Tutorial on Intermittency and the multifractal formalism

Edinburgh, LMS-EPSRC Short Course

Following previous works [6, 7], the (Eulerian) velocity increment $\delta_\ell u(x) = u(x+\ell) - u(x)$ is modeled as the product of two **independent** random variables (the scale ℓ is assumed smaller than the integral length scale L):

$$\delta_\ell u \overset{\text{law}}{=} \beta_\ell \times \delta \ ,$$

where δ is a Gaussian zero-average unit-variance random variable, and β_ℓ a *stochastic variance*, namely a positive random variable, assumed of the form

$$\beta_\ell = \sigma \left(\frac{\ell}{L}\right)^h \ ,$$

with $\sigma^2 = \langle (\delta_L u)^2 \rangle$ the large-scale velocity increment variance and h a random exponent (called the Hölder exponent in multifractal terminology [6]) of density

$$\mathcal{P}_h^{(\ell)}(h) = \frac{1}{\mathcal{Z}(\ell)} \left(\frac{\ell}{L}\right)^{1-\mathcal{D}(h)} \ ,$$

where $\mathcal{D}(h)$ is a (scale independent) parameter function called the singularity spectrum and $\mathcal{Z}(\ell)$ a normalizing function.

- Compute the q^{th}-order structure function $M_q(\ell) = \langle |\delta_\ell u|^q \rangle$.
  hint: $\langle |\delta|^p \rangle = \Gamma\left(\frac{p+1}{2}\right) / \sqrt{2^p \pi}$ where Γ is the Gamma function.
- assuming that $\min_h [1 - \mathcal{D}(h)] = 0$ and considering vanishing scales, i.e. $\ell / L \to 0$, show that structure functions behave as power-laws,

  $$M_q(\ell) \sim \left(\frac{\ell}{L}\right)^{\zeta_q} \ ,$$

  with

  $$\zeta_q = \min_h \left[qh + 1 - \mathcal{D}(h)\right] \ .$$

  hint: apply a steepest-descent approximation.

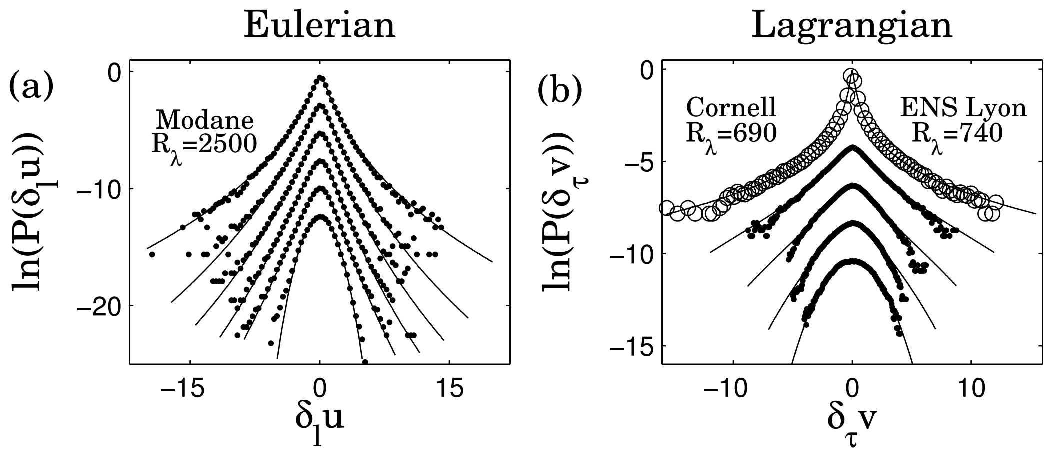

Figure 1: (a) PDFs of signed longitudinal velocity increments in Modane data [1]. Represented scales (from top to bottom): $\ln(\ell/L)$ = -6.4137, -5.6028, -4.6645, -3.6411, -2.7501, -1.8598, -0.8685, 0.1226. All curves are arbitrarily vertically shifted for the sake of clarity and their variance is set to unity. The solid curves correspond to theoretical predictions (see Ref. [2]). (b) PDFs of Lagrangian temporal increments from the ENS-Lyon experiment, for time lags (from bottom to top, symbols $\bullet$) τ/T = 0.07, 0.16, 0.35, 1 and from Cornell acceleration data (symbols $\circ$, from Ref. [4]). Also, the curves are displayed with an arbitrary vertical shift for clarity, the variance is set to unity at any scales, and the original axis for the acceleration PDF ($\circ$) has been shrunk by a factor 4. Solid lines correspond to theoretical predictions (see Ref. [5]).

- Lognormal model:

$$\mathcal{D}^{\mathrm{LN}}(h) = 1 - \frac{(h-c_1)^2}{2c_2},$$

  the parameter c_2 is called the intermittency coefficient. Compute ζ_q^{LN}.

- She-Lévêque model:

$$\mathcal{D}^{\mathrm{SL}}(h) = -1 + 3\left[\frac{1+\ln(\ln(3/2))}{\ln(3/2)} - 1\right](h-1/9) - \frac{3}{\ln(3/2)}(h-1/9)\ln(h-1/9)\ .$$

  Compute ζ_q^{SL}.

- Finally, show that the velocity increment probability density function in the general case reads:

$$\mathcal{P}_{\delta_\ell u}(\delta_\ell u) = \int_{h_{\min}}^{h_{\max}} \frac{1}{\sigma}\left(\frac{\ell}{L}\right)^{-h} \mathcal{P}_\delta\left[\frac{\delta_\ell u}{\sigma}\left(\frac{\ell}{L}\right)^{-h}\right] \mathcal{P}_h^{(\ell)}(h) dh \ ,$$

where $\mathcal{P}_\delta(x) = \exp(-x^2/2)/\sqrt{2\pi}$ is the PDF of a unit-variance zero-mean Gaussian variable.

References

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[4] N. Mordant, A. M. Crawford and E. Bodenschatz, Experimental Lagrangian acceleration probability density function measurement, Physica D 193 (2004), 245.

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