi_j)\hat{u}_{k,i}\hat{u}_{k,l} = 0. \qquad (1.13)$$

Note that the third term is essentially quadratic in the $\hat{u}_{k,j}$s – so this is a set of locally Lipschitz ODEs. The theory of existence and uniqueness for such ODEs is well developed: a unique solution exists while the solution stays bounded, i.e. unique solutions for the $\hat{u}_{k,j}(t)$ exist while

$$\sum_{j=1}^{k}|\hat{u}_{k,j}(t)|^2 < \infty \qquad (1.14)$$

(this is just the norm in $\mathbb{R}^k$ of the vector $(\hat{u}_{k,1},\ldots,\hat{u}_{k,k})$ of coefficients). How can we show that this quantity stays bounded?

The easiest way is to remember that in fact the $\{\hat{u}_{k,j}\}_{j=1}^{k}$ are the coefficients in an expansion of a function $u_k(x,t)$, and look at the equation satisfied by this function. If we multiply (1.13) by ψ_j, $j=1,\ldots,k$ and sum, we obtain

$$\frac{\partial u_k}{\partial t} - \Delta u_k + P_k((u_k\cdot\nabla)u_k) = 0,$$

where P_k denotes the orthogonal projection (in L^2) onto the space spanned by the $\{\psi_1,\ldots,\psi_k\}$ (this was defined in passing above in (1.12)). Note that if P_k did not appear the nonlinearity would generate terms that could not be expressed as a sum of $\{\psi_1,\ldots,\psi_k\}$.

We can now easily estimate norms of the solution u_k. Note that since all the ψ_j are smooth (in space), so is u_k, and therefore we can perform the manipulations that were 'formal' at the beginning of this chapter entirely rigorously: we take the inner product with u_k and integrate in time. Once again the nonlinear term vanishes, since

$$(P_k[(u_k\cdot\nabla)u_k],u_k)=((u_k\cdot\nabla)u_k,P_ku_k)=((u_k\cdot\nabla)u_k,u_k)=0,$$

and we obtain

$$\frac{1}{2}\|u_k\|^2+\|Du_k\|^2=0.$$

Integrating this in time gives an energy equality for the Galerkin solutions,

$$\frac{1}{2}\|u_k(t)\|^2+\int_0^t\|Du_k(s)\|^2\,\mathrm{d}s=\frac{1}{2}\|u_k(0)\|^2.$$

Since $\|u_k(t)\|^2=\sum_{j=1}^k|\hat{u}_{k,j}|^2$, (1.14) is satisfied for all $t\geq 0$, the approximate solution u_k exists for all $t\geq 0$.

Now, since $u_k(0)=P_ku_0$, we have $\|u_k(0)\|\leq\|u_0\|$, and so we have a uniform bound (with respect to k) on the approximate solutions:

$$\frac{1}{2}\|u_k(t)\|^2+\int_0^t\|Du_k(s)\|^2\,\mathrm{d}s\leq\frac{1}{2}\|u_0\|^2.$$

In other words, for any $T>0$, u_k is uniformly bounded in $L^\infty(0,T;H)$ and in $L^2(0,T;H^1)$.

We can therefore use weak and weak-* compactness to find a subsequence (which we relabel) such that u_k converges to u weakly-* in $L^\infty(0,T;H)$ and weakly in $L^2(0,T;H^1)$. It follows that the limit enjoys the same bounds as the approximations,

$$\frac{1}{2}\|u(t)\|^2+\int_0^t\|Du(s)\|^2\,\mathrm{d}s\leq\frac{1}{2}\|u_0\|^2;$$

but note that the energy equality for the approximations has now turned into an energy inequality for the (candidate) weak solution.

We may have shown that the approximations converge to a limit, but does this limit satisfy the right equation? As a first step, we need to show that for each ψ_j the terms in the equation

$$(u_k(t),\psi_j)-(u_0,\psi_j)+\int_0^t(\nabla u_k,\psi_j)+((u_k\cdot\nabla)u_k,\psi_j)=0 \tag{1.15}$$

converge to the same thing but with u_k replaced by u. We will need some better convergence of u_k to u before we can guarantee the convergence of the nonlinear term.

In fact we will show that $\mathrm{d}u_k/\mathrm{d}t$ is uniformly bounded in $L^{4/3}(0,T;H^{-1})$, which coupled with the bounds we already have in $L^2(0,T;H^1)$ will be enough to prove that there is a subsequence that converges that converges strongly in $L^2(0,T;L^2)$: this is the content of the Aubin Lemma (for a proof of a more general result[1] see Temam, 2001).

Lemma 1.4 *Let u_k be a sequence that is bounded in $L^2(0,T;H^1)$, and has $\mathrm{d}u_k/\mathrm{d}t$ bounded in $L^p(0,T;H^{-1})$ for some $p>1$. Then u_k has a subsequence that converges strongly in $L^2(0,T;L^2)$.*

You can think of this lemma as being a version of the familiar result that a bounded sequence in H^1 has a subsequence that converges strongly in L^2 (i.e. that H^1 is compact in L^2); things are more complicated here since for each t, $u(t)$ lies in a space of functions instead of being a real number.

We already have the bound on u_k in $L^2(0,T;H^1)$, so we need to look at the derivative of u_k. We have

$$\frac{\mathrm{d}u_k}{\mathrm{d}t} = -\Delta u_k - P_k((u_k\cdot\nabla)u_k).$$

Since $u_k\in L^2(0,T;H^1)$, we have $\Delta u_k \in L^2(0,T;H^{-1})$, so we have to estimate the nonlinear term. If we take the inner product with some $v\in H^1$ we obtain

$$\begin{aligned}
|(P_k[(u_k\cdot\nabla)u_k],v)| = |((u_k\cdot\nabla)u_k,P_kv)| &= \left|\int (u_k\cdot\nabla)u_k\cdot P_kv\right| \\
&\le \int |u_k||Du_k||P_kv| \\
&\le \|u_k\|_{L^3}\|Du_k\|_{L^2}\|P_kv\|_{L^6} \\
&\le c\|u_k\|^{1/2}\|Du_k\|^{3/2}\|P_kv\|_{H^1} \\
&\le \left(c\|u_k\|^{1/2}\|Du_k\|^{3/2}\right)\|v\|_{H^1},
\end{aligned}$$

and so

$$\|P_k(u_k\cdot\nabla)u_k\|_{H^{-1}} \le c\|u_k\|^{1/2}\|Du_k\|^{3/2}.$$

It follows that the nonlinear term is bounded in $L^{4/3}(0,T;H^{-1})$. So $\mathrm{d}u_k/\mathrm{d}t$ is uniformly bounded in $L^{4/3}(0,T;H^{-1})$, and we can use the Aubin Lemma to find a subsequence that converges strongly in $L^2(0,T;L^2)$.

We are now in a position to show the convergence of all the terms in (1.15).

For the first term, note that if $u_k\to u$ in $L^2(0,T;L^2)$, then $u_k(t)\to u(t)$

[1] We will need this later: take three separable, reflexive, Banach spaces $X_1 \subset\subset X_0 \subset X_{-1}$; if $u_k\in L^p(0,T;X_1)$ and $\dot u_k \in L^q(0,T;X_{-1})$, with $p,q>1$, then there is a subsequence that converges strongly in $L^p(0,T;X_0)$.

in L^2 for almost every $t \in (0,T)$. So the first term converges for almost every $t \in (0,T)$.

The second term doesn't depend on k, so converges trivially.

The third term converges since $u_k \rightharpoonup u$ in $L^2(0,T;H^1)$: in particular, $\int_0^t (\nabla u_k, v) \to \int_0^t (\nabla u, v)$ for any $v \in L^2$ and all $t \in (0,T)$.

The nonlinear term requires the strong convergence in $L^2(0,T;L^2)$ and a little work. We have

$$\begin{aligned}
&\left| \int_0^t ((u_k \cdot \nabla)u_k, \psi_j) - ((u \cdot \nabla)u, \psi_j) \, \mathrm{d}s \right| \\
&\quad = \left| \int_0^t (((u_k - u) \cdot \nabla)u_k, \psi_j) + ((u \cdot \nabla)(u - u_k), \psi_j) \, \mathrm{d}s \right| \\
&\quad \le \|\psi_j\|_\infty \left(\int_0^t \|u_k - u\|^2 \right)^{1/2} \left(\int_0^t \|\nabla u_k\|^2 \right)^{1/2} \\
&\qquad + \sum_{i=1}^3 \left| \int_0^t (\partial_i (u - u_k), u_i \psi_j) \right|.
\end{aligned}$$

For the first term on the RHS we use the fact that u_k is uniformly bounded in $L^2(0,T;H^1)$ and that u_k converges strongly to u in $L^2(0,T;L^2)$; for the second term we use the fact that $u_i \psi_j \in L^2$ and the weak convergence of u_k to u in $L^2(0,T;H^1)$.

So for each j we have

$$(u(t), \psi_j) - (u_0, \psi_j) + \int_0^t (\nabla u, \nabla \psi_j) + ((u \cdot \nabla)u, \psi_j) = 0$$

for all $t \ge 0$ (we get all t by adjusting on a set of measure zero). We want this equation to hold for any $\psi \in \mathscr{D}(\Omega)$: this follows using the fact that finite linear combinations of the ψ_j are dense in $\mathscr{D}(\Omega)$.

We have therefore obtained the existence of a weak solution, and we have already seen that this weak solution satisfies the energy inequality (1.11). To see that $u(t) \to u_0$ as $t \to 0$, we have from the weak continuity

$$\|u_0\| \le \liminf_{t \to 0} \|u(t)\|,$$

while from the energy inequality we obtain

$$\|u_0\| \ge \limsup_{t \to 0} \|u(t)\|.$$

It follows that $\|u(t)\| \to \|u_0\|$ as $t \to 0$, and since H is a Hilbert space it follows from this plus the weak continuity that $u(t) \to u_0$. □

2

Partial regularity

In this chapter we give an idea of the argument of Ladyzhenskaya & Seregin (1999) to prove one part of the local regularity theory developed by Caffarelli, Kohn, & Nirenberg (1982). (For other variant proofs see Kukavica, 2009a, and Lin, 1998.)

We consider 'suitable weak solutions': these are weak solutions that satisfy a local form of the energy inequality and for which $p \in L^{5/3}(\Omega \times (0,T))$ (or more often the weaker integrability criterion $p \in L^{3/2}(\Omega \times (0,T))$, depending on which paper you read).

We write $Q_T = \Omega \times (0,T)$, the space-time domain of definition of the solution.

We consider solutions that satisfy the NSE locally: this means that we have a solution that satisfies the equation in a weak form,

$$\int_{Q_T} -u \cdot \frac{\partial \varphi}{\partial t} + \nabla u \cdot \nabla \varphi - [(u \cdot \nabla)u]\varphi \, \mathrm{d}x \, \mathrm{d}t = \int_Q p \, \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t$$

for every $\varphi \in C_0^\infty(Q_T)$, and for which the local energy inequality

$$\int_\Omega |u(t)|^2 \varphi(t) \, \mathrm{d}x + 2 \int_0^t \int_\Omega |\nabla u|^2 \varphi \, \mathrm{d}x \, \mathrm{d}s \\ \leq \int_0^t \int_\Omega |u|^2 \left\{ \frac{\partial \varphi}{\partial t} + \Delta \varphi \right\} \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Omega (|u|^2 + 2p)(u \cdot \nabla)\varphi \, \mathrm{d}x \, \mathrm{d}s$$

holds for almost all $0 < t < \infty$ and all non-negative $\varphi \in C_0^\infty(Q_T)$ (multiply the equation by $u\varphi$ and integrate by parts).

The weak form of the equation here is fact equivalent to the definition in the previous chapter. The existence of solutions that satisfy the local energy inequality is a new fact that has to be proved (it can be). It is conceivable that there are weak solutions that satisfy the energy inequality but do not satisfy the local energy inequality.

In what follows I will ignore the pressure and the divergence-free condition. YOU CANNOT DO THIS! The most involved part of the analysis is the part that deals with the pressure. But it will make it possible to give a fairly quick sketch of the argument that retains the main ideas. You can find the full argument in the paper by Ladyzhenskaya & Seregin (1999).

We will prove the following theorem, where

$$Q_r(z) = Q_r(x,t) = B_r(x) \times (t - r^2, t)$$

is a 'parabolic cylinder'. If $z = 0$ we write

$$Q_r = Q_r(0).$$

Theorem 2.1 (Local regularity result) *There exist $R > 0$ and $\epsilon' > 0$ such that if $r < R$, $Q_r(z) \subset \Omega_T$, and*

$$r\left[\left(\fint_{Q_r(z)} |u|^3\right)^{1/3} + \left(\fint_{Q_r(z)} |p|^{3/2}\right)^{2/3}\right] < \epsilon'$$

then u is Hölder in a neighbourhood of z.

A key tool is the Campanato Lemma, which we recall in the following section.

2.1 Integral characterisation of Hölder spaces: the Campanato Lemma.

For any $f \in L^1(Q_r(z))$, define

$$(f)_{z,r} = \fint_{Q_r(z)} f(z)\,\mathrm{d}z.$$

Lemma 2.2 (Campanato) *Let $f \in L^1(Q_R(0))$ and suppose that there exist positive constants $\alpha \in (0,1]$, $M > 0$, such that*

$$\fint_{Q_r(z)} |f(z) - (f)_{z,r}|^3 \,\mathrm{d}y \leq M^3 r^{3\alpha} \tag{2.1}$$

for any $x \in Q_{R/2}(0)$ and any $r \in (0, R/2)$. Then f is Hölder continuous in $Q_{R/2}(0)$: for any $z, w \in Q_{R/2}(0)$,

$$|f(x,t) - f(y,s)| \leq cM(|x-y| + |t-s|^{1/2})^\alpha.$$

Proof Choose $z \in Q_{R/2}(0)$ and $r < R/2$. We first compare $(f)_{z,r/2}$ with $(f)_{z,r}$. We have

$$\begin{aligned}
\left|(f)_{z,r/2} - (f)_{z,r}\right|^3 &= \left|\frac{1}{\omega_3 (r/2)^5} \int_{Q_{r/2}(z)} f(y) - (f)_{z,r} \,\mathrm{d}y\right|^3 \\
&\leq \frac{1}{\omega_3^3 (r/2)^{15}} \left(\int_{Q_{r/2}(z)} |f(y) - (f)_{z,r}| \,\mathrm{d}y\right)^3 \\
&\leq \frac{2^{15}}{\omega_3^3 r^{15}} \left(\int_{Q_r(z)} |f(y) - (f)_{z,r}| \,\mathrm{d}y\right)^3 \\
&\leq \frac{2^{15}}{\omega_3^3 r^{15}} \left[\left(\int_{Q_r(z)} |f(y) - (f)_{z,r}|^3 \,\mathrm{d}y\right)^{1/3} \left(\int_{Q_r(z)} \mathrm{d}y\right)^{2/3}\right]^3 \\
&= \frac{2^{15}}{\omega_3^3 r^{15}} \left(\int_{Q_r(z)} |f(y) - (f)_{z,r}|^3 \,\mathrm{d}y\right) \omega_3^2 r^{10} \\
&= \frac{2^{15}}{\omega_3 r^5} \left(\int_{Q_r(x)} |f(y) - (f)_{z,r}|^3 \,\mathrm{d}y\right) \\
&= 2^{15} \fint_{Q_r(x)} |f(y) - (f)_{z,r}|^3 \,\mathrm{d}y \leq cM^3 r^{3\alpha}.
\end{aligned}$$

In particular,

$$|(f)_{z,r/2} - (f)_{z,r}| \leq cMr^\alpha.$$

Now consider $(f)_{z,2^{-k}} - (f)_{z,r}$. Since

$$(f)_{z,r2^{-k}} - (f)_{z,r} = \sum_{j=1}^{k} (f)_{z,r2^{-k}} - (f)_{z,r2^{-(k-1)}},$$

it follows that

$$\left|(f)_{z,r2^{-k}} - (f)_{z,r}\right| \leq \sum_{j=1}^{k} cMr^{\alpha}2^{-(j-1)\alpha} \leq \sum_{j=0}^{\infty} cMr^{\alpha}2^{-j\alpha} = cMr^{\alpha}. \quad (2.2)$$

This shows that $(f)_{z,r2^{-k}}$ forms a Cauchy sequence, and hence the averages converge for every $z \in Q_{r/2}(0)$. By the Lebesgue Theorem, these averages converge to $f(z)$, and so if we let $k \to \infty$ in (2.2) we obtain an estimate for the difference between $f(z)$ and its average,

$$|f(z) - (f)_{z,r}| \leq c_1 M r^{\alpha}.$$

Now take another point $y \in Q_{R/2}(0)$; we compare $(f)_{z,r}$ with $(f)_{y,2r}$:

$$\begin{aligned} |(f)_{z,r} - (f)_{y,2r}|^3 &= \left|\frac{1}{\omega_3 r^5}\int_{Q_r(z)} f(w) - (f)_{y,2r}\,\mathrm{d}w\right|^3 \\ &\leq \frac{1}{\omega_3 r^5}\int_{Q_r(z)} |f(w) - (f)_{y,2r}|^3\,\mathrm{d}w, \end{aligned}$$

arguing as before. Now if $z = (x,t)$ and $y = (\xi, s)$, choose

$$r = |x - \xi| + |t - s|^{1/2};$$

then $Q_r(x,t) \subset Q_{2r}(\xi, s)$, and so

$$\begin{aligned} |(f)_{z,r} - (f)_{y,2r}|^3 &\leq 2^5\,\frac{1}{\omega_3 (2r)^5}\int_{Q_{2r}(y)} |f(w) - (f)_{y,2r}|^3\,\mathrm{d}w \\ &= 2^5 ⨍_{Q_{2r}(y)} |f(w) - (f)_{y,2r}|^3\,\mathrm{d}w \\ &\leq 2^5 M^3 (2r)^{3\alpha} = 2^{5+3\alpha} M^3 (|x-y| + |t-s|^{1/2})^{3\alpha}, \end{aligned}$$

i.e.

$$|(f)_{z,r} - (f)_{y,2r}| \leq c_2 M(|x-y| + |t-s|^{1/2})^{\alpha}.$$

So now (still with r chosen as above)

$$\begin{aligned} |f(z) - f(y)| &\leq |f(z) - (f)_{z,r}| + |(f)_{z,r} - (f)_{y,2r}| + |(f)_{y,2r} - f(y)| \\ &\leq c_1 M r^{\alpha} + c_2 M r^{\alpha} + c_1 M (2r)^{\alpha} \\ &= cM(|x-y| + |t-s|^{1/2})^{\alpha}. \end{aligned}$$

□

2.2 Proof of a local regularity criterion

We now give an outline of Ladyzhenskaya & Seregin's proof of the CKN regularity criterion.

Lemma 2.3 (Decay estimate) *Fix* θ, M *with* $0 < \theta \leq \frac{1}{2}$ *and* $M \geq 3$*. Then there exist* $\varepsilon(\theta, M)$*,* $R(\theta, M)$*, and* $c(M)$ *such that if, for some* $0 < r < R$*,*

$$r \left| \fint_{Q_r(z)} u(z)\,\mathrm{d}z \right| \leq M$$

and

$$Y(z,r) := \left\{ \fint_{Q_r(z)} |u - (u)_{z,r}|^3\,\mathrm{d}z \right\}^{1/3} < \varepsilon,$$

then

$$Y(z, \theta r) \leq c\theta^{2/3} Y(z,r).$$

Proof If the result is not true then there are $\{z_m, r_m\}$ with $r_m \to 0$ such that

$$Q(z_m, r_m) \subset\subset Q_T,$$

with $\bar{u}_m := (u)_{z_m, r_m}$

$$r_m |\bar{u}_m| \leq M,$$

and

$$Y(z_m, r_m) =: \varepsilon_m \to 0 \qquad \text{but} \qquad Y(z_m, \theta r_m) \geq c Y(z_m, r_m).$$

We will choose c to ensure a contradiction.

We consider various rescalings of the variables (x,t) and the solution functions (u,p): for the mth element of the sequence, we rescale according to

$$\tilde{x} = \frac{x - x_m}{r_m}, \qquad \tilde{t} = \frac{t - t_m}{r_m^2} \qquad \Rightarrow \qquad \tilde{z} = (\tilde{x}, \tilde{t}) \in B(0,1) \times (-1, 0) = Q_1$$

and

$$\tilde{u}_m = \frac{u(z) - \bar{u}_m}{\varepsilon_m},$$

so that $(\tilde{u}_m)_{0,1} = 0$.

In particular, it follows that for this rescaling

$$\left(\fint_{Q_1} |\tilde{u}_m|^3 \, \mathrm{d}\tilde{z}\right)^{1/3} = 1,$$

so that $\tilde{u}_m$ is uniformly bounded (w.r.t. m) in $L^3(Q_1)$. The fact that will lead us to a contradiction (the lower bound) becomes

$$\left(\fint_{Q_\theta} |\tilde{u}_m|^3 \, \mathrm{d}\tilde{z}\right)^{1/3} \geq c\theta^{2/3}. \tag{2.3}$$

Now we drop the tildes, and write the equation for the rescaled functions (in the rescaled variables) within Q_1:

$$\partial_t u_m - \Delta u + r_m(\bar{u}_m \cdot \nabla)u_m + \varepsilon_m r_m (u \cdot \nabla)u = 0.$$

(Remember, we are neglecting the divergence-free condition and the pressure.) Since u_m is uniformly bounded, it has a subsequence (which we relabel) that converges weakly:

$$u_m \rightharpoonup u \text{ in } L^3(Q_1).$$

It follows from the weak convergence that

$$(u)_{0,1} = 0 \qquad \text{and} \qquad \left(\fint_{Q_1} |u|^3 \, \mathrm{d}\tilde{z}\right)^{1/3} \leq 1.$$

Since by assumption $r_m|\bar{u}_m| \leq M$, we can also extract a further subsequence such that $r_m\bar{u}_m \to a \in \mathbb{R}^3$, with $|a| \leq M$.

Now let $m \to \infty$; it follows with some standard argumentation that the limits (u, p) satisfy the *linear* equation

$$\partial_t u - \Delta u + (a \cdot \nabla)u = 0$$

in Q_1. One can show that solutions of this linear equation satisfy

$$\fint_{Q_\theta} |u|^3 \, \mathrm{d}\tilde{z} \leq c_1(M)\theta^{2/3}.$$

[The argument is, very roughly, a 'standard' parabolic one giving regularity of u_t and $\nabla^2 u$; one can then use a 'parabolic' version of the Poincaré inequality to prove the integral estimate - the argument combines ideas from Proposition 3.3 in Seregin (2003) and Proposition 2.2 in Seregin (2007).]

This would give a contradiction with (2.3), simply by choosing $c > c_1(M)$, if only $u_m \to u$ in L^3 (strongly). To show this we simply use estimates on

$\partial_t u_m$, which follow from the equation, along with the same Aubin Compactness Lemma we used to prove existence of weak solutions. Indeed, it follows from the governing equation that

$$\partial_t u_m \qquad \text{is uniformly bounded in} \quad L^{3/2}(0,T;H^{-2});$$

using the local energy inequality we can also bound u_m uniformly in

$$L^\infty((-(3/4)^2,0);L^2(B_{3/4}(0)))$$

and ∇u_m uniformly in $L^2(Q_{3/4})$, we obtain a uniform bound in $L^{10/3}(Q_{3/4})$ (see page 23). From the Aubin lemma we can obtain $u_m \to u$ in $L^2(0,T;L^2)$.

Now, from strong convergence in L^2 and boundedness in $L^{10/3}$ we can deduce strong convergence in L^3, since by Hölder's inequality

$$\begin{aligned}\int |u-u_m|^3 &= \int |u-u_m|^{1/2}|u-u_m|^{5/2} \\ &\leq \left(\int |u-u_m|^2\right)^{1/4}\left(\int |u-u_m|^{10/3}\right)^{3/4}.\end{aligned}$$

In particular, therefore

$$⨍_{Q_\theta} |u_m|^3\,\mathrm{d}z \to ⨍_{Q_\theta} |u|^3\,\mathrm{d}z$$

as $m \to \infty$, and so choosing any $c > c_1(M)$ we obtain a contradiction. □

Iterating this yields a local regularity result with the 'wrong' scaling.

Proposition 2.4 (Bad local regularity result) *There exist $R > 0$ and $\epsilon > 0$ such that if $r < R$, $Q_{z,r} \subset \Omega_T$, and*

$$\left(⨍_{Q_r(z)} |u|^3\right)^{1/3} + \left(⨍_{Q_r(z)} |p|^{3/2}\right)^{2/3} < \epsilon$$

then u is Hölder in a neighbourhood of z.

Proof One need only verify that

$$r\left|⨍_{Q_r(z)} u\right| \leq r\left(⨍_{Q_r(z)} |u|^3\right)^{1/3}$$

and

$$Y(z,r) \leq \left(⨍_{Q_r(z)} |u|^3\right)^{1/3}.$$

Then, with an appropriate choice of constants (any $\beta < 2/3$), we can iterate the decay estimate to show that then for any $k \in \mathbb{N}$

$$Y(z, \theta^k r) \le c\theta^{k\beta} Y(z, r).$$

By choosing a k such that $\theta^{k+1} < \rho/r < \theta^k$ one can deduce (this is not immediate, and uses steps similar to those used to prove Campanato's Lemma) that

$$Y(z, \rho) \le c \left(\frac{\rho}{r}\right)^{\beta} Y(z, r) \tag{2.4}$$

for all $0 < \rho < r$.

Since $Y(z, r)$ depends continuously on r, (2.4) holds uniformly in some small neighbourhood of z; then by Campanato's Lemma, u is Hölder in a neighbourhood of z. □

We now rescale the solutions and improve on this.

Theorem 2.5 (Good local regularity result) *There exist $R > 0$ and $\epsilon' > 0$ such that if $r < R$, $Q_r(z) \subset \Omega_T$, and*

$$r \left[\left(\fint_{Q_r(z)} |u|^3 \right)^{1/3} + \left(\fint_{Q_r(z)} |p|^{3/2} \right)^{2/3} \right] < \epsilon' \tag{2.5}$$

then u is Hölder in a neighbourhood of z.

Proof For simplicity suppose that $z = 0$. We rescale by a factor $\lambda = 2r/R_0$: consider $\tilde{u}(y, s) = \lambda u(\lambda y, \lambda^2 s)$; $\tilde{u}$ is still a suitable weak solution of the Navier–Stokes equations, but now

$$\begin{aligned}
\bar{Y}(0, R_0/2; \tilde{u})^3 &= \frac{1}{(R_0/2)^5} \int_{Q_{R_0/2}} |\tilde{u}(y, s)|^3 \, \mathrm{d}y \, \mathrm{d}s \\
&= \frac{1}{(R_0/2)^5} \int_{Q_{R_0/2}} |\lambda u(\lambda y, \lambda^2 s)|^3 \, \mathrm{d}y \, \mathrm{d}s \\
&= \frac{1}{(R_0/2)^5} \lambda^3 \lambda^{-5} \int_{Q_r} |u(x, t)|^3 \, \mathrm{d}x \, \mathrm{d}t \\
&= \left(\frac{2}{R_0}\right)^3 r^3 \frac{1}{r^5} \int_{Q_r} |u(x, t)|^3 \, \mathrm{d}x \, \mathrm{d}t \\
&= c[r\overline{Y}(0, r; u)]^3.
\end{aligned}$$

□

3

More on partial regularity

We deduce two important consequences of the result we have so far – a local conditional regularity result, and a partial regularity result. Both of these can be deduced from the following simple application of Hölder's inequality. For simplicity we again neglect the pressure. This is mostly harmless...

The following simple lemma, and the subsequent application to deduce the Serrin criterion, can be found in Robinson & Sadowski (2011), see also Wolf (2010).

Lemma 3.1 *Suppose that u is a suitable weak solution, and $u \in L^r(0,T;L^s)$ with $3 \le r, s < \infty$. Then there exists an absolute constant $\varepsilon > 0$ such that if $z = (x,t)$ is a singular point of u then*

$$\varepsilon \rho^{r(\frac{3}{r}+\frac{2}{s}-1)} \le \int_{t-\rho^2}^{t} \left(\int_{B_\rho(x)} |u|^s \,\mathrm{d}x \right)^{r/s} \mathrm{d}t$$

for all $\rho > 0$ sufficiently small that $Q_\rho(z) \subset Q_T$.

Proof From the local regularity theorem, if z is a singular point then

$$\frac{1}{\rho^2} \int_{Q_\rho(z)} |u|^3 \,\mathrm{d}x \,\mathrm{d}t \ge \epsilon_0$$

whenever $Q_\rho \subset Q_T$.

Using Hölder's inequality with exponent $s/3$ and its conjugate, for any

$s \geq 3$ we have

$$\rho^{-2}\int_{Q_\rho}|u|^3\,\mathrm{d}x\,\mathrm{d}t \leq \rho^{-2}\int_{t-\rho^2}^{t}\left(\int_{B_\rho(x)}|u|^s\,\mathrm{d}x\right)^{3/s}\left(\int_{B_\rho}\mathrm{d}x\right)^{1-\frac{3}{s}}\mathrm{d}t$$
$$= \rho^{1-\frac{9}{s}}\int_{t-\rho^2}^{t}\left(\int_{B_\rho}|u|^s\,\mathrm{d}x\right)^{3/s}\mathrm{d}t.$$

Now use Hölder again, with exponent $r/3$ and its conjugate, so that for any $r \geq 3$ we have

$$\rho^{-2}\int_{Q_\rho}|u|^3\,\mathrm{d}x\,\mathrm{d}t \leq \rho^{1-\frac{9}{s}}\left[\int_{t-\rho^2}^{t}\left(\int_{B_\rho}|u|^s\,\mathrm{d}x\right)^{r/s}\mathrm{d}t\right]^{3/r}\left[\int_{t-\rho^2}^{t}\mathrm{d}t\right]^{1-\frac{3}{r}}$$
$$= \rho^{3-\frac{6}{r}-\frac{9}{s}}\int_{t-\rho^2}\left[\left(\int_{B_\rho}|u|^s\,\mathrm{d}x\right)^{r/s}\mathrm{d}t\right]^{3/r}.$$

□

3.1 Serrin's regularity condition

We now show that if $u \in L^r(t, t-\varrho^2; L^s(B_\varrho(x))$ for some $\varrho > 0$ then u is regular at (Hölder in a neighbourhood of) (x,t) provided that

$$\frac{2}{r}+\frac{3}{s}=1 \tag{3.1}$$

with $3 \leq r, s < \infty$. Serrin (1962) proved this for $2 < r < \infty$, $3 < s < \infty$, under the slightly stronger condition that $2/r + 3/s < 1$; this was then improved by a number of authors; for an up-to-date summary see Galdi's notes, for example.

Indeed, if (x,t) is not regular then it follows from Lemma 3.1 that there is a sequence $\rho_n \to 0$ such that for each n

$$\varepsilon \leq \int_{t-\rho_n^2}^{t}\left(\int_{B_{\rho_n}}|u|^s\,\mathrm{d}x\right)^{r/s}\mathrm{d}t$$
$$\leq \int_{t-\rho_n^2}^{t}\|u\|_{L^s(U)}^{r/s}\;dt,$$

which contradicts the assumed integrability of u.

[Given the theorem that we have proved in the previous chapter this argument doesn't quite work, since the result also requires the pressure; our desired 'contradiction' should in fact come from

$$\varepsilon \leq \int_{t-\rho_n^2}^{t} \left(\int_{B_{\rho_n}} |u|^s \, \mathrm{d}x \right)^{r/s} \mathrm{d}t + \int_{t-\rho_n^2}^{t} \left(\int_{B_{\rho_n}} |p|^{s/2} \, \mathrm{d}x \right)^{r/s} \mathrm{d}t,$$

and for the above argument to work we need some local estimates on the pressure, $u \in L^r(0,T;L^s) \Rightarrow p \in L^{r/2}(0,T;L^{s/2})$. This is known if the domain is $\mathbb{R}^3$, but not for a bounded subdomain. (In fact, though, it is true that the local regularity result holds without the pressure; this is a recent theorem of Wolf, 2010.)]

3.2 Partial regularity

We now show that the set S of singular points in space-time of any suitable weak solution cannot be too large; we call a point 'regular' if u is Hölder in a neighbourhood of z, and singular if it is not regular.

First we can show that $d_{\mathrm{box}}(S) \leq 5/3$ (Scheffer found the same bound on the Hausdorff dimension in 1976). The argument is simple (see Robinson & Sadowski, 2009, who go on to use this result to discuss the uniqueness of Lagrangian particle paths). With some serious effort the bound can be improved slightly (Kukavica, 2009b).

While the (upper) box-counting dimension of a set X is usually defined as

$$\limsup_{\epsilon \to 0} \frac{\log N(X,\epsilon)}{-\log \epsilon},$$

where $N(X,\epsilon)$ is the minimum number of balls of radius ϵ required to cover X, one obtains the same quantity if $N(X,\epsilon)$ denotes instead the maximum number of ϵ-separated points in X (see 'equivalent definitions 3.1' in Falconer (1990), for example). We adopt this form of the definition here, since it is well adapted to our argument. Indeed, the dimension bound follows easily from this definition using the local regularity theorem and the fact that

$u \in L^{10/3}$. This follows since $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$:

$$\begin{aligned}\int_\Omega |u|^{10/3} &= \int_\Omega |u|^2 |u|^{4/3} \\ &\le \left(\int_\Omega |u|^6\right)^{1/3} \left(\int_\Omega |u|^2\right)^{2/3} \\ &= \|u\|_{L^6}^2 \|u\|_{L^2}^{4/3},\end{aligned}$$

and so

$$\int_0^T |u|^{10/3} \le \|u\|_{L^\infty(0,T;L^2)}^{4/3} \|u\|_{L^2(0,T;H^1)}.$$

Theorem 3.2 *If S denotes the singular set of a suitable weak solution then $d_{\text{box}}(S) \le 5/3$.*

(In fact one has to be a little more careful: we require $p \in L^{5/3}(\Omega \times (0,T))$ to deal with the pressure term (which we have neglected), and strictly the result is that $d_{\text{box}}(S \cap K) \le 5/3$ for any compact subset K of $\Omega \times (0,T)$, since one has to be able to take balls of a fixed radius about any singular point, and these must lie entirely within $\Omega \times (0,T)$.)

Proof Using Lemma 3.1 with $r = s = 10/3$, it follow that if $(x,t) \in S$ then

$$\int_{t-\rho^2}^t \int_{B_\rho(x)} |u|^{10/3} \, \mathrm{d}x \, \mathrm{d}t \ge \varepsilon \rho^{5/3}.$$

Now suppose that $d_{\text{box}}(S) > 5/3$. Fix d with $5/3 < d < d_{\text{box}}(S \cap K)$. It follows from the definition of the box-counting dimension that there exists a decreasing sequence $\epsilon_j \to 0$ such that $N(S,\epsilon_j) \ge \epsilon_j^{-d}$. Let $\{(x_i,t_i)\}_{i=1}^{N(S,\epsilon_j)}$ be a collection of ϵ_j-separated points in S. Take j large enough that $\epsilon_j < r_0$, and then

$$\int_{\Omega\times(0,T)} |u|^{10/3} \ge \sum_{i=1}^{N(S,\epsilon_j)} \int_{Q_\epsilon(x_i,t_i)} |u|^{10/3} \ge \epsilon_j^{-d} \times c_3 \epsilon_j^{5/3} = c_3 \epsilon_j^{5/3-d}.$$

The left-hand side is finite, but the right-hand side tends to infinity as $j \to \infty$ since $d > 5/3$, and we obtain a contradiction. It follows that $d_{\text{box}}(S) \le 5/3$ as claimed. □

3.3 Better partial regularity

In order to prove a better partial regularity result, CKN, L&S, etc., in fact show that the condition on the average of $|u|^3$ is a consequence of an integral condition on $|\nabla u|^2$.

Theorem 3.3 *There exists a* $\delta^\star > 0$ *such that*

$$\liminf_{r\to 0} \frac{1}{r}\int_{Q_{z,r}} |\nabla u|^2 < \delta^\star$$

implies that (2.5) holds for some r *sufficiently small.*

Proof (Very sketchy!) Some idea of the argument, neglecting the pressure... Set

$$A(r) = \sup_{t-r^2\le s\le t} \frac{1}{r}\int_{B_r(x)} |u|^2 \,\mathrm{d}x, \qquad E(r) = \frac{1}{r}\int_{Q_r(z)} |\nabla u|^2 \,\mathrm{d}z,$$

and

$$C(r) = \frac{1}{r^2}\int_{Q_r(z)} |u|^3 \,\mathrm{d}z.$$

Basic inequalities (Sobolev, Poincaré, interpolation, nothing to do with Navier–Stokes) give, for $\theta \in (0,1)$,

$$C(\theta\rho) \le c\left[\theta^3 A^{3/2}(\rho) + \frac{1}{\theta^3}A^{3/4}(\rho)E^{3/4}(\rho)\right]. \tag{3.2}$$

The LEI gives

$$A(R/2) + E(R/2) \le c[C^{2/3}(R) + A(R)E(R)].$$

With $R = \theta\rho$,

$$\begin{aligned} A\left(\frac{\theta\rho}{2}\right) &\le c[C^{2/3}(\theta\rho) + A(\theta\rho)E(\theta\rho)] \\ &\le c[\theta^2 A(\rho) + \frac{1}{\theta^2}A^{1/2}(\rho)E^{1/2}(\rho) + \frac{1}{\theta^2}A(\rho)E(\rho)] \\ &\le c[(\theta^2 + \frac{1}{\theta^2}E(\rho))A(\rho) + \frac{1}{\theta^6}E(\rho)]. \end{aligned}$$

If θ is small and $E(\rho)$ is small then can make $A(\cdot)$ small. By (3.2) this bounds $C(\cdot)$. □

For a set $X \subset \mathbb{R}^3 \times \mathbb{R}$ and $k \geq 0$, we define $P^k(X) = \lim_{\delta \to 0^+} P^k_\delta(X)$, where

$$P^k_\delta(X) = \inf\left\{\sum_{i=1}^\infty r_i^k : X \subset \bigcup_i Q_{r_i} : r_i < \delta\right\},$$

and $Q_r(x,t) = B_r(x) \times (t - r^2, t + r^2)$. We have $P^k(X) = 0$ if and only if for every $\delta > 0$ the set X can be covered by a collection $\{Q_{r_i}\}$ such that $\sum_i r_i^k < \delta$.

Theorem 3.4 *Let S denote the singular set of a suitable weak solution of the Navier–Stokes equations. Then $P^1(S) = 0$.*

We will need the following fact: given a family of parabolic cylinders $Q_r(x,t)$, there exists a finite or countable disjoint subfamily $\{Q_{r_i}(x_i,t_i)\}$ such that for any cylinder $Q_r(x,t)$ in the original family there exists an i such that $Q_r(x,t) \subset Q_{5r_i}(x_i,t_i)$. (For a proof see CKN.)

Proof Let V be any neighbourhood of S, and choose $\delta > 0$.

For each $(x,t) \in S$, choose a cylinder $Q_r(x,t)$ such that $Q_r(x,t) \subset V$, $r < \delta$, and

$$\frac{1}{r}\iint_{Q_r(x,t)} |\nabla u|^2 > \delta^\star.$$

(This must be possible, for otherwise the point (x,t) would be regular.) We now find a disjoint subcollection of these cylinders $\{Q_{r_i}(x_i,t_i)\}$ such that the singular set is still covered by $\{Q_{5r_i}(x_i,t_i)\}$. Since these cylinders are disjoint,

$$\iint_V |\nabla u|^2 \geq \sum_i \iint_{Q_{r_i}(x_i,t_i)} |\nabla u|^2 \geq \delta^\star \sum_i r_i.$$

Since $\nabla u \in L^2((0,T) \times \Omega)$, the left-hand side is finite, so $\sum_i r_i \leq C$. Since S is contained in the union of $\{Q_{5r_i}(x_i,t_i)\}$, and $r_i < \delta$ for every i, we must have

$$\mu(S) \leq c \sum_{(5r_i)}^{5} \leq c\delta^4 \sum_{r_i} \leq K\delta^4.$$

Since $\delta > 0$ we arbitrary, it follows that $\mu(S) = 0$.

Since $|\nabla u|^2$ is integrable and V is an arbitrary neighbourhood of S (which has zero measure), we can make

$$\frac{1}{\delta^{\star}} \iint_V |\nabla u|^2$$

as small as we wish by choosing V suitably. The above construction then furnishes a cover with $\sum_i (5r_i)$ arbitrarily small, and so $P^1(S) = 0$ as claimed. □

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