Lagrangian Dynamics and Statistical Geometry in Turbulence

...and Intermittency

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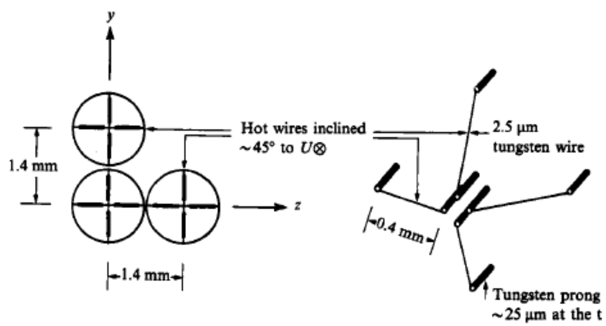
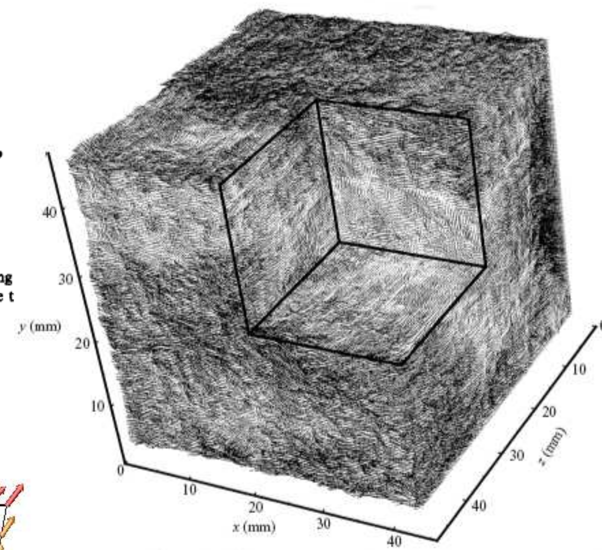
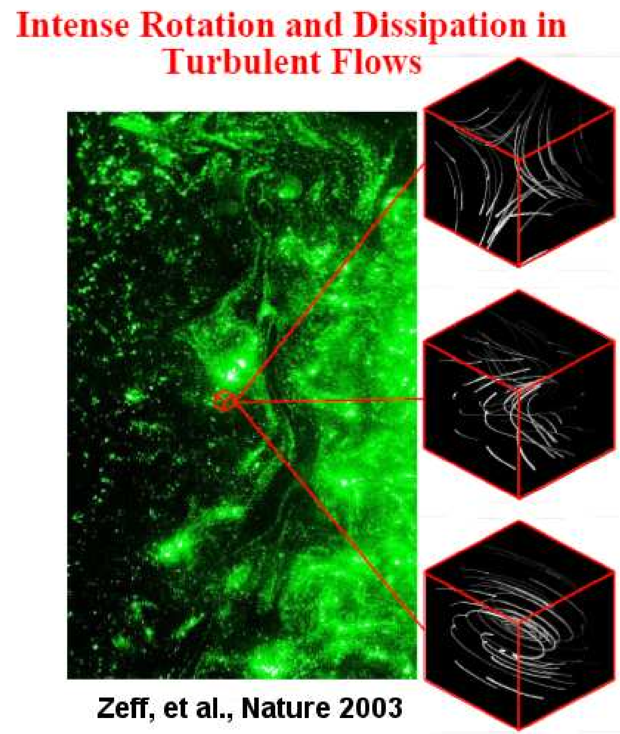
*3D Fluid **Turbulence**: Full velocity gradients*



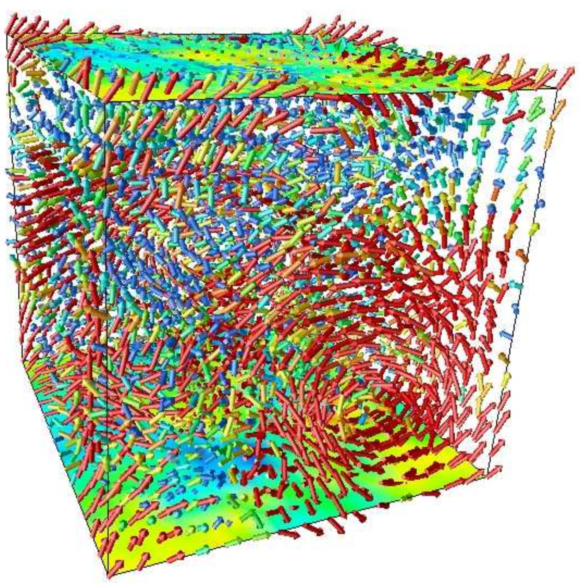
Tsinober, Kit, Dracos J.F.M. 1992



Tao, Katz, Meneveau J.F.M. 2002



Zeff, et al., Nature 2003



Direct Numerical Simulations
(picture by Toschi)

$$\mathbf{A} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix}$$

3D Nature of Turbulence and Non Locality of Pressure

$$\overbrace{a_i}^{\text{Acceleration}} = \frac{du_i}{dt} = \underbrace{\frac{\partial u_i}{\partial t} + \overbrace{(\mathbf{u}.\boldsymbol{\nabla})u_i}^{\text{Recquire 3D}}}_{\text{Anti-Correlated}} = \underbrace{\overbrace{-\nabla_i p}^{\text{nonlinear}}}_{\text{non local}} + \nu \boldsymbol{\nabla}^2 u_i$$

Incompressibility $\leftrightarrow$ Poisson equation: $\boldsymbol{\nabla}^2 p = -\text{tr}\left(\mathbf{A}^2\right)$

Velocity Gradient Tensor: $\mathbf{A} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix}$

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Incompressibility $\leftrightarrow$ Poisson equation: $\boldsymbol{\nabla}^2 p = -\mathrm{tr}\left(\mathbf{A}^2\right)$

$\rightarrow$ Pressure Gradient: $\nabla_i p(\mathbf{x}) = \int d\mathbf{y}\ \underbrace{\overbrace{\mathcal{G}_i(|\mathbf{y}-\mathbf{x}|)}^{\text{Green fct.}}}_{\sim \frac{1}{|\mathbf{y}-\mathbf{x}|^2}} \mathrm{tr}\left(\mathbf{A}^2(\mathbf{y})\right)$ **Non** Local!!

3D Nature of Turbulence and Non Locality of Pressure

$$\overbrace{a_i}^{\text{Acceleration}} = \frac{du_i}{dt} = \underbrace{\frac{\partial u_i}{\partial t} + \overbrace{(\mathbf{u}.\boldsymbol{\nabla})u_i}^{\text{Recquire 3D}}}_{\text{Anti-Correlated}} = \underbrace{\overbrace{-\nabla_i p}^{\text{nonlinear}}}_{\text{non local}} + \nu \boldsymbol{\nabla}^2 u_i$$

Incompressibility $\leftrightarrow$ Poisson equation: $\boldsymbol{\nabla}^2 p = -\text{tr}\left(\mathbf{A}^2\right)$

$\rightarrow$ Pressure **Hessian**: $P_{ij} = \nabla_{ij} p(\mathbf{x}) = \overbrace{-\text{tr}\left(\mathbf{A}^2(\mathbf{x})\right)\frac{\delta_{ij}}{3}}^{\text{Local-Isotropic}} + \overbrace{\text{P.V.}\int d\mathbf{y}\; \underbrace{\mathcal{G}_{ij}(|\mathbf{y}-\mathbf{x}|)}_{\sim \frac{1}{|\mathbf{y}-\mathbf{x}|^3}} \text{tr}\left(\mathbf{A}^2(\mathbf{y})\right)}^{\text{Non Local-Anisotropic}}$

See Ohkitani & Kishiba (PoF,95) and Majda, Bertozzi (CUP,01)

3D *Nature of Turbulence and Non Locality of* ***Pressure***

$$\overbrace{a_i}^{\text{Acceleration}} = \frac{du_i}{dt} = \underbrace{\frac{\partial u_i}{\partial t} + \overbrace{(\mathbf{u}.\boldsymbol{\nabla})u_i}^{\text{Recquire 3D}}}_{\text{Anti-Correlated}} = \underbrace{\overbrace{-\nabla_i p}^{\text{nonlinear}}}_{\text{non local}} + \nu \boldsymbol{\nabla}^2 u_i$$

$$\Downarrow$$
$$\frac{\partial}{\partial x_j}$$
$$\Downarrow$$

Time evolution of the velocity gradient tensor $\mathbf{A} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix}$

3D Nature of Turbulence and Non Locality of Pressure

$$\overbrace{a_i}^{\text{Acceleration}} = \frac{du_i}{dt} = \underbrace{\frac{\partial u_i}{\partial t} + \overbrace{(\mathbf{u}.\boldsymbol{\nabla})u_i}^{\text{Recquire 3D}}}_{\text{Anti-Correlated}} = \underbrace{\overbrace{-\nabla_i p}^{\text{nonlinear}}}_{\text{non local}} + \nu \boldsymbol{\nabla}^2 u_i$$

$$\Downarrow \quad \frac{\partial}{\partial x_j} \quad \Downarrow$$

$$\frac{d}{dt}A_{ij} = \underbrace{-A_{iq}A_{qj}}_{\text{self-stretching term}} \underbrace{- \overbrace{\frac{\partial^2 p}{\partial x_i \partial x_j}}^{\text{Pressure Hessian}} + \underbrace{\nu \frac{\partial^2 A_{ij}}{\partial x_q \partial x_q}}_{\text{Viscous term}}}_{\text{need to be \textbf{modeled}}}$$

The Lagrangian evolution of the Eulerian velocity gradient tensor

See the review C. Meneveau, *Lagrangian dynamics and models of the velocity gradient tensor in Turbulent flows*, Ann. Rev. Fluid Mech. (2011)

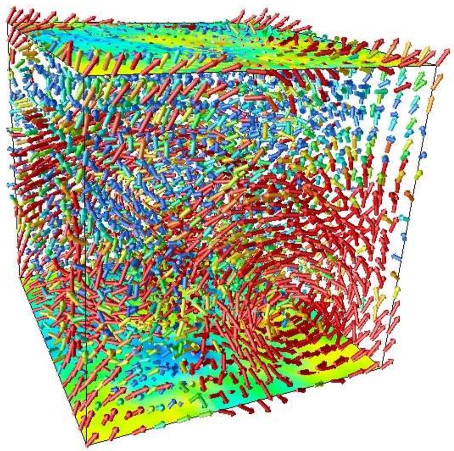
Let $A_{ij} = \dfrac{\partial u_i}{\partial x_j}$ and $\dfrac{d}{dt} \equiv \dfrac{\partial}{\partial t} + u_q \dfrac{\partial}{\partial x_q}$

$$A = \begin{pmatrix} \text{Long}_{11} & \text{Trans}_{12} & \text{Trans}_{13} \\ \text{Trans}_{21} & \text{Long}_{22} & \text{Trans}_{23} \\ \text{Trans}_{31} & \text{Trans}_{32} & \text{Long}_{33} \end{pmatrix}$$

Then, along a fluid trajectory (Léorat 75, Vieillefosse 82):

$$\nabla(\text{Navier-Stokes}) \Rightarrow \frac{d}{dt}A_{ij} = \underbrace{-A_{iq}A_{qj}}_{\text{self-stretching term}} - \underbrace{\overbrace{\frac{\partial^2 p}{\partial x_i \partial x_j}}^{\text{Pressure Hessian}} + \underbrace{\nu \frac{\partial^2 A_{ij}}{\partial x_q \partial x_q}}_{\text{Viscous term}}}_{\text{need to be } \textbf{modeled}}$$

Tracking Velocity Gradients along Lagrangian trajectories



DNS

- Yeung & Pope (89).
- Girimaji & Pope, J.F.M. (90).
- Pope & Chen , Phys. Fluids (90).

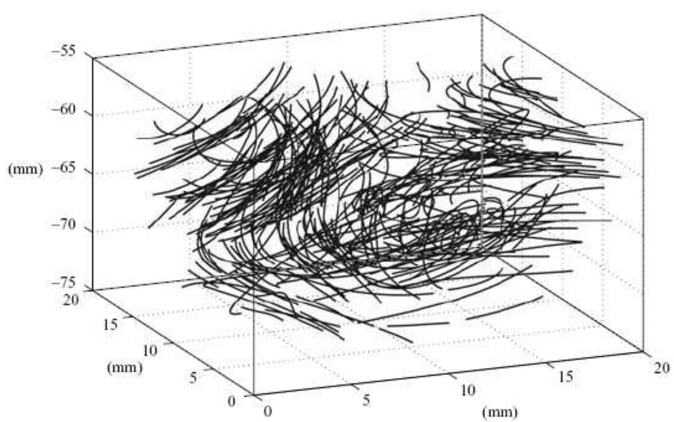


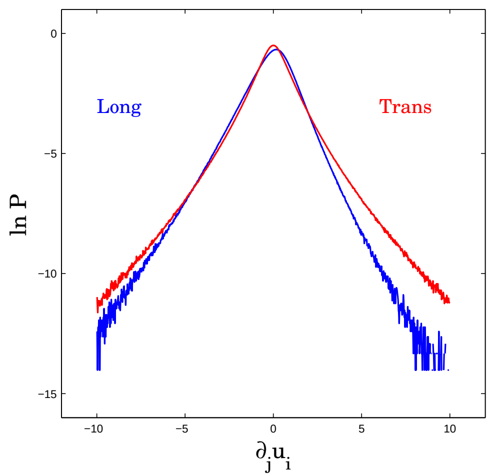
FIGURE 2. Selected particle trajectories as obtained from 3D-PTV.

Experimental

- Zeff *et al.*, Nature (2003).
- Lüthi, Tsinober & Kinzelbach, J.F.M. (2005).

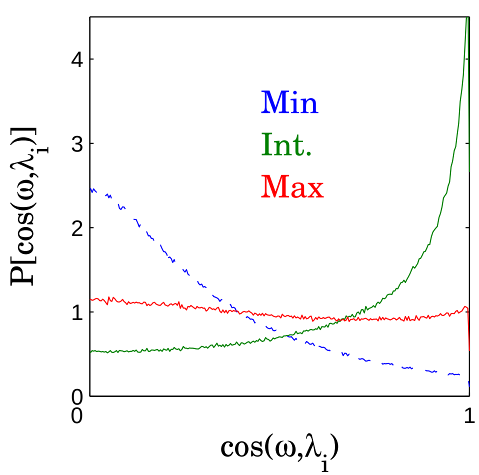
Statistical ***Intermittency*** *and* ***Geometry*** *in turbulence*

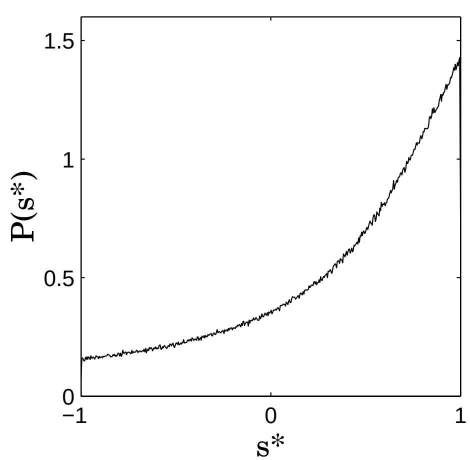
DNS Results $\mathcal{R}_\lambda = 150$



Intermittency

- Non-Gaussianity
- Skewness
- Anomalous scaling with Reynolds number



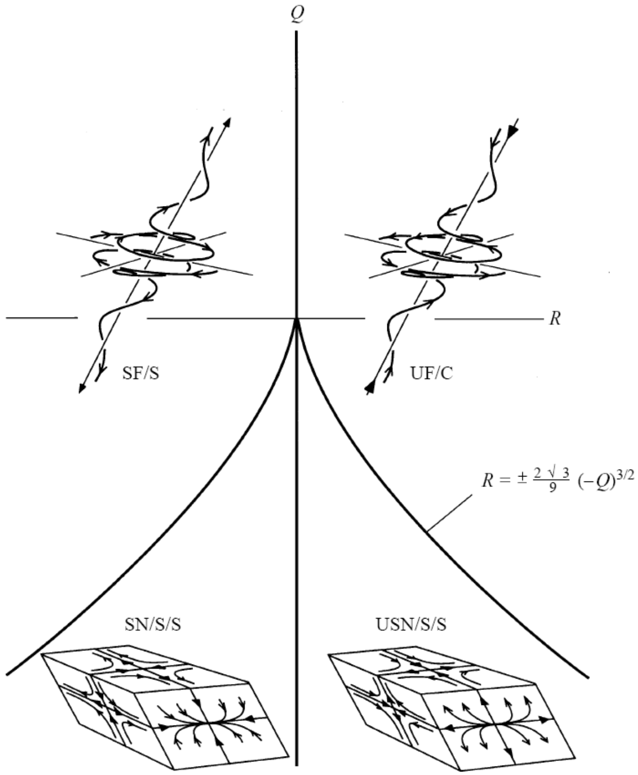


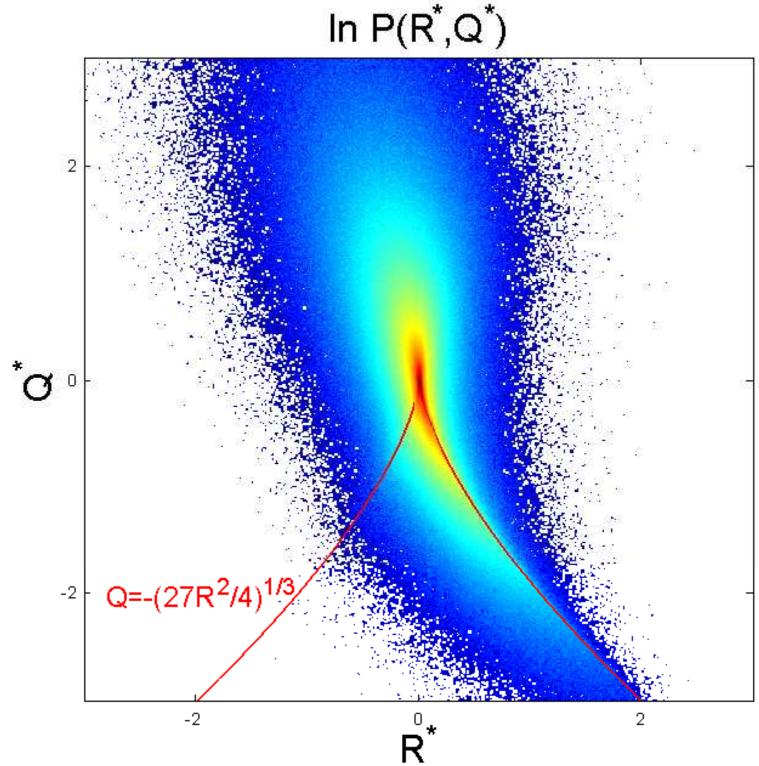
Geometry

- Preferential alignment of vorticity
- Preferential axisymmetric expansion

The RQ plane - Local Topology

See Chong, Perry and Cantwell (90) and Cantwell (93)





$\lambda_i = f(R,Q) \in \mathbb{C}$: Eigenvalues of **A**

- Second invariant: $Q = -\frac{1}{2}\text{Tr}(\mathbf{A}^2) = \frac{1}{4}\overbrace{|\omega|^2}^{\text{Enstrophy}} - \frac{1}{2}\overbrace{\text{Tr}(\mathbf{S}^2)}^{\text{Dissipation}}$
- Third invariant: $R = -\frac{1}{3}\text{Tr}(\mathbf{A}^3) = -\frac{1}{4}\underbrace{\omega_i S_{ij}\omega_j}_{\text{Enstrophy Production}} - \frac{1}{3}\underbrace{\text{Tr}(\mathbf{S}^3)}_{\text{Strain Skewness}}$

Review of the Restricted Euler approximation (I)

$$\frac{d}{dt}A_{ij} = -A_{iq}A_{qj} - \frac{\partial^2 p}{\partial x_i \partial x_j} + \nu \frac{\partial^2 A_{ij}}{\partial x_q \partial x_q}$$

- Restricted Euler Dynamics (Léorat 75-Vieillefosse 84-Cantwell 92)
  $\frac{\partial^2 p}{\partial x_i \partial x_j} = -\frac{\delta_{ij}}{3}\text{Tr}(\mathbf{A}^2)$ and $\nu = 0$

$$\frac{d}{dt}\mathbf{A} = -\left(\mathbf{A}^2 - \frac{\delta_{ij}}{3}\text{Tr}(\mathbf{A}^2)\right)$$

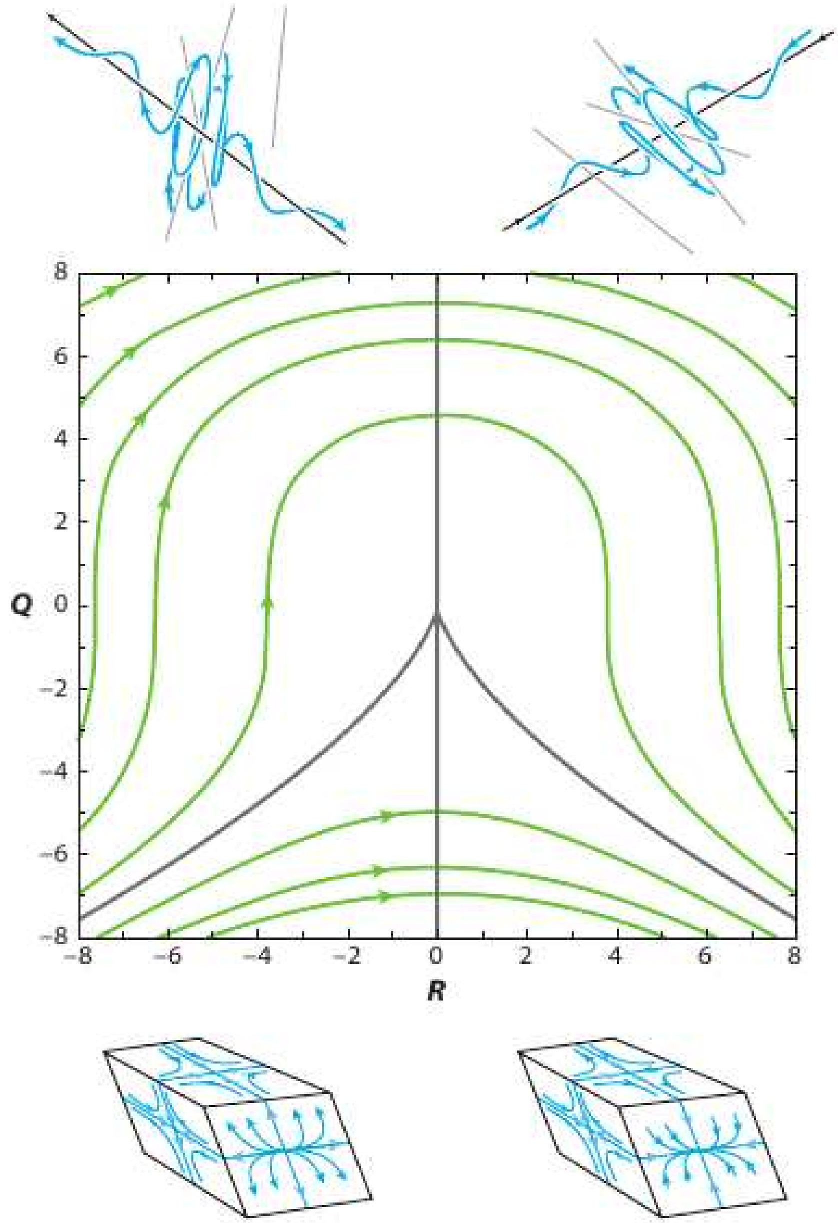
$\rightarrow$ Non stationary

- Evolution equations for Q and R:

$$\frac{dQ}{dt} = -3R \quad \text{and} \quad \frac{dR}{dt} = \frac{2}{3}Q^2$$

- $\frac{27}{4}R^2(t) + Q^3(t)$ is time invariant

Review of the Restricted Euler approximation (II)



Review of the Restricted Euler approximation (III)

$$\frac{d}{dt}A_{ij} = -A_{iq}A_{qj} - \frac{\partial^2 p}{\partial x_i \partial x_j} + \nu \frac{\partial^2 A_{ij}}{\partial x_q \partial x_q}$$

- Restricted Euler Dynamics (Léorat 75-Vieillefosse 84-Cantwell 92)
  $\frac{\partial^2 p}{\partial x_i \partial x_j} = -\frac{\delta_{ij}}{3}\text{Tr}(\mathbf{A}^2)$ and $\nu = 0$

$$\frac{d}{dt}\mathbf{A} = -\left(\mathbf{A}^2 - \frac{\delta_{ij}}{3}\text{Tr}(\mathbf{A}^2)\right)$$

$\rightarrow$ Non stationary

- Finite time singularity (t^*) of **A** for any initial condition
- BUT, at $t \lesssim t^*$
  - Second eigenvalue λ of **S** is positive
    $\rightarrow$ Preferential axisymmetric expansion
  - Vorticity gets aligned with the associated eigenvector u_λ

Review of various models

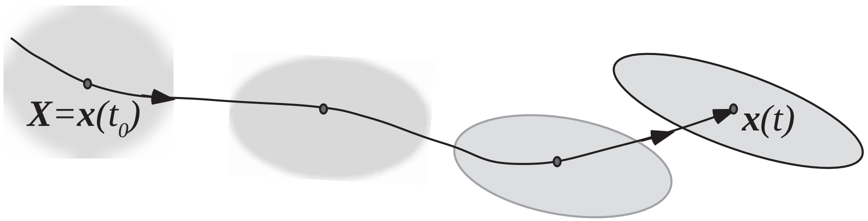
$$\frac{d}{dt}A_{ij} = -A_{iq}A_{qj} - \frac{\partial^2 p}{\partial x_i \partial x_j} + \nu \frac{\partial^2 A_{ij}}{\partial x_q \partial x_q}$$

- Restricted Euler Dynamics (Vieillefosse 84-Cantwell 92)
  $\frac{\partial^2 p}{\partial x_i \partial x_j} = -\frac{\delta_{ij}}{3}\text{Tr}(\mathbf{A}^2)$ and $\nu = 0 \rightarrow$ Finite time singularity
- Lognormality of Pseudo-dissipation $\varphi = \text{Tr}(\mathbf{A}\mathbf{A}^T)$ (Girimaji-Pope 90)
  $\rightarrow$ Strong *a-priori* assumption
- Linear damping term (Martin *et al.* 98)
  $\frac{\partial^2 p}{\partial x_i \partial x_j} = -\frac{\delta_{ij}}{3}\text{Tr}(\mathbf{A}^2)$ and $\nu \frac{\partial^2 \mathbf{A}}{\partial x_q \partial x_q} = -\frac{1}{\tau}\mathbf{A} \rightarrow$ Finite time singularity
- Delta-vee system (Yi-Meneveau (05))
  Projection on Longitudinal $\delta_\ell u$ and Transverse $\delta_\ell v$ increments

Using the material **Deformation** (**Cauchy-Green** Tensor **C**)

- Tetrad's model (Chertkov-Pumir-Shraiman 99)
  $\frac{\partial^2 p}{\partial x_i \partial x_j} = -\frac{\text{Tr}(\mathbf{A}^2)}{\text{Tr}(\mathbf{C}^{-1})} C_{ij}^{-1}$ and $\nu = 0 \rightarrow$ Non stationary
- Differential damping term (Jeong-Girimaji 03)
  $\frac{\partial^2 p}{\partial x_i \partial x_j} = -\frac{\delta_{ij}}{3}\text{Tr}(\mathbf{A}^2)$ and $\nu \frac{\partial^2 \mathbf{A}}{\partial x_q \partial x_q} = -\frac{\text{Tr}(\mathbf{C}^{-1})}{3\tau}\mathbf{A}$
  $\rightarrow$ Non stationary

Cauchy-Green Tensor: Tracking the volume deformation



Deformation gradient: $D_{ij}(t) = \frac{\partial x_i}{\partial X_j}(t)$

Dyn.: $\frac{d}{dt}\mathbf{D} = \mathbf{A}\mathbf{D}$

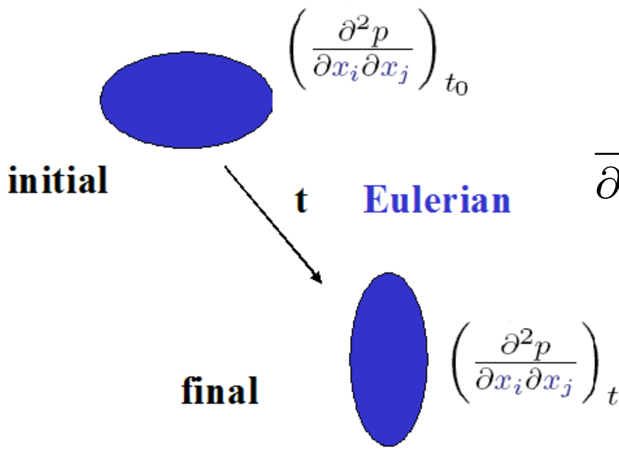
$$\leftrightarrow \quad \underbrace{\mathbf{D}(t) = \prod_{t_0}^{t} e^{\mathbf{A}(\xi)d\xi}}_{\text{Time ordered exponential}}$$

Cauchy-Green Tensor: $\boxed{\mathbf{C} = \mathbf{D}\mathbf{D}^T \text{ or } C_{ij}^{-1} = \frac{\partial X_p}{\partial x_i}\frac{\partial X_p}{\partial x_j}}$

Non Stationary

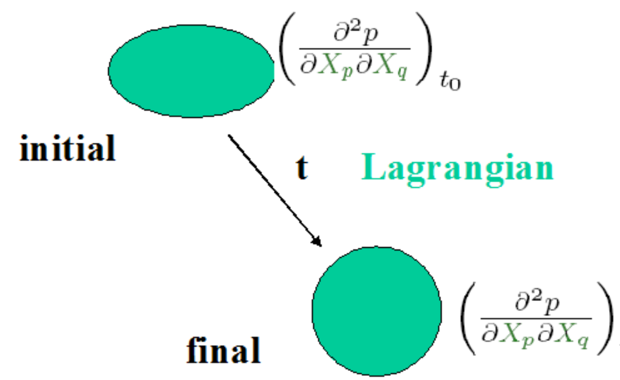
- Monin-Yaglom 75
- Girimaji-Pope 90: DNS $\mathcal{R}_\lambda = 90$
- Lüthi, Tsinober, Kinzelbach 05: Exp. $\mathcal{R}_\lambda = 50$

Re-Interpretation of the Chertkov et al. Tetrad Model



Eulerian $\rightarrow$ Lagrangian

$$\frac{\partial^2 p}{\partial x_i \partial x_j} \approx \frac{\partial X_p}{\partial x_i} \frac{\partial X_q}{\partial x_j} \frac{\partial^2 p}{\partial X_p \partial X_q}$$



$$\text{hyp.:} \quad \overbrace{\frac{\partial^2 p}{\partial X_p \partial X_q}}^{\text{Isotropic}} = \frac{\delta_{pq}}{3} \frac{\partial^2 p}{\partial X_m \partial X_m} \overset{\text{Poisson}}{\underset{\text{equation}}{=}} -\delta_{pq} \frac{\mathrm{Tr}(\mathbf{A}^2)}{\mathrm{Tr}(\mathbf{C}^{-1})}$$

Chertkov, Pumir and Shraiman (99)

$$\frac{\partial^2 p}{\partial x_i \partial x_j} = -\frac{\mathrm{Tr}(\mathbf{A}^2)}{\mathrm{Tr}(\mathbf{C}^{-1})} C^{-1}_{ij}$$

Non Stationary

The Jeong et al. Lagrangian Linear Diffusion Model

Eulerian $\rightarrow$ Lagrangian

$$\frac{\partial^2 \mathbf{A}}{\partial x_m \partial x_m} = \frac{\partial X_p}{\partial x_m}\frac{\partial X_q}{\partial x_m}\frac{\partial^2 \mathbf{A}}{\partial X_p \partial X_q}$$

hyp.: **Linear damping** in the Lagrangian frame $\nu \frac{\partial^2 \mathbf{A}}{\partial X_p \partial X_q} = -\frac{\delta_{pq}}{3}\frac{\mathbf{A}}{\Theta}$

$$\nu \frac{\partial^2 \mathbf{A}}{\partial x_m \partial x_m} = -\frac{\text{Tr}(\mathbf{C}^{-1})}{3\Theta}\mathbf{A} \rightarrow \Theta \text{ ??}$$

Jeong and Girimaji (03) Non Stationary

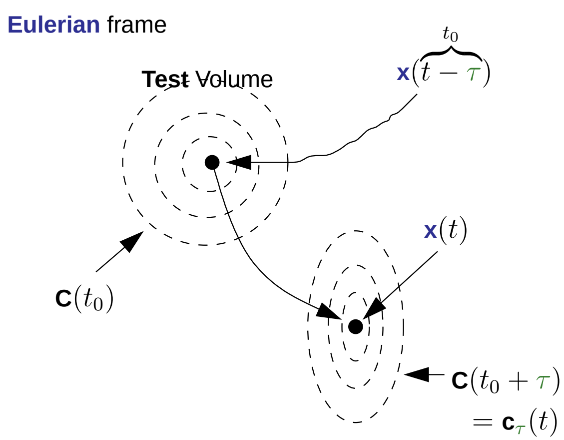
Chevillard-Meneveau (06)$\rightarrow$Self-consistent time-scale estimation:

$$\frac{1}{\Theta} \sim \nu \frac{\delta^2}{\delta X^2} \sim \frac{\nu}{(\text{distance traveled during } \tau_K)^2} \sim \frac{\nu}{\underbrace{\lambda^2}_{\textbf{Taylor}}} \sim \frac{1}{T} \textbf{ Integral time scale}^{-1}$$

Stationary *Cauchy-Green Tensor*

$$\mathbf{D}(t) = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}(t) = \prod_{t_0}^{t} e^{\mathbf{A}(\xi)d\xi} = \underbrace{\mathbf{d}_\tau(t)}_{\text{present}} \overbrace{\mathbf{D}(t-\tau)}^{\text{old history}}$$

Over a **dissipative** time scale $\tau = \begin{cases} \tau_K & \text{Kolmogorov} \\ 1/\sqrt{\text{Tr}(2\mathbf{S}^2)} & \text{Local} \end{cases}$



with $\mathbf{d}_\tau(t) = \prod_{t-\tau}^{t} e^{\mathbf{A}(\xi)d\xi} \approx e^{\int_{t-\tau}^{t} \mathbf{A}(\xi)d\xi} \approx e^{\tau \mathbf{A}(t)}$

Recent deformation

Let $\mathbf{c}_\tau$ the stationary "Cauchy-Green" Tensor

$$\boxed{\mathbf{c}_\tau = \mathbf{d}_\tau \mathbf{d}_\tau^T}$$

See Chevillard-Meneveau 06

A *Stochastic* model for the *velocity gradient tensor*

Chevillard-Meneveau, Physical Review Letters 97, 174501 (2006)

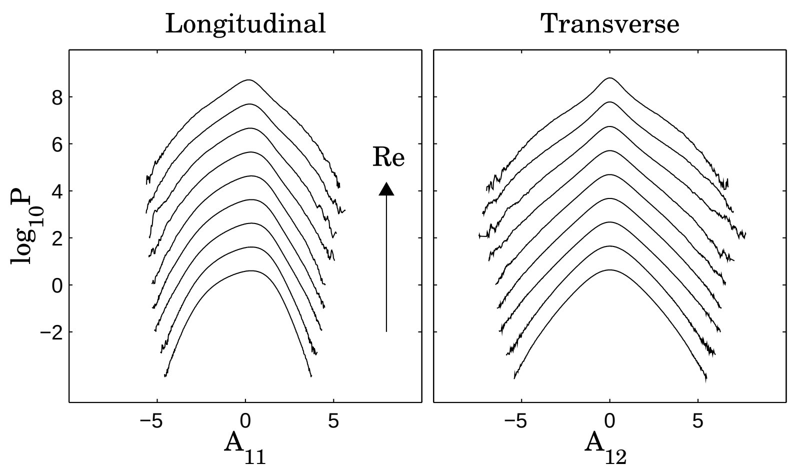
$$d\mathbf{A} = \left(\underbrace{-\mathbf{A}^2}_{\text{Self-stretching}} + \overbrace{\frac{\mathbf{c}_\tau^{-1}}{\mathrm{Tr}(\mathbf{c}_\tau^{-1})}\mathrm{Tr}(\mathbf{A}^2)}^{\text{Pressure Hessian}} - \underbrace{\frac{\mathrm{Tr}(\mathbf{c}_\tau^{-1})}{3T}\mathbf{A}}_{\text{Viscous}} \right) dt + \overbrace{d\mathbf{W}}^{\text{Forcing}}$$

- Simplest **white-in-time Gaussian** forcing
  $\rightarrow$ Tracefree-Isotropic-Homogeneous-Unit variance
- **Explicit** Reynolds number $\mathcal{R}_e$ dependence when $\tau = \tau_K$
- **Fluctuating** (Local) dissipative time scale $\tau = \Gamma(\mathcal{R}_e)/\sqrt{\mathrm{Tr}(2\mathbf{S}^2)}$

$$\mathcal{R}_e \text{ effects} \rightarrow \begin{cases} \text{Isotropization of Pressure Hessian} \\ \text{Weakening Viscous term} \end{cases}$$

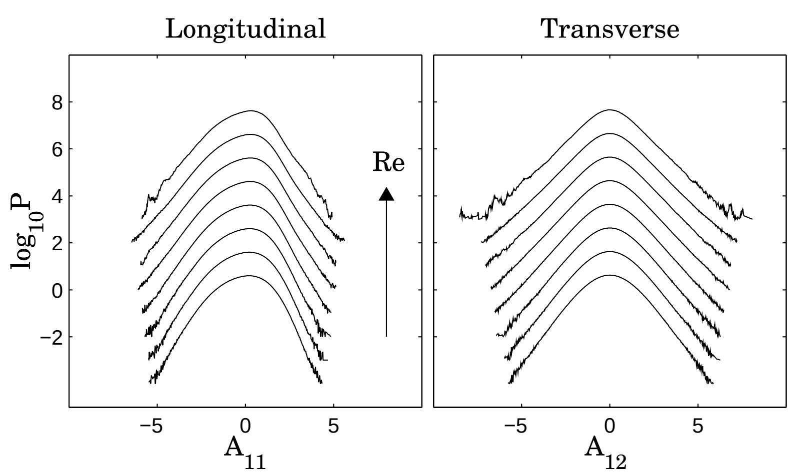
***Prediction** of **Intermittency** (I): Deformation of PDFs*

Chevillard & Meneveau, C.R. Mécanique **335**, 187 (2007).



$$\tau = \tau_K$$

- **Continuous** deformation $\leftrightarrow$ **Intermittency**
- At **high** $\mathcal{R}_e \rightarrow$ Not realistic

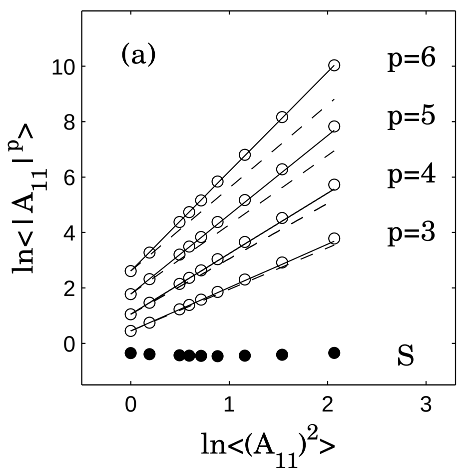


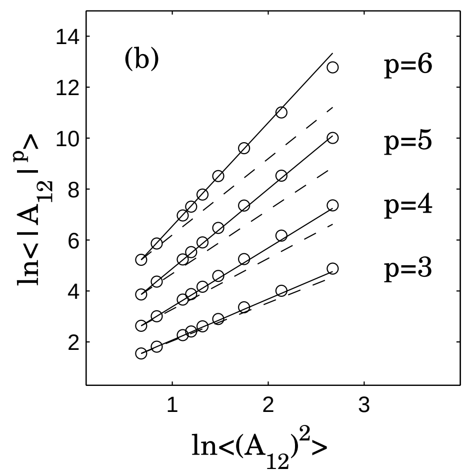
$$\tau = \Gamma(\mathcal{R}_e)/\sqrt{\text{Tr}(2\mathbf{S}^2)}$$

- **Continuous** deformation $\leftrightarrow$ **Intermittency**
- Very realistic but $\Gamma > \Gamma_c$ for regularization

***Prediction** of **Intermittency** (II): Relative scalings*

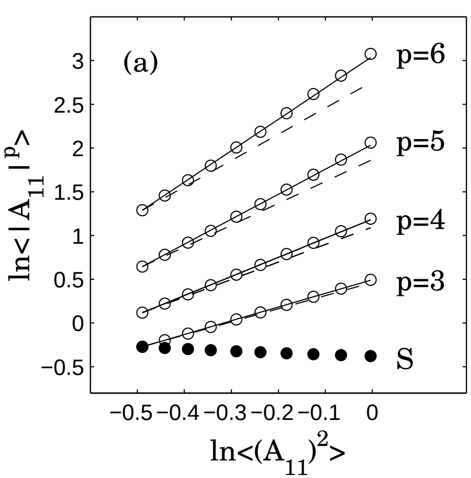
- - (K41 **Monofractal**) and — Nelkin's **Multifractal** predictions (Lognormal $\rightarrow c_2$)

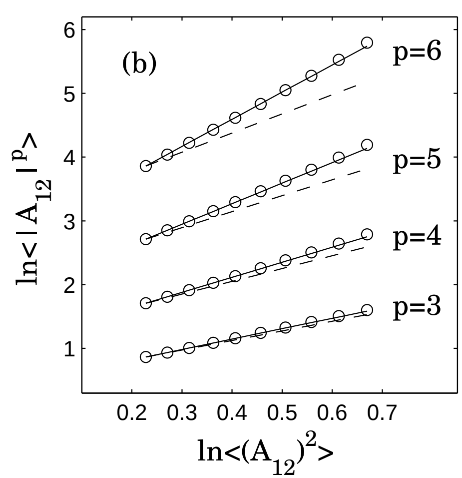




$\tau = \tau_K$

- c_2 Long $\approx 0.025 \rightarrow \mu \approx 0.22$
- c_2 Trans ≈ 0.040
- Skewness $\approx -0.35 \rightarrow -0.5$





$\tau = \Gamma(\mathcal{R}_e)/\sqrt{\text{Tr}(2\mathbf{S}^2)}$

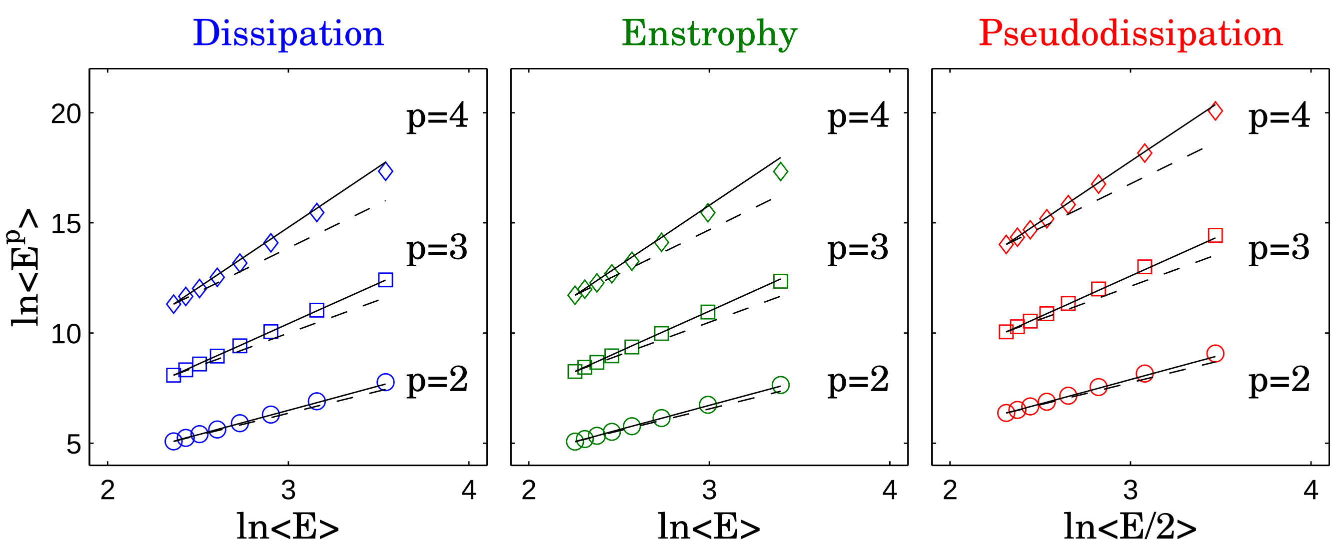
- c_2 Long ≈ 0.025
- c_2 Trans ≈ 0.045
- Skewness $\approx -0.35 \rightarrow -0.5$

$\Rightarrow$ **Robustness** (Universality)

Prediction *of Intermittency (III): Relative scalings of Measures*

- - (K41) and — Nelkin-Meneveau's **Multifractal** predictions (Lognormal $\to \mu$)

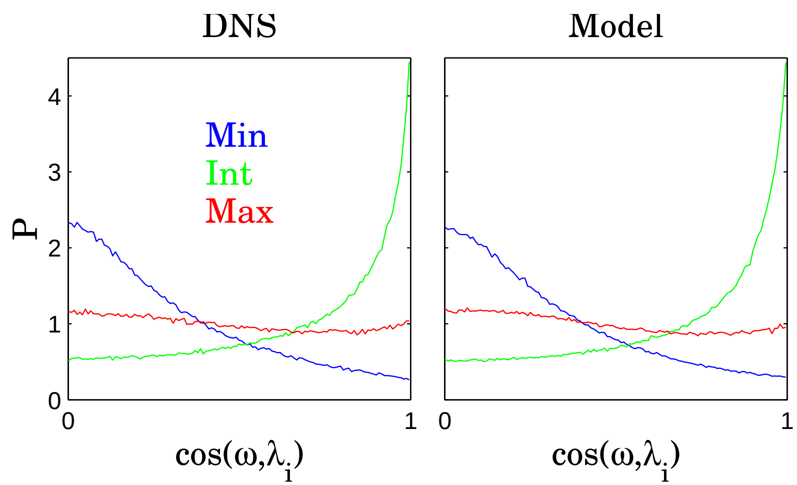
$$\mathbf{A} = \overbrace{\mathbf{S}}^{\text{Deformation}} + \underbrace{\mathbf{\Omega}}_{\text{Rotation}} \left\{ \begin{array}{l} \text{Dissipation } \epsilon = \text{Tr}(\mathbf{S}^2) \to \mu^\epsilon \\ \text{Enstrophy } \zeta = -\text{Tr}(\mathbf{\Omega}^2) \to \mu^\zeta \\ \text{Pseudodissipation } \varphi = \text{Tr}(\mathbf{A}\mathbf{A}^T) \to \mu^\varphi \end{array} \right.$$



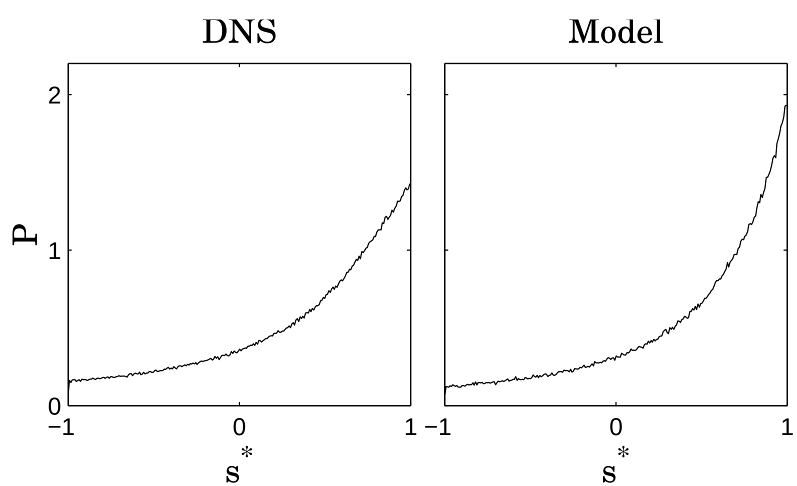
Conclusions: $\overbrace{\mu^\epsilon = \mu^\zeta = \mu^\varphi}^{\textbf{Same intermittency}} = \mu \approx 0.25 \approx \underbrace{c_2^{\text{Long}} \times 9}_{\textbf{Refined Similarity Hypothesis}}$

DNS comparisons (I)

- **DNS** 256^3: $\mathcal{R}_\lambda = 150$
- **Model**: $\tau_K/T = 0.1$ (Consistent with Yeung *et al.* JoT 06)



Alignment of vorticity with eigenvectors of strain **S**
$\rightarrow$ Preferential **alignment**



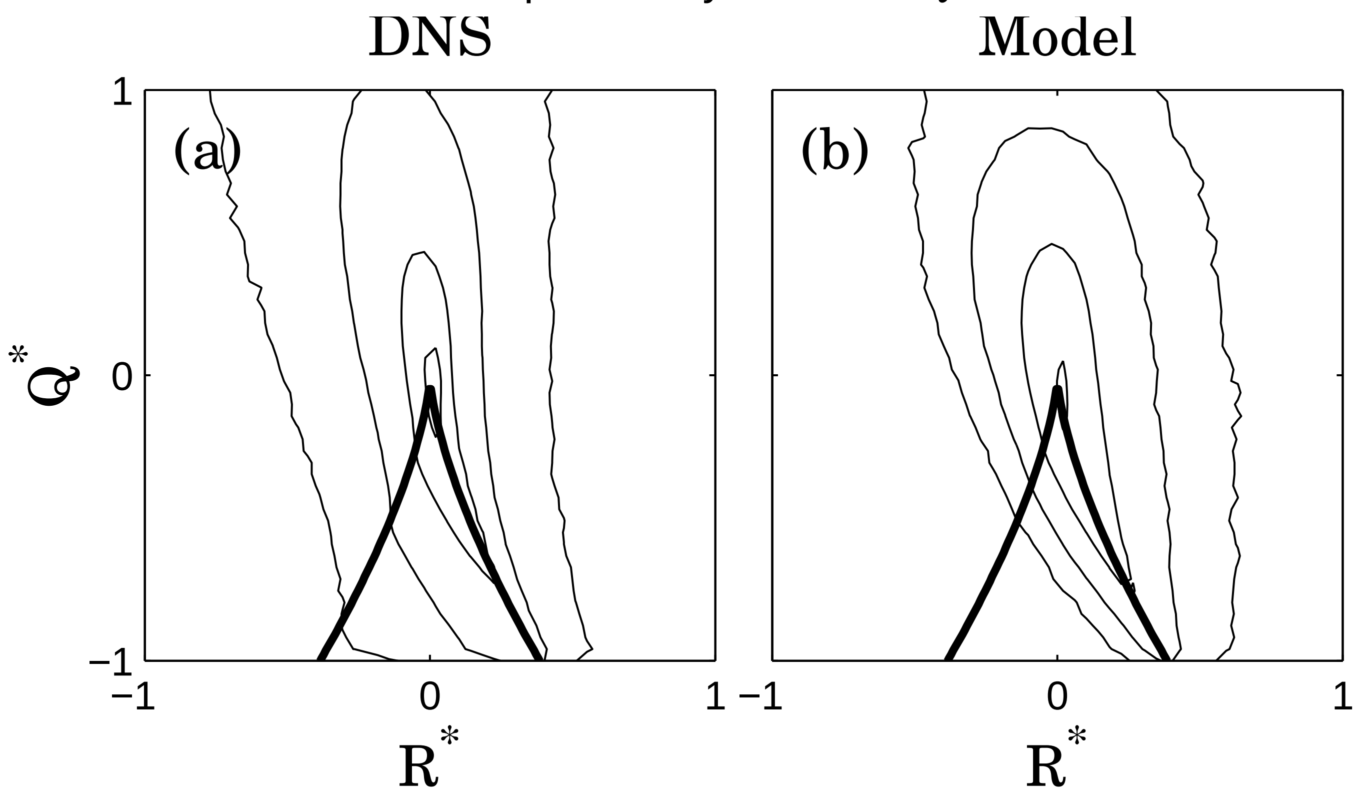
PDF of rate of Strain s^*:

$$s^* = \frac{-3\sqrt{6}\alpha\beta\gamma}{(\alpha^2+\beta^2+\gamma^2)^{3/2}}$$

$\rightarrow$ Preferential **axisymmetric expansion**

***DNS** comparisons (II)*

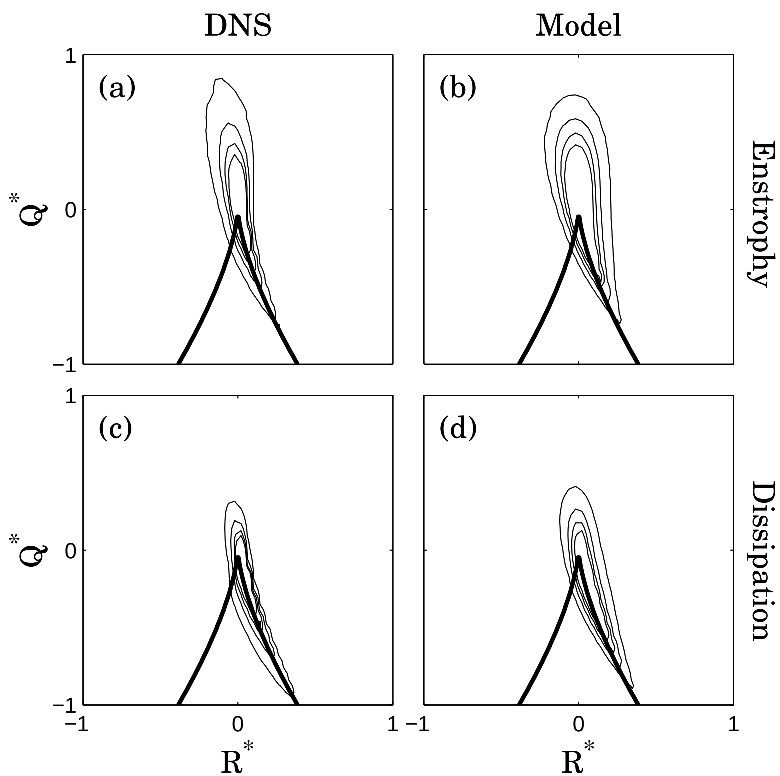
Joint probability of R and Q



***DNS** comparisons (III)*

Focussing on Enstrophy-Dissipation dominated regions (see Chertkov *et al.* 99):

$$Q = -\frac{1}{2}\mathrm{Tr}(\mathbf{A}^2) = \frac{1}{4}\overbrace{|\omega|^2}^{\text{Enstrophy}} - \frac{1}{2}\overbrace{\mathrm{Tr}(\mathbf{S}^2)}^{\text{Dissipation}}$$



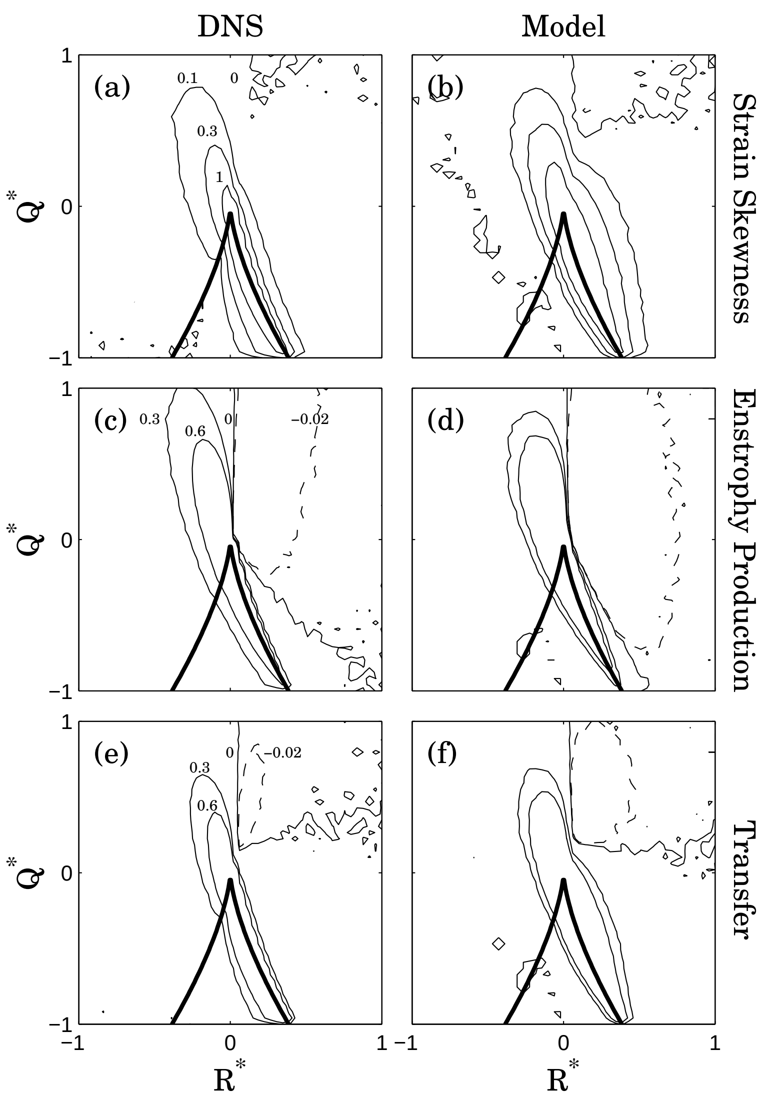
Conditional average:

$$\langle |\omega|^2 | R, Q\rangle \mathcal{P}(R,Q)$$

$$\langle \mathrm{Tr}(\mathbf{S}^2) | R, Q\rangle \mathcal{P}(R,Q)$$

DNS comparisons (IV)

$$R = -\frac{1}{3}\mathrm{Tr}(\mathbf{A}^3) = -\frac{1}{4}\underbrace{\omega_i S_{ij}\omega_j}_{\text{Enstrophy Production}} - \frac{1}{3}\underbrace{\mathrm{Tr}(\mathbf{S}^3)}_{\text{Strain Skewness}}$$



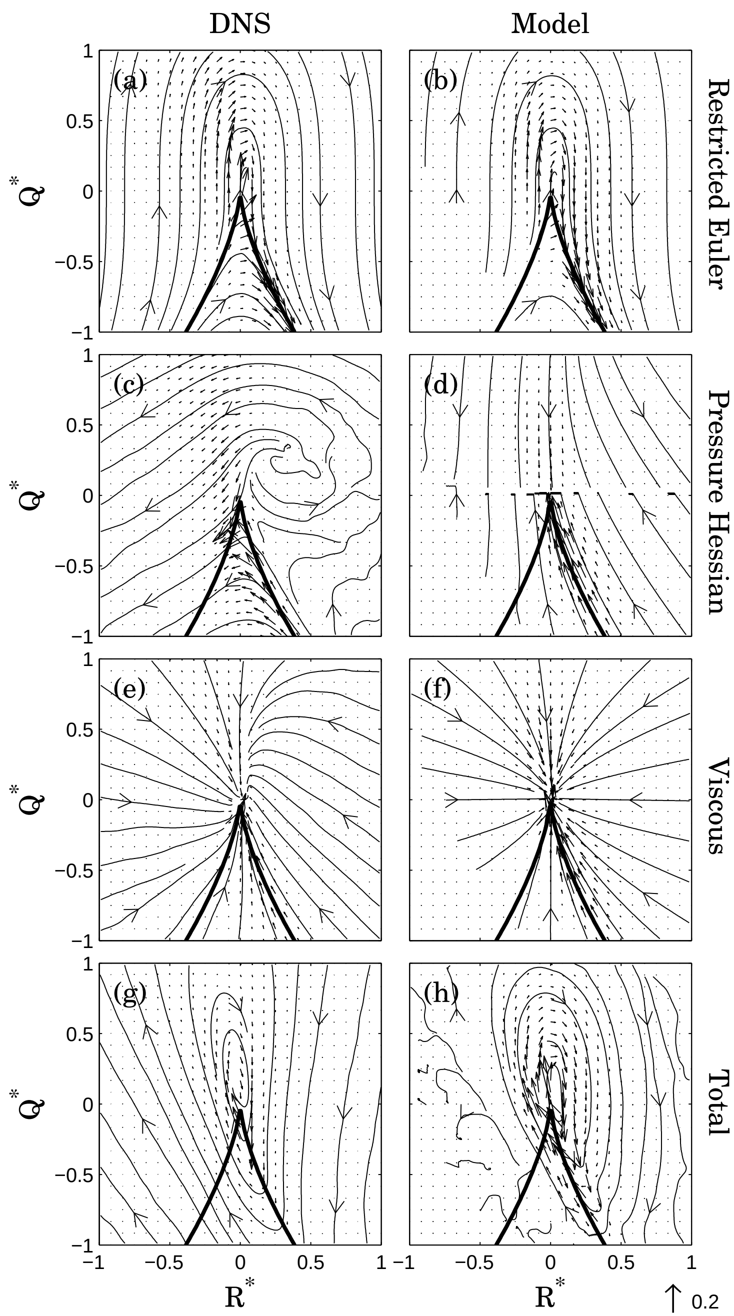
Conditional average (Chertkov *et al.* 99):

$$\langle -\mathrm{Tr}(\mathbf{S}^3)|R,Q\rangle \mathcal{P}(R,Q)$$

$$\langle \omega_i S_{ij}\omega_j|R,Q\rangle \mathcal{P}(R,Q)$$

$$\langle -\mathrm{Tr}(\mathbf{A}^T\mathbf{A}^2)|R,Q\rangle \mathcal{P}(R,Q)$$

***DNS** comparisons (V): Focussing on Pressure Hessian and Viscous effects*



Cond. average (van der Bos *et al.* 02):

$$\left\langle \begin{pmatrix} dR/dt \\ dQ/dt \end{pmatrix}_{\mathrm{RE}} \middle| R, Q \right\rangle \mathcal{P}(R,Q)$$

$$\left\langle \begin{pmatrix} dR/dt \\ dQ/dt \end{pmatrix}_{\mathrm{PH}} \middle| R, Q \right\rangle \mathcal{P}(R,Q)$$

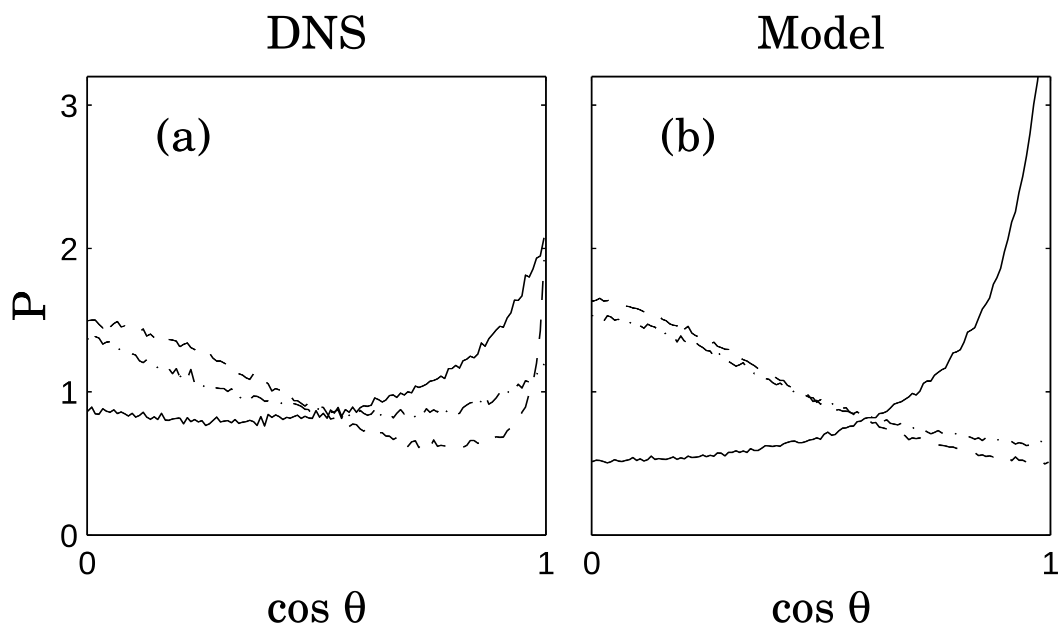
$$\left\langle \begin{pmatrix} dR/dt \\ dQ/dt \end{pmatrix}_{\mathrm{Viscous}} \middle| R, Q \right\rangle \mathcal{P}(R,Q)$$

$$\left\langle \begin{pmatrix} dR/dt \\ dQ/dt \end{pmatrix}_{\mathrm{Total}} \middle| R, Q \right\rangle \mathcal{P}(R,Q)$$

Alignment of Vorticity with Pressure Hessian eigenvectors

Euler equations

Ohkitani (93), Gibbon et al. (97, 06, 07) $\Rightarrow \begin{cases} \frac{d\omega_i}{dt} = S_{ij}\omega_j & \text{Vorticity streching} \\ \frac{d^2\omega_i}{dt^2} = -P_{ij}\omega_j & \text{Ertel's Theorem (42)} \end{cases}$



Alignments with "intermediate" eigenvector reproduced

Alignments with "smallest" eigenvector NOT reproduced

Conclusions

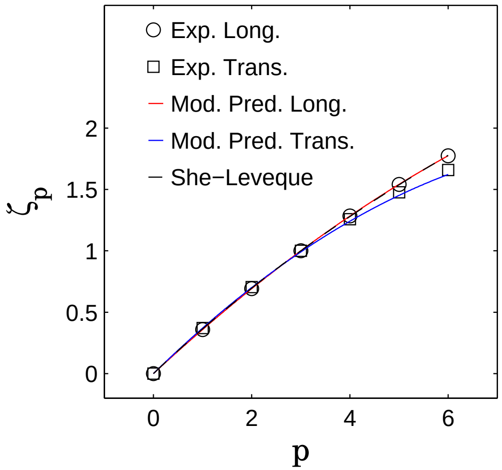
8 independent **ODEs** $d\mathbf{A}/dt = -\mathbf{A}^2 - \mathbf{P} + \nu\Delta\mathbf{A}$

- A **new** stationary stochastic model for $\mathbf{A}$ including closures for
  - Pressure Hessian $\mathbf{P}$
  - Velocity gradient Laplacian $\nu\Delta\mathbf{A}$

$\rightarrow$ Physics of Recent deformation

- Well-known properties of turbulence (vorticity alignments, RQ-plane, skewness) well reproduced. Discrepancies in **Enstrophy** dominated regions.

- Prediction of Intermittency
  - **quantitative** agreement with standard data
    - Transverse more intermittent than Longitudinal
    - Dissipation and Enstrophy scale the same



Perspectives
- Improving **rotation-vorticity stretching** dominated regions
- Reaching **very high** $\mathcal{R}_e$ (see Biferale et al., PRL **98**, 214501 (2007).)
- Modeling **Subgrid**-scale stress tensor (See Chevillard, Li, Eyink, Meneveau