Length scales in weak solutions of the 3D Navier-Stokes equations

Guest Lecture: LMS-EPSRC Course at Heriot-Watt

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Summary of this talk

(i) **Mathematically :** What are the issues? What is their history?

(ii) **Physically :** Can NS-analysis tell us anything interesting physically about 3D NS-flows without first having the full solution of the regularity problem?

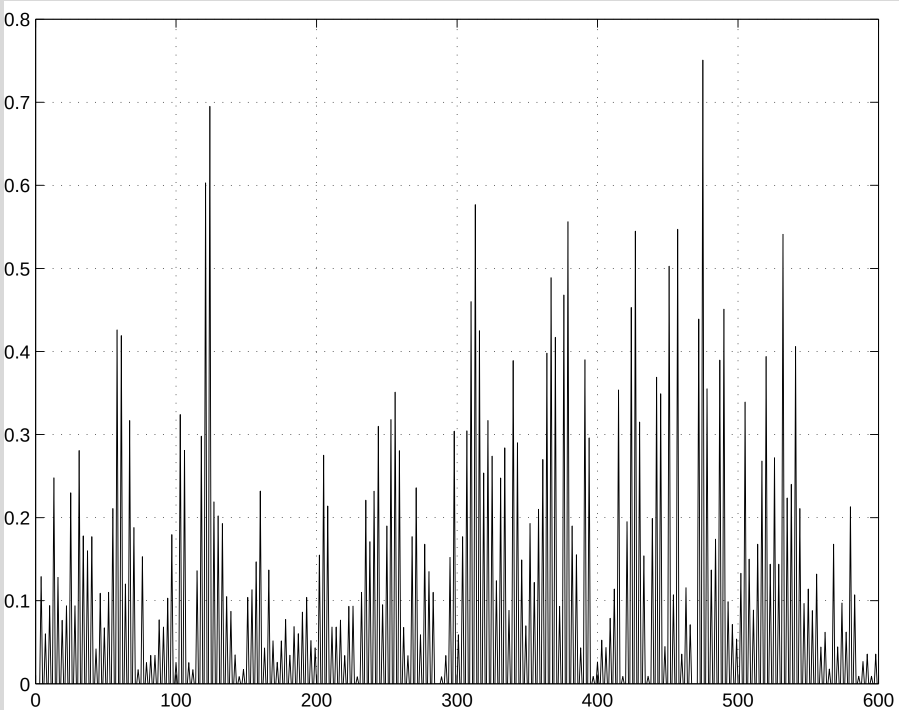
We will address this problem by looking at resolution lengths.

Intermittency may be the key to understanding the NSE :

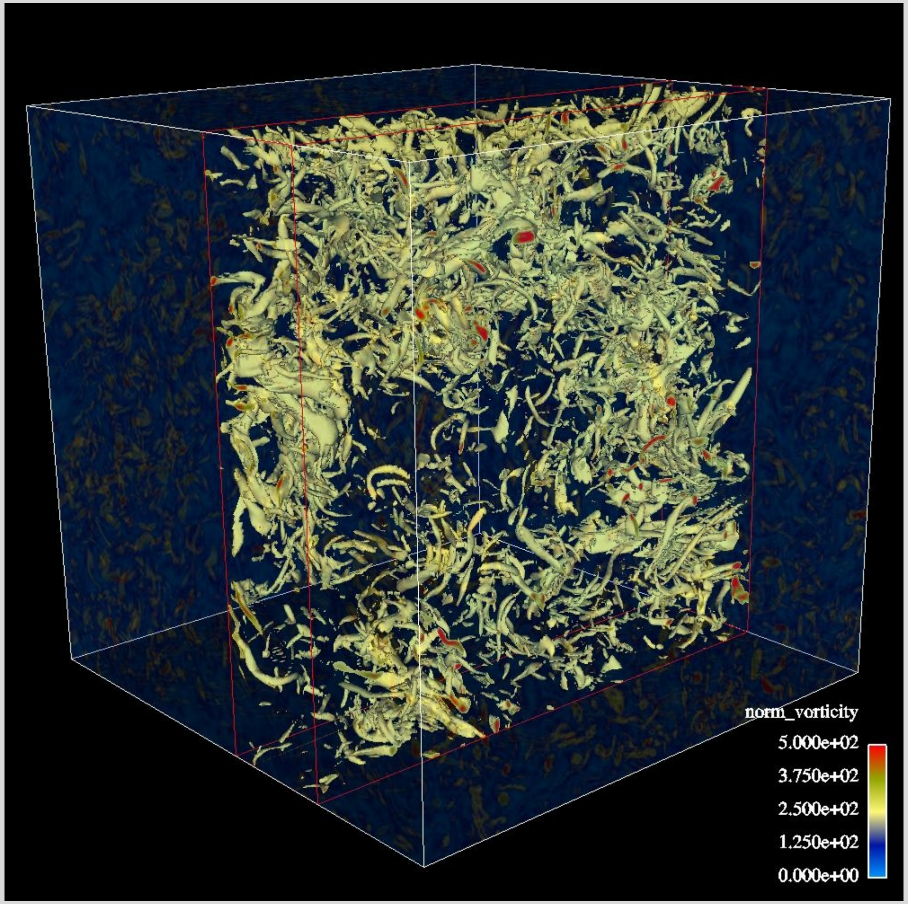
(a) Intermittent events are manifest as violent spiky surges away from space-time averages in both vorticity & strain. Spectra have non-Gaussian characteristics – see Batchelor & Townsend 1949.

(b) This raises the question : Are the spikes smooth down to some small scale or does vorticity cascade down to small scales where the NSE are invalid?

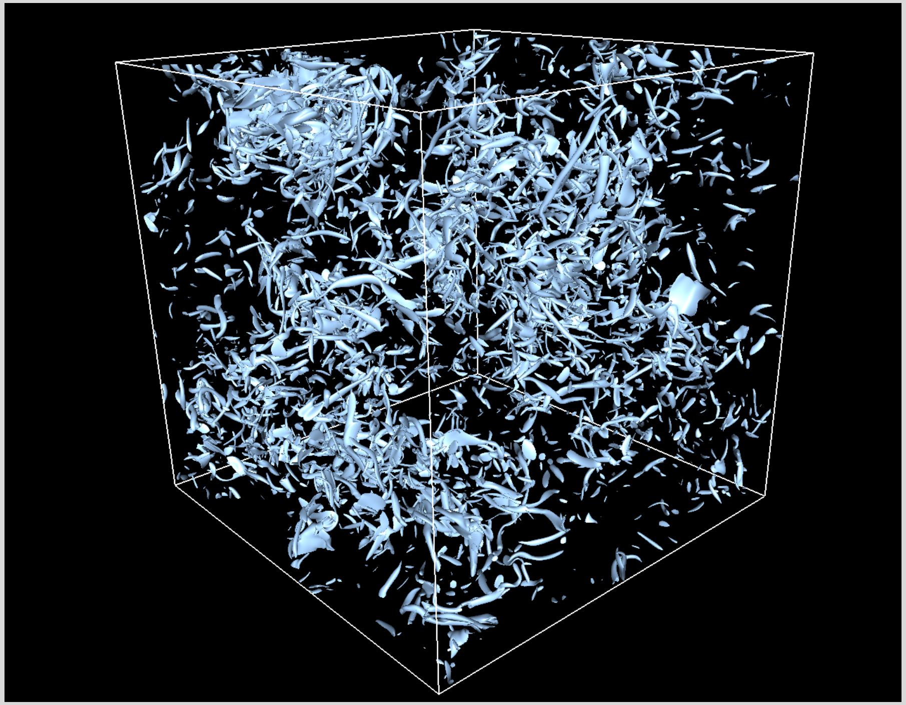
Later in this talk : some very new results on intermittency.



Dissipation-range intermittency from wind tunnel turbulence where hot wire anemometry has been used to measure the longitudinal velocity derivative at a single point (D. Hurst & J. C. Vassilicos). The horizontal axis spans 8 integral time scales with $Re_\lambda \sim 200$.



Vorticity iso-surfaces in a 512^3 sub-domain of the LANL decaying 2048^3 NS-simulation at $Re_\lambda \sim 200$. Uneven clustering results in **intermittency**; Batchelor & Townsend (1949); Kuo & Corrsin (1971), Sreenivasan & Meneveau (1988, 1991); Frisch (1995): courtesy of Darryl Holm.



A $3D$ statistically stationary homogeneous isotropic NS-flow at $Re_\lambda \sim 107$ showing iso-surfaces of ω^2 at $10 \times \omega_{av}^2$ in a $(2\pi)^3$ cube resolved with 2048^3 points : courtesy of Jörg Schumacher of TU Ilmenau.

1. Morphological changes from quasi-2D sheets $\rightarrow$ quasi-1D tubes is typical.
2. Why do the 3D-NS equations produce these topologically **thin sets** in the vorticity field? – Frisch & Orszag (1990), Karniadakis & Orszag (1993), Vincent & Meneguzzi (1994); Schumacher, Eckhardt & Doering (2010).

What do statistical physicists calculate in turbulence?

Based on Kolmogorov's axioms: see Uriel Frisch, *Turbulence: the legacy of A. N. Kolmogorov*, CUP, 1995 or Davidson, *Turbulence*, OUP (2004).

In K41 statistical theory the standard $-5/3$ inertial-range energy-spectrum has a cut-off appearing at $Lk_c \sim Re^{3/4}$: **the Kolmogorov length.**

The objects that are used to study intermittency are the **ensemble-averaged velocity structure functions**

$$\langle\, |u(\boldsymbol{x}+\boldsymbol{r}) - u(\boldsymbol{x})|^p \,\rangle_{ens.av.} \sim r^{\zeta_p}$$

Kolmogorov predicted a linear relation between ζ_p and p: the two coincide for $p = 3$. **Departure from this is called anomalous scaling & is usually manifest by ζ_p lying on a concave curve below linear for $p > 3$.**

Problem: the idea is not amenable to NS analysis.

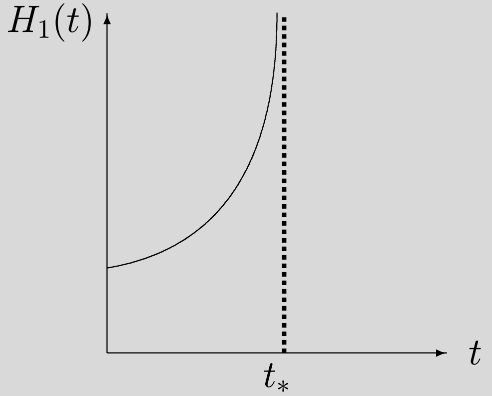
Conditional 3D-NS regularity : a very brief history

1. **Leray (1934)** ; Prodi (1959), Serrin (1963) & Ladyzhenskaya (1964) : every Leray-Hopf solution $\boldsymbol{u}$ of the $3D$-NSE with $\boldsymbol{u} \in L^r\left((0,\,T)\,;L^s\right)$ is regular on $(0,\,T]$ provided $2/r + 3/s = 1$ with $s \in (3,\,\infty]$ or if $\boldsymbol{u} \in L^\infty\left((0,\,T)\,;L^p\right)$ with $p > 3$.
2. For the case $s = 3$: von Wahl (1983) & Giga (1986) first proved the regularity in the space $C\left((0,\,T]\,;L^3\right)$: see also Kozono & Sohr (1997) & Escauriaza, Seregin & Sverák (2003).
3. Various regularity results involving the pressure or one velocity derivative : Kukavic & Ziane (2006, 2007), Zhou (2002), Cao & Titi (2008, 2010), Cao (2010), Cao, Qin & Titi (for channel flows) (2008), & Chen & Gala (2011), or on the direction of vorticity (Constantin & Fefferman (1993) & Vasseur (2008), or with the use of Besov spaces (see Cheskidov & Shvydkoy (2011).
4. Books by Constantin & Foias 1988 & Foias, Manley, Rosa & Temam 2001.

Physical assumptions corresponding to conditional regularity?

- $\|\boldsymbol{u}\|_p$ $(p \geq 3)$ assumed to be bounded. **Drawback :** no physical interpretation.
- Likewise, assumptions on bounds on the pressure or single derivatives of the velocity field have no physical interpretation.
- The global enstrophy $H_1 = \int_{\mathcal{V}} |\boldsymbol{\omega}|^2 dV$ is assumed to be bounded pointwise in time : **Drawback :** assumes the answer we're seeking. It also ignores intermittency, which is important.

- Short time regularity : the upper bound on H_1 blows up at t^*.



Use of higher moments of vorticity for the NS equations

How might we pick up intermittent behaviour? Consider higher moments of vorticity ($m \geq 1$) as the "frequencies"

$$\Omega_m(t) = \left(L^{-3}\int_{\mathcal{V}} |\boldsymbol{\omega}|^{2m} dV\right)^{1/2m} + \varpi_0$$

The basic frequency associated with the domain is given by $\varpi_0 = \nu L^{-2}$.

$$\Omega_1^2 = L^{-3}\int_{\mathcal{V}} |\boldsymbol{\omega}|^2 dV + \varpi_0 \qquad\qquad H_1\text{-norm}$$

is the enstrophy/unit volume which is related to the energy dissipation rate. **The higher moments will naturally pick up events at smaller scales**

$$\varpi_0 \leq \Omega_1(t) \leq \Omega_2(t) \leq \ldots \leq \Omega_m(t) \leq \Omega_{m+1}(t) \leq \ldots$$

Estimates in terms of Re

Traditionally, most NS-estimates have been found in terms of the Grashof number Gr ($f_{rms}^2 = L^{-3}\|\boldsymbol{f}\|_2^2$) of the divergence-free forcing $\boldsymbol{f}(\boldsymbol{x})$ but it would be more helpful to express these in terms of the Reynolds number Re to facilitate comparison with the results of statistical physics.

$$Gr = L^3 f_{rms}\nu^{-2}\,, \qquad\qquad Re = U_0 L\nu^{-1}\,.$$

Doering and Foias 2002 used the idea of defining U_0 as

$$U_0^2 = L^{-3}\left\langle\|\boldsymbol{u}\|_2^2\right\rangle_T$$

where the time average $\left\langle\,\cdot\,\right\rangle_T$ over an interval $[0,\,T]$ is defined by

$$\langle g(\cdot)\rangle_T = \limsup_{g(0)}\frac{1}{T}\int_0^T g(\tau)\,d\tau\,.$$

Gr is fixed provided $\boldsymbol{f}$ is L^2-bounded, while Re is the system response.

Leray's energy inequality shows that

$$\tfrac{1}{2}\frac{d}{dt}\int_{\mathcal{V}}|\boldsymbol{u}|^2\,dV \le -\nu\int_{\mathcal{V}}|\boldsymbol{\omega}|^2\,dV + \|\boldsymbol{f}\|_2\|\boldsymbol{u}\|_2\,,$$

$$\left<\Omega_1^2\right>_T \le \varpi_0^2 Gr\,Re + O\left(T^{-1}\right)\,.$$

Doering & Foias (2002) showed that NS-solutions obey $Gr \le c\,Re^2$ provided the spectrum of $\boldsymbol{f}$ is concentrated in a narrow-band around a single frequency or its spectrum is bounded above & below

$$\left<\Omega_1^2\right>_T \le c\,\varpi_0^2 Re^3 + O\left(T^{-1}\right)\,.$$

$\nu\left<\Omega_1^2\right>_T$ is the time-averaged energy dissipation rate over $[0,\,T]$ and the Kolmogorov length scale λ_k^{-1} is estimated as

$$\lambda_k^{-4} = \frac{\nu\left<\Omega_1^2\right>_T}{\nu^3} \qquad \Rightarrow \qquad L\lambda_k^{-1} \le c\,Re^{3/4} + O\big(T^{-1/4}\big)\,.$$

Weak solution result

Theorem 1 : *Weak solutions of the 3D-Navier-Stokes equations satisfy*

$$\left\langle \left(\varpi_0^{-1}\Omega_m\right)^{\alpha_m}\right\rangle_T \leq c\,Re^3 + O\left(T^{-1}\right), \qquad\qquad 1 \leq m \leq \infty\,,$$

where $\varpi_0 = \nu L^{-2}$*,* c *is a uniform constant and*

$$\alpha_m = \frac{2m}{4m-3}\,.$$

Remark : The exponent $\alpha_m = \frac{2m}{4m-3}$ appears to be a natural scaling, consistent with the application of Hölder & Sobolev inequalities.

Proof : (JDG 2011, CMS) The proof is based on a result of Foias, Guillopé and Temam (1983), *New a priori estimates for Navier-Stokes equations in Dimension 3*, Comm. Partial Diff. Equat., 6, 329–359, 1981 (their Theorem 3.1) for weak solutions.

When modified in the manner of Doering & Foias (02) the FGT result becomes

$$\left\langle H_N^{\frac{1}{2N-1}} \right\rangle_T \leq c_N L^{-1} \nu^{\frac{2}{2N-1}} Re^3 + O\left(T^{-1}\right) ,$$

where

$$H_N = \int_{\mathcal{V}} \left|\nabla^N \boldsymbol{u}\right|^2 dV = \int_{\mathcal{V}_k} k^{2N} \left|\hat{\boldsymbol{u}}\right|^2 d^3k ,$$

where $H_1 = \int_{\mathcal{V}} |\nabla \boldsymbol{u}|^2 \, dV = \int_{\mathcal{V}} |\boldsymbol{\omega}|^2 \, dV$. An interpolation between $\|\boldsymbol{\omega}\|_{2m}$ and $\|\boldsymbol{\omega}\|_2$ is found using H_N

$$\|\boldsymbol{\omega}\|_{2m} \leq c_{N,m} \|\nabla^{N-1}\boldsymbol{\omega}\|_2^a \, \|\boldsymbol{\omega}\|_2^{1-a} , \qquad\qquad a = \frac{3(m-1)}{2m(N-1)} ,$$

for $N \geq 3$. $\|\boldsymbol{\omega}\|_{2m}$ is now raised to the power α_m, to be determined.

In effect, we are translating from derivatives to L^{2m}-norms.

$$
\begin{aligned}
\left\langle \|\boldsymbol{\omega}\|_{2m}^{\alpha_m} \right\rangle_T &\leq c_{N,m}^{\alpha_m} \left\langle \|\nabla^{N-1}\boldsymbol{\omega}\|_2^{a\alpha_m} \|\boldsymbol{\omega}\|_2^{(1-a)\alpha_m} \right\rangle_T \\
&\leq c_{N,m}^{\alpha_m} \left\langle H_N^{\frac{1}{2N-1}} \right\rangle_T^{\frac{1}{2}a\alpha_m(2N-1)} \left\langle H_1^{\frac{(1-a)\alpha_m}{2-a\alpha_m(2N-1)}} \right\rangle_T^{1-\frac{1}{2}a\alpha_m(2N-1)}
\end{aligned}
$$

An explicit upper bound in terms of Re is available only if the exponent of H_1 within the average is unity ; that is

$$
\frac{(1-a)\alpha_m}{2-a\alpha_m(2N-1)} = 1\,.
$$

This determines α_m, uniformly in N, as

$$
\alpha_m = \frac{2m}{4m-3}\,.
$$

The constant $c_{N,m}$ can be minimized by choosing $N=3$ which is finite even when $m=\infty$; thus we take the largest value of $c_{3,m}^{\alpha_m}$ and call this c. ∎

A continuum of length scales

Based on the definition of the inverse Kolmogorov length λ_k^{-1}, **a generalization of this to a hierarchy of inverse lengths λ_m^{-1} suggests :**

$$\left(L\lambda_m^{-1}\right)^{2\alpha_m} := \left\langle\left(\varpi_0^{-1}\Omega_m\right)^{\alpha_m}\right\rangle_T$$

with $\alpha_m = \frac{2m}{4m-3}$ and where $\varpi_0 = \nu L^{-2}$:

$$L\lambda_m^{-1} \leq c\, Re^{3/2\alpha_m} + O\left(T^{-1/2\alpha_m}\right) \qquad 1 \leq m \leq \infty$$

m	1	9/8	3/2	2	3	...	∞
$3/2\alpha_m$	3/4	1	3/2	15/8	9/4	...	3

Values of the Re-exponent $3/2\alpha_m = 3\left(1 - \frac{3}{4m}\right)$.

1. For $m > 1$ the λ_m are interpreted here as the length scales corresponding to ever deep intermittent events.

2. Computationally it is hard to get beyond $m = 1$. $m = 9/8$ (corresponding to Re^1) is close to modern resolutions. **Two alternative views : either**

   - Flow resolution difficulties could be a symptom of the lack of uniqueness of weak solutions :

   - Or, these difficulties may simply be caused by the practical challenges of computing a system where even the naturally largest scale (other than L) lies close to the limit of what can currently be resolved.

3. As $m \to \infty$, the Re^3 bound has an exponent $4\times$ greater than the Kolmogorov length ; this lies below molecular scales where the NSE are invalid.

Another look at conditional regularity: arXiv/1108.4651 [nlin.CD]

Lemma 1: *With* $1 \leq m < \infty$, *and define* $\alpha_m = \frac{2m}{4m-3}$; $\beta_m = \frac{4}{3}m(m+1)$ *and* $n = \frac{1}{2}(m+1)$, $\Omega_m(t)$ *formally satisfies*

$$\dot{\Omega}_m \leq \varpi_0 \Omega_m \left\{ -\frac{1}{c_{1,m}} \left(\frac{\Omega_{m+1}}{\Omega_m} \right)^{\beta_m} + c_{2,m} \left(\varpi_0^{-1} \Omega_n \right)^{2\alpha_n} + c_{3,m} Gr \right\} \qquad (*)$$

The proof requires just two remarks: see JDG (2010) [?]

(i) In the Laplacian term, need to show that ($A_m = \omega^m$)

$$\frac{d}{dt} \int_{\mathcal{V}} |\boldsymbol{\omega}|^{2m} \, dV \leq -c_m \int_{\mathcal{V}} |\nabla A_m|^2 \, dV$$

plus a Sobolev inequality $\|A_m\|_{\frac{2(m+1)}{m}} \leq c_m \, \|\nabla A_m\|_2^{3/2(m+1)} \|A_m\|_2^{(2m-1)/2(m+1)}$

(ii) For the nonlinear term a Hölder inequality gives ($n = \frac{1}{2}(m+1)$)

$$\frac{d}{dt} \int_{\mathcal{V}} |\boldsymbol{\omega}|^{2m} \, dV \leq \Omega_{m+1}^{2m} \|\nabla \boldsymbol{u}\|_{m+1} \leq c_m \, \Omega_{m+1}^{2m} \, \Omega_n$$

$$y_m = \left(\varpi_0^{-1}\Omega_m\right)^{-2\alpha_n} \qquad\qquad F_m = \left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{\beta_m} - c_{1,m}c_{3,m}Gr$$

Using the fact that $\Omega_n \leq \Omega_m$, (*) linearizes to $(\tau = 2\varpi_0\alpha_n c_{1,m}^{-1}t)$

$$\frac{dy_m}{d\tau} \geq F_m\, y_m - c_{2,m}\,,$$

and integrates to

$$\left\{\varpi_0^{-1}\Omega_m(\tau)\right\}^{2\alpha_n} \leq \frac{\exp\left(-\int_0^\tau F_m d\xi\right)}{y_{m,0} - c_{2,m}\int_0^\tau \exp\left(-\int_0^\xi F_m d\xi'\right)\, d\xi} \qquad (**)$$

where the initial value $y_{m,0} = y_m(0)$.

1. Can the denominator develop a zero in a finite time?
2. Control of $\Omega_m(\tau)$ from above for **any** $m \geq 1$ will control the H_1-norm.

Lower bounds on the dissipation

Choose a set of parameters μ_m such that with $\alpha_m = \frac{2m}{4m-3}$ & $\beta_m = \frac{4}{3}m(m+1)$

$$\alpha_m\left(\frac{1-\mu_m}{\mu_m}\right) = \beta_m\,, \qquad \Rightarrow \qquad \mu_m = \frac{3}{(2m-1)(4m+3)}$$

Lemma 2 :

α_m	$\frac{2m}{4m-3}$
$\alpha_m - \alpha_{m+1}$	$\frac{6}{(4m+1)(4m-3)}$
β_m	$\frac{4}{3}m(m+1)$
μ_m	$\frac{3}{(2m-1)(4m+3)}$
$\frac{1-\mu_m}{\mu_m} = \frac{\beta_m}{\alpha_m}$	$\frac{2}{3}(m+1)(4m-3)$
$\frac{\alpha_m}{\alpha_{m+1}}$	$\frac{1+\mu_m}{1-\mu_m}$
$\left(\frac{\alpha_m}{\alpha_{m+1}}-1\right)\left(\frac{1-\mu_m}{\mu_m}\right)$	2

$$\int_0^\tau \left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{\beta_m} d\xi \geq \tau^{-1}\left(\frac{\left(\int_0^\tau \left(\varpi_0^{-1}\Omega_{m+1}\right)^{\alpha_{m+1}} d\xi\right)^{\frac{\alpha_m}{\alpha_{m+1}}}}{\int_0^\tau \left(\varpi_0^{-1}\Omega_m\right)^{\alpha_m} d\xi}\right)^{\beta_m/\alpha_m}.$$

Corollary : *Using the average notation* $\langle\cdot\rangle_{(\tau)} = \frac{1}{\tau}\int_0^\tau \cdot\, d\xi$, *Lemma 2 can be re-written as*

$$\left\langle \left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{\beta_m} \right\rangle_{(\tau)} \geq \left(\frac{\left\langle \left(\varpi_0^{-1}\Omega_{m+1}\right)^{\alpha_{m+1}} \right\rangle_{(\tau)}^{\frac{\alpha_m}{\alpha_{m+1}}}}{\left\langle \left(\varpi_0^{-1}\Omega_m\right)^{\alpha_m} \right\rangle_{(\tau)}} \right)^{\frac{\beta_m}{\alpha_m}} .$$

Proof of Lemma 2 : Clearly μ_m lies in the range $0 < \mu_m < 1$ so a Hölder inequality gives

$$\begin{aligned} \int_0^\tau \Omega_{m+1}^{\alpha_m(1-\mu_m)} d\xi &= \int_0^\tau \left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{\alpha_m(1-\mu_m)} \Omega_m^{\alpha_m(1-\mu_m)} d\xi \\ &\leq \left(\int_0^\tau \left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{\alpha_m\left(\frac{1-\mu_m}{\mu_m}\right)} d\xi \right)^{\mu_m} \left(\int_0^\tau \Omega_m^{\alpha_m} d\xi \right)^{1-\mu_m} \end{aligned}$$

thus leading to

$$\int_0^\tau \left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{\beta_m} d\xi \geq \left(\frac{\int_0^\tau \Omega_{m+1}^{\alpha_m(1-\mu_m)} d\xi}{\left(\int_0^\tau \Omega_m^{\alpha_m} d\xi\right)^{1-\mu_m}}\right)^{1/\mu_m} .$$

It is easily checked that

$$\begin{aligned}\alpha_m(1-\mu_m) = \mu_m\beta_m &= \frac{4m(m+1)}{8m^2+2m-3}\\ &> \frac{2(m+1)}{4m+1} = \alpha_{m+1}\end{aligned}$$

so a further Hölder inequality on the numerator of gives

$$\int_0^\tau \Omega_{m+1}^{\alpha_{m+1}} d\xi \leq \tau^{\frac{\mu_m}{1+\mu_m}} \left(\int_0^\tau \Omega_{m+1}^{\alpha_m(1-\mu_m)} d\xi\right)^{\frac{1}{1+\mu_m}} .$$

The following theorem formally expresses the main result of the paper :

Theorem 2 : *If there exists a value of m lying in the range $1 \leq m < \infty$, with initial data $\Omega_{m,0} < C_m \varpi_0 Gr^{\Delta_m/2\alpha_n}$, for which the integral lies on or above the critical value*

$$c\,\left(\tau\, Gr^{2\delta_{m+1}} + \eta_2\right) \leq \int_0^\tau \left(\varpi_0^{-1}\Omega_{m+1}\right)^{\alpha_{m+1}} d\xi\,,$$

where $\eta_2 \geq \eta_1 Gr^{2(\delta_{m+1}-1)}$, and where δ_{m+1} lies in the range

$$\frac{\alpha_{m+1}}{\alpha_m}\left(1 + \frac{\alpha_m}{2\beta_m}\right) < \delta_{m+1} < 1\,,$$

then $\Omega_m(\tau)$ remains bounded for all time.

Proof : The lower bound on the dissipation is

$$\int_0^\tau \left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{\beta_m} d\xi \;\geq\; c_m\tau^{-1}\left\{\frac{\left(\tau Gr^{2\delta_{m+1}} + \eta_1 Gr^{2(\delta_{m+1}-1)}\right)^{\alpha_m/\alpha_{m+1}}}{\left(\tau Gr^2 + \eta_1\right)}\right\}^{\beta_m/\alpha_m}$$

$$\geq\; c_m\tau^{\left(\frac{\alpha_m}{\alpha_{m+1}}-1\right)\frac{\beta_m}{\alpha_m}-1} Gr^{2\left\{\delta_{m+1}\frac{\alpha_m}{\alpha_{m+1}}-1\right\}\frac{\beta_m}{\alpha_m}}$$

$$=\; c_m\,\tau\,Gr^{\Delta_m}$$

where

$$\Delta_m = 2\left\{\delta_{m+1}\frac{\alpha_m}{\alpha_{m+1}} - 1\right\}\frac{\beta_m}{\alpha_m}\,.$$

Therefore

$$\int_0^\tau F_m d\xi \geq \left\{c_m Gr^{\Delta_m} - c_3 Gr\right\}\tau\,.$$

To have the dissipation greater than forcing ($\Delta_m > 1$) raises the lower bound on δ_{m+1} away from $\frac{\alpha_{m+1}}{\alpha_m}$.

$$\int_0^\tau \exp\left(-\int_0^\xi F_m d\xi'\right) d\xi \;\leq\; Gr^{-\Delta_m}\left[1-\exp\left(-\tau Gr^{\Delta_m}\right)\right] \;,$$

and so the denominator of (**) satisfies

$$\text{Denominator of (**)} \geq y_{m,0} - c_{1,m}c_{2,m}Gr^{-\Delta_m}\left(1-e^{-\tau Gr^{\Delta_m}}\right)$$

This can never go negative if $y_{m,0} > c_{1,m}c_{2,m}Gr^{-\Delta_m}$, which means large initial data is restricted by

$$\left(\varpi_0^{-1}\Omega_{m,0}\right)^{2\alpha_n} < C_m Gr^{\Delta_m}$$

where $1 < \Delta_m \leq 4$. ■

Intermittency

1. A feature of intermittent flows lies in the strong excursions of the vorticity away from average with periods of inactivity between the spikes. How does the critical lower bound imposed as an assumption in Theorem 2 lead to this?

2. If $\left\langle \Omega_{m+1}^{\alpha_{m+1}} \right\rangle_{(\tau)}$ lies above critical then not only Ω_m cannot blow up but it actually collapses exponentially. Curiously, & counter-intuitively, if the value of this integral drops below critical then the occurrence of a singular event must still formally be considered.

3. Experimentally, signals go through cycles of growth/collapse : **thus it is not realistic to expect the critical lower bound to hold for τ.**

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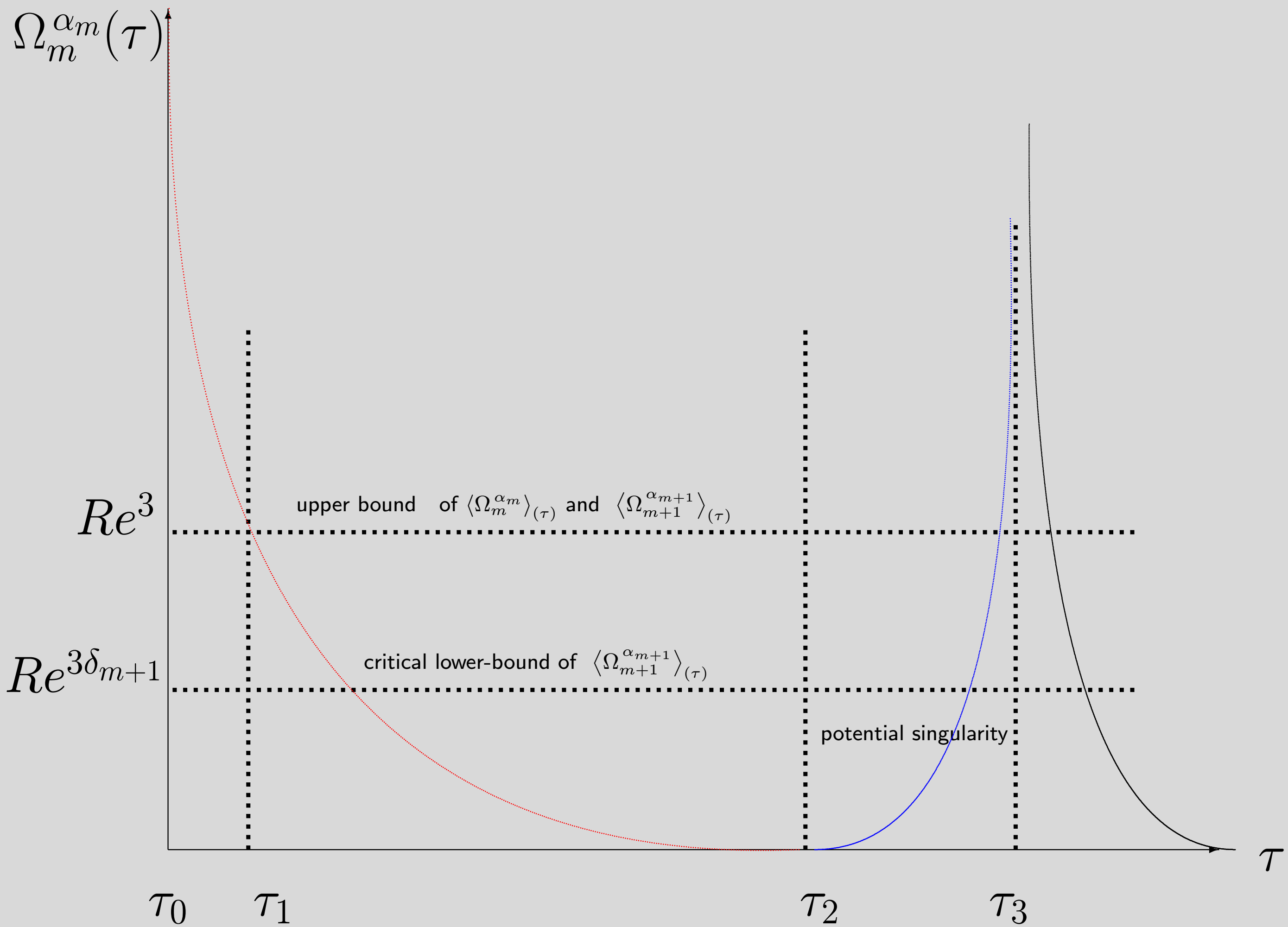
$\Omega_m^{\alpha_m}(\tau)$
Re^3
upper bound of $\langle \Omega_m^{\alpha_m} \rangle_{(\tau)}$ and $\langle \Omega_{m+1}^{\alpha_{m+1}} \rangle_{(\tau)}$
$Re^{3\delta_{m+1}}$
critical lower-bound of $\langle \Omega_{m+1}^{\alpha_{m+1}} \rangle_{(\tau)}$
potential singularity
τ
τ_0
τ_1
τ_2
τ_3

Sequence of events

1. For $\tau_0 \leq \tau \leq \tau_1$, if $\left\langle \Omega_{m+1}^{\alpha_{m+1}} \right\rangle_{(\tau)}$ lies above the critical lower bound then $\Omega_m(\tau)$ collapses. The H_1-norm (Ω_1^2) (which controls the regularity of all variables) is suppressed. In turn, $\Omega_{m+1}(\tau)$ collapses.
2. Because of the collapse in the point-wise value of $\Omega_{m+1}(\tau)$, the magnitude of $\left\langle \Omega_{m+1}^{\alpha_{m+1}} \right\rangle_{(\tau)}$ in the region $\tau_1 \leq \tau \leq \tau_2$ decays but remains above the critical lower bound for a period of time, thus leading to both $\Omega_m(\tau)$ and $\Omega_{m+1}(\tau)$ remaining small.
3. In the region $\tau_2 \leq \tau \leq \tau_3$ the continuing smallness of $\Omega_{m+1}(\tau)$ finally causes $\left\langle \Omega_{m+1}^{\alpha_{m+1}} \right\rangle_{(\tau)}$ to drop below the critical lower bound, in which case $\Omega_m(\tau)$ is free to grow. At this point there are two options :
   (a) Rapid growth in Ω_m (and thus in Ω_{m+1}) leads to $\left\langle \Omega_{m+1}^{\alpha_{m+1}} \right\rangle_{(\tau)}$ rising back through the critical lower bound leading to a renewed cycle of collapse

at τ_3: this requires a re-setting and a conformity of the initial conditions at $\tau = \tau_3$. The dynamics thus behave like a relaxation oscillator.

(b) There is still the possibility that pointwise growth in Ω_{m+1}, but without significant growth in $\left\langle \Omega_{m+1}^{\alpha_{m+1}} \right\rangle_{(\tau)}$, could lead to the formation of an integrable singularity. This would restrict any singular event to $\Omega_{m+1}^{\alpha_{m+1}} \sim (t_0 - t)^{-p}$ for $0 < p < 1$.

While option 3a) is the most attractive, the possibility of singularity formation implied by option 3b) cannot formally be ruled out. The intriguing *mathematical* question remains concerning what happens to solutions when $\left\langle \Omega_{m+1}^{\alpha_{m+1}} \right\rangle_{(\tau)}$ drops below critical. Can regularity be proved in this case?

Cascade?? (JDG – arXiv/1108.4651, 2011)

It is always true that $\Omega_m \leq \Omega_{m+1}$ but $\alpha_m > \alpha_{m+1}$.

$$\Omega_m^{\alpha_m} \lessgtr \Omega_{m+1}^{\alpha_{m+1}} \qquad\qquad ??$$

Now use our definition of a length scale

$$\left(L\lambda_m^{-1}\right)^{2\alpha_m} = \left\langle\left(\varpi_0^{-1}\Omega_m\right)^{\alpha_m}\right\rangle_{(\tau)}$$

although $\lambda_m \lessgtr \lambda_{m+1}$?? Our lower bound becomes

$$\left\langle\left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{\beta_m}\right\rangle_{(\tau)} \geq \left(\frac{\lambda_m}{\lambda_{m+1}}\right)^{2\beta_m} .$$

We could enforce the assumption that there exists an ordered cascade of length scales $\lambda_m > \lambda_{m+1}$, decreasing sufficiently fast with m to take advantage of the quadratic nature of β_m.

References

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