Noether's theorem for ideal fluids

Darryl D Holm
Imperial College London

In collaboration with Colin J Cotter

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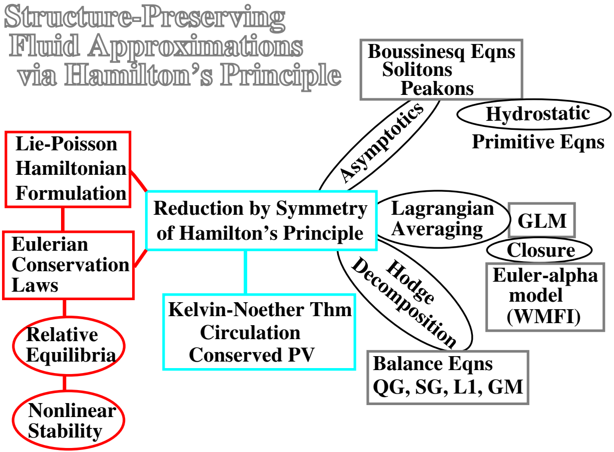
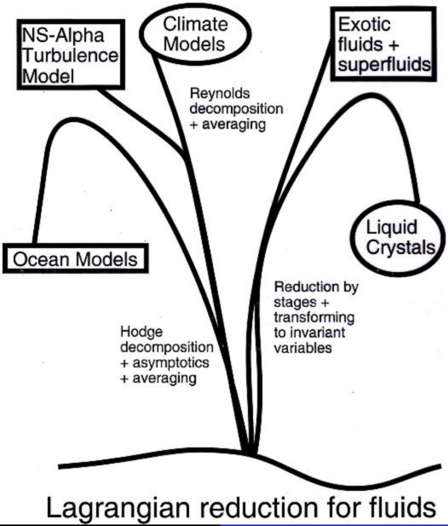
Abstract

We show how Noether conservation laws of ideal fluids with advected quantities can be obtained from flows of Eulerian vector fields for Lie symmetries.

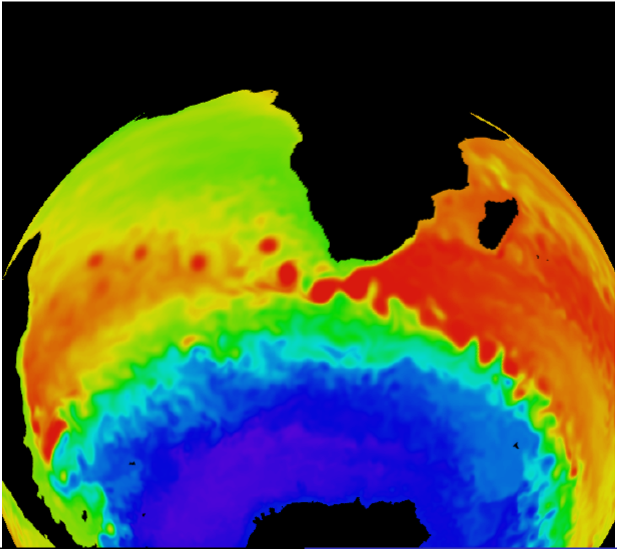
Hamilton's principle gives a framework for ideal fluids



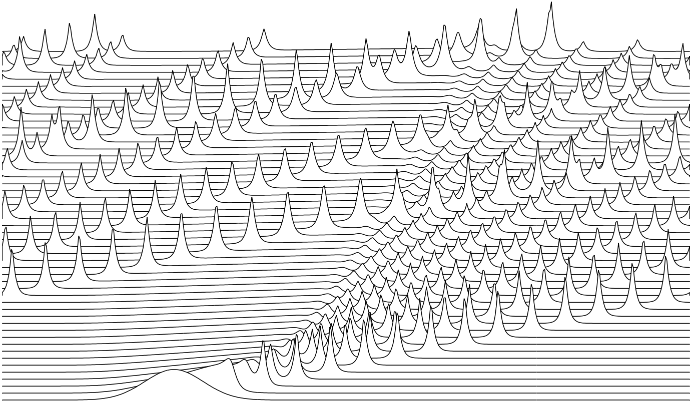
This framework has many applications



For example, in ocean circulation . . .



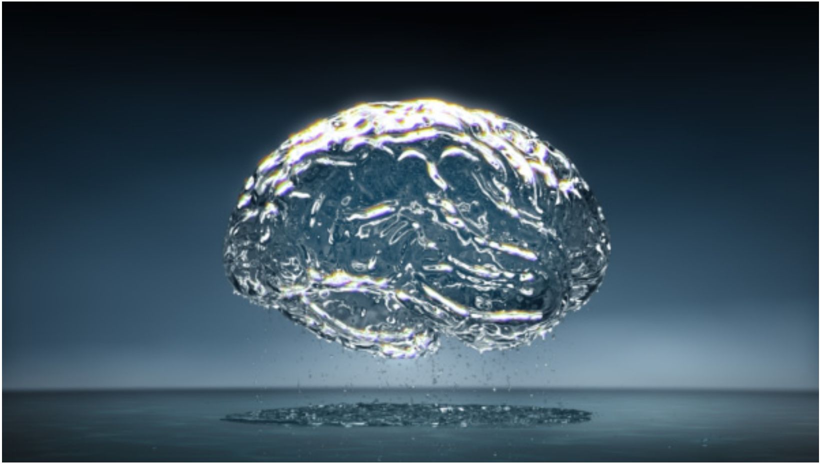
The applications also include solitons



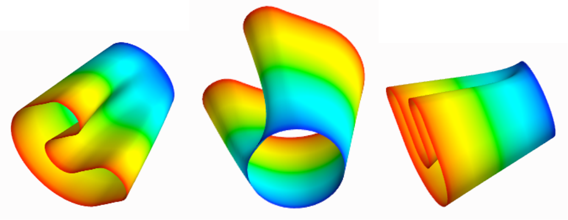
- R Camassa, DDH,

  An integrable shallow water equation with peaked solitons. Phys Rev Lett 71: 1661–1664 (1993).
- DDH, M Staley, *Interaction dynamics of singular wave fronts*. See M Staley, webpage.

Fluid flow is a sort of shape morphing . . .



so these applications even extend to Image Matching



- CJ Cotter & DDH, *Geodesic boundary value problems with symmetry*, J Geom Mech 2:1, 417–444 (2010)
- CJ Cotter & DDH, *Continuous and discrete Clebsch variational principles*, FoCM, 9:2, 221–242, (2009)
- CJ Cotter, *The variational particle-mesh method for matching curves*, J Phys A, 41:34, 340301–18 (2008)

. . . since image analysis relates to the flow of shape!

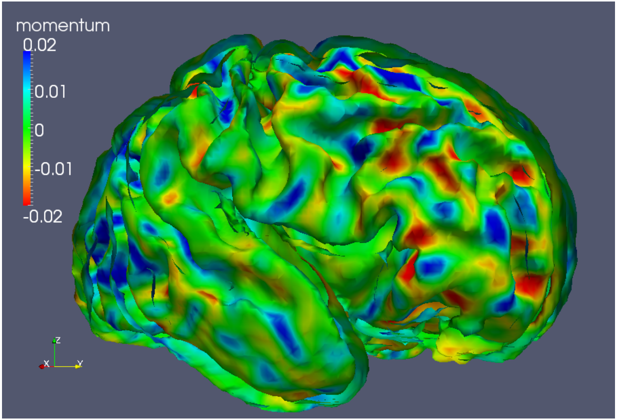


Figure: A segmented brain image from our 3D diffeomorphic shooting algorithm. The colours represent the initial momentum, which is a scalar valued function on the surface, associated with the registration problem.

Key points of the lecture

Point #1:

Ideal fluid equations follow from Hamilton's principle

$$\delta S = 0 \quad \text{with} \quad S = \int \ell(u, a)dt$$

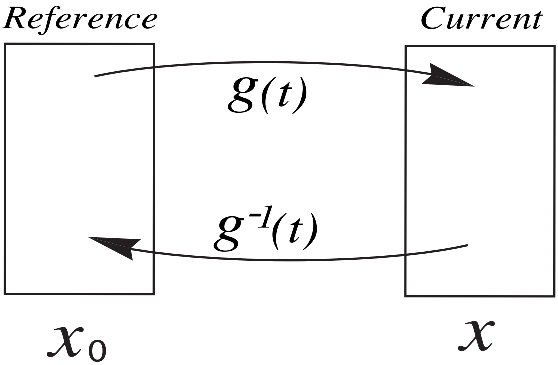
Point #2:

The geometric approach reveals the symmetry vector fields responsible for the conservation laws for ideal fluids.

Point #3:

The result is Noether's theorem for ideal fluids in Eulerian form.

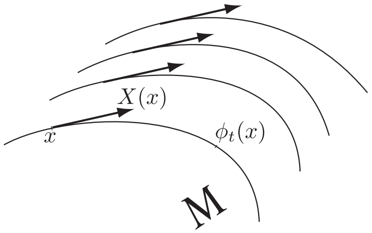
Fluid flows & Lie symmetries are related . . .



. . . because fluid flows are flows of vector fields

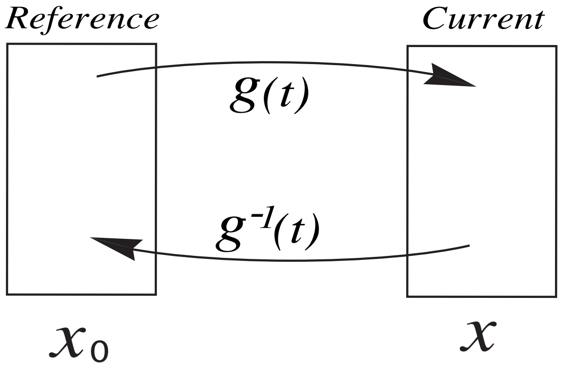
Flows of vector fields arise,
whenever a Lie group *G* acts on a manifold *M*.

Thus, fluid flows & Lie symmetry are related.



Thus, fluid flows are Lie group actions on manifolds

Fluid flows for the Lie group action $G = \mathrm{Diff}(\mathbb{R}^3)$



Lie group actions summon differential-form operations

- Exterior derivative d *raises* the degree of a k-form:

$$d\Lambda^k \mapsto \Lambda^{k+1}\,.$$

- Contraction $\lrcorner$ with a vector field $X \in \mathfrak{X}$ *lowers* the degree:

$$X \lrcorner \Lambda^k \mapsto \Lambda^{k-1}\,.$$

- Lie derivative $\mathcal{L}_X$ by vector field X *preserves* the degree:

$$\mathcal{L}_X\Lambda^k \mapsto \Lambda^k\,, \quad \text{where} \quad \mathcal{L}_X\Lambda^k := \frac{d}{dt}\Big|_{t=0}\phi_t^*\Lambda^k\,,$$

  in which ϕ_t is the flow of the vector field X.

- Lie derivative $\mathcal{L}_X$ satisfies Cartan's formula:

$$\mathcal{L}_X\alpha = X \lrcorner d\alpha + d(X \lrcorner \alpha) \quad \text{for} \quad \alpha \in \Lambda^k\,.$$

Differential & contraction relations in vector notation

Exterior derivative

$$
\begin{aligned}
\mathrm{d}f &= \nabla f \cdot \mathrm{d}\mathbf{x}\,, \\
\mathrm{d}(\mathbf{v}\cdot \mathrm{d}\mathbf{x}) &= (\mathrm{curl}\,\mathbf{v})\cdot \mathrm{d}\mathbf{S}\,, \\
\mathrm{d}(\mathbf{A}\cdot \mathrm{d}\mathbf{S}) &= (\mathrm{div}\,\mathbf{A})\,\mathrm{d}V\,.
\end{aligned}
$$

$$
\begin{aligned}
0 = \mathrm{d}^2 f = \mathrm{d}(\nabla f\cdot \mathrm{d}\mathbf{x}) &= (\mathrm{curl}\,\mathrm{grad}\,f)\cdot \mathrm{d}\mathbf{S}\,, \\
0 = \mathrm{d}^2(\mathbf{v}\cdot \mathrm{d}\mathbf{x}) = \mathrm{d}\big((\mathrm{curl}\,\mathbf{v})\cdot \mathrm{d}\mathbf{S}\big) &= (\mathrm{div}\,\mathrm{curl}\,\mathbf{v})\,\mathrm{d}V\,.
\end{aligned}
$$

Contraction with $X = \mathbf{X}\cdot\nabla$

$$
\begin{aligned}
X \,\lrcorner\, \mathbf{v}\cdot \mathrm{d}\mathbf{x} &= \mathbf{v}\cdot\mathbf{X}\,, \\
X \,\lrcorner\, \mathbf{B}\cdot \mathrm{d}\mathbf{S} &= -\mathbf{X}\times\mathbf{B}\cdot \mathrm{d}\mathbf{x}\,, \\
X \,\lrcorner\, \mathrm{d}V &= \mathbf{X}\cdot \mathrm{d}\mathbf{S}\,, \\
\mathrm{d}(X \,\lrcorner\, \mathrm{d}V) &= \mathrm{d}(\mathbf{X}\cdot \mathrm{d}\mathbf{S}) = (\mathrm{div}\mathbf{X})\,\mathrm{d}V\,.
\end{aligned}
$$

Lie derivative relations in vector notation

(a) $\mathcal{L}_X f = X \,\lrcorner\, \mathrm{d}f = \mathbf{X}\cdot\nabla f\,,$

(b) $\mathcal{L}_X\,(\mathbf{v}\cdot \mathrm{d}\mathbf{x}) = \big(-\ \mathbf{X}\times\mathrm{curl}\,\mathbf{v} + \nabla(\mathbf{X}\cdot\mathbf{v})\big)\cdot \mathrm{d}\mathbf{x}\,,$

(c) $\mathcal{L}_X(\omega\cdot \mathrm{d}\mathbf{S}) = \big(-\ \mathrm{curl}\,(\mathbf{X}\times\omega) + \mathbf{X}\,\mathrm{div}\,\omega\big)\cdot \mathrm{d}\mathbf{S}\,,$

(d) $\mathcal{L}_X(f\ \mathrm{d}V) = (\mathrm{div}\,f\mathbf{X})\ \mathrm{d}V\,,$

(e) For vector fields X and Y

$$\mathcal{L}_X Y = [X,\ Y] := \big(\mathbf{X}\cdot\nabla\mathbf{Y} - \mathbf{Y}\cdot\nabla\mathbf{X}\big)\cdot\nabla =: \mathrm{ad}_X Y,$$

where $[X,\ Y] = \mathrm{ad}_X Y$ is the commutator of X and $Y \in \mathfrak{X}$.

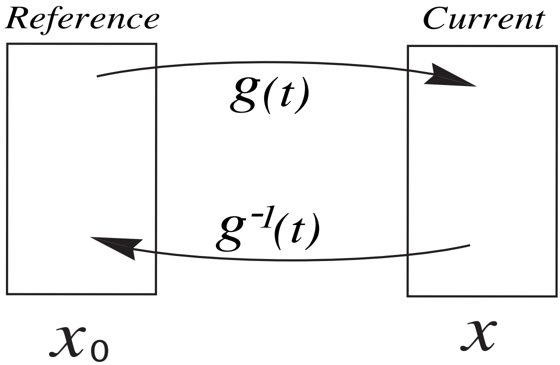
(f) For a 1-form density $m = \mathbf{m}\cdot \mathrm{d}\mathbf{x}\otimes \mathrm{d}V \in \mathfrak{X}^*$

$$\mathcal{L}_X\, m = \big(\nabla\cdot(\mathbf{X}\otimes\mathbf{m}) + (\nabla\mathbf{X})^T\cdot\mathbf{m}\big)\cdot \mathrm{d}\mathbf{x}\otimes \mathrm{d}V =: \mathrm{ad}^*_X\, m\,.$$

(g) Pairing $\mathfrak{X}^*\times\mathfrak{X}\to\mathbb{R}$ with $\langle\, m\,,\ u\,\rangle := \int_\Omega u \,\lrcorner\, m\ \mathrm{d}V$

Fluid flows arise as Lie group actions on manifolds

Fluid flows for the Lie group action $G = \mathrm{Diff}(\mathbb{R}^3)$



Fluid flows of the Lie group action $G = \mathrm{Diff}(\mathbb{R}^3)$

- For fluids, the Lie group $G = \mathrm{Diff}(\mathbb{R}^3)$ is the group of smooth invertible maps of $\mathbb{R}^3$ with smooth inverses.
- At time t, the mapping $g(t)$ takes the *label space* to the physical domain so that the path

$$x(t) = g(t)x_0, \text{ with velocity } \dot{x}(t) = \dot{g}g(t)^{-1}x(t) =: u(x,t)$$

describes Lagrangian particle trajectories for each label x_0.

- Recalling $\frac{d}{dt}(g(t)^{-1}) = -g^{-1}\dot{g}g^{-1}$, the solution

$$a(t) = a_0 g(t)^{-1} \quad \text{of} \quad \dot{a}(t) = -au(t) = -\mathcal{L}_u a$$

is called an advected quantity for fluids, with, e.g., $a \in \Lambda^k$.

- Under the *relabelling transformation* $g \to gh$ for any $h \in G$

$$a(t) \to (a_0 h^{-1})g(t)^{-1} \text{ and } u(t) := \dot{g}g(t)^{-1} \in \mathfrak{X}(\mathbb{R}^3)$$

that is, the spatial fluid velocity vector field $u(t)$ is right-invariant.

Hamilton's principle

$$0 = \delta S = \delta \int_0^T \int_\Omega l(u, a)\, \mathrm{d}V\, \mathrm{d}t$$

and Noether's Theorem

Hamilton's principle and Noether's Theorem

Hamilton's principle for Lagrangian $l(u, a)$ in *spatial* variables is

$$0 = \delta S = \delta \int_0^T \int_\Omega l(u, a)\, \mathrm{d}V\, \mathrm{d}t = \int_0^T \int_\Omega \left(\frac{\delta l}{\delta u} \cdot \delta u + \frac{\delta l}{\delta a}\, \delta a \right) \mathrm{d}V\, \mathrm{d}t,$$

where the variations $\delta u = u'(\epsilon, t)|_{\epsilon=0}$ and $\delta a = a'(\epsilon, t)|_{\epsilon=0}$ are given via the vector field $w = \delta g g^{-1} = g' g^{-1}(\epsilon, t)|_{\epsilon=0}$ as

$$\delta u - \dot{w} = [u, w] =: \mathrm{ad}_u w \,, \quad \delta a = -\mathcal{L}_w a \,.$$

Here a denotes any quantity that is advected with the flow, and

$$[u, w] = u \cdot \nabla w - w \cdot \nabla u$$

is the commutator of vector fields u & w, and finally

$$-\mathcal{L}_w a(t) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} a_0 g^{-1}(\epsilon, t) = -a_0 \Big[g^{-1}(\epsilon, t) g' g^{-1}(\epsilon, t) \Big]_{\epsilon=0}$$

denotes the derivative of a along the flow $g(\epsilon, t)$ of w.

Hamilton's principle implies the ideal fluid equations

Hamilton's principle for $l(u, a)$ in *spatial* variables yields

$$0 = -\int_0^T \int_\Omega \left(\frac{\partial}{\partial t} \frac{\delta l}{\delta u} + \mathrm{ad}_u^* \frac{\delta l}{\delta u} - \frac{\delta l}{\delta a} \diamond a \right) \cdot w \, dV + \left[\int_\Omega \frac{\delta l}{\delta u} \cdot w \, dV \right]_0^T .$$

Here ad_u^* is the dual operator to ad_u defined for a vector field w,

$$\int_\Omega w \cdot \mathrm{ad}_u^* m \, dV = \int_\Omega m \cdot \mathrm{ad}_u w \, dV ,$$

ad_u^* has an explicit formula, given for $m = \delta l / \delta u$ as

$$\mathrm{ad}_u^* m = \nabla \cdot (u \otimes m) + (\nabla u)^T \cdot m = \mathcal{L}_u m ,$$

and the diamond operation $\diamond$ is defined by

$$\int_\Omega \left(\frac{\delta l}{\delta a} \diamond a \right) \cdot w \, dV := \int_\Omega \frac{\delta l}{\delta a} \cdot (-\mathcal{L}_w a) \, dV .$$

Noether's Theorem for EP with advected quantities

A vector field η is a symmetry of Hamilton's principle if it obeys

$$\delta u = \dot\eta + [u,\eta] = 0 \quad \text{and} \quad \delta a = -\mathcal{L}_\eta a = 0\,.$$

Hamilton's principle for $l(u,a)$ in *spatial* variables then yields

$$0 = -\int_0^T\int_\Omega \underbrace{\left(\frac{\partial}{\partial t}\frac{\delta l}{\delta u} + \mathrm{ad}^*_u\frac{\delta l}{\delta u} - \frac{\delta l}{\delta a}\diamond a\right)}_{=\,0\ \text{(EP equation)}}\cdot\eta\,\mathrm{d}V + \left[\int_\Omega\frac{\delta l}{\delta u}\cdot\eta\,\mathrm{d}V\right]_0^T.$$

Theorem (Noether theorem for EP with advected quantities)

For solutions of the EP equation, each symmetry vector field η of the EP Lagrangian yields an integral of the motion satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\delta l}{\delta u}\cdot\eta\,\mathrm{d}V = 0\,. \qquad (1)$$

Kelvin's circulation theorem

The Euler-Poincaré equation

$$\left(\frac{\partial}{\partial t}+\mathcal{L}_u\right)\frac{\delta l}{\delta u}-\frac{\delta l}{\delta a}\diamond a=0\,,$$

in combination with the mass conservation law

$$\left(\frac{\partial}{\partial t}+\mathcal{L}_u\right)\rho\,\mathrm{d}V=0\,,$$

yields Kelvin's circulation theorem in terms of $(\mathbf{v}\cdot\mathbf{dx}):=\rho^{-1}\frac{\delta l}{\delta u}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c(\mathbf{u})}\mathbf{v}\cdot\mathbf{dx}=\oint_{c(\mathbf{u})}\left(\frac{\partial}{\partial t}+\mathcal{L}_u\right)(\mathbf{v}\cdot\mathbf{dx})=\oint_{c(\mathbf{u})}\frac{1}{\rho}\frac{\delta l}{\delta a}\diamond a$$

Theorems for advected quantities

Theorem (Commutator)

A commutation relation holds among the Lie derivatives,

$$\left[\partial_t + \mathcal{L}_{u(t)}\,,\,\mathcal{L}_{\eta(t)}\right] a(t) = \mathcal{L}_{(\dot{\eta}+[u,\eta])}\, a(t)\,. \tag{2}$$

Proof.

By the product rule for Lie derivatives

$$\left(\partial_t + \mathcal{L}_{u(t)}\right)\mathcal{L}_\eta a(t) \;=\; \mathcal{L}_{(\dot{\eta}+[u,\eta])} a(t) + \mathcal{L}_\eta \left(\partial_t + \mathcal{L}_{u(t)}\right) a(t)\,.$$

Hence, commutation relation (2) holds, and because $a(t) \in V$ is arbitrary, the Lie derivative commutation relation holds

$$\left[\partial_t + \mathcal{L}_{u(t)}\,,\,\mathcal{L}_\eta\right] = \mathcal{L}_{(\dot{\eta}+[u,\eta])}\,. \tag{3}$$

Corollary and Ertel's Theorem for advected quantities

Corollary (Symmetry)

If a vector field η is a symmetry, then the Lie derivative $\mathcal{L}_\eta$ commutes with the evolution operator, $\big(\partial_t + \mathcal{L}_{u(t)}\big)$.

$$\textit{By (3)} \quad \big[\partial_t + \mathcal{L}_{u(t)}\,,\,\mathcal{L}_{\eta(t)}\big]\,a(t) = 0 \quad \textit{for} \quad \dot\eta + [u,\eta] = 0\,. \qquad (4)$$

Theorem (Ertel theorem)

If a is an advected quantity so that $\big(\partial_t + \mathcal{L}_{u(t)}\big)\,a(t) = 0$ and the vector field η is a symmetry, then $\mathcal{L}_\eta a$ is also advected.

Proof.

Relation (4) implies the advection relation for $\mathcal{L}_\eta a$,

$$\big(\partial_t + \mathcal{L}_{u(t)}\big)\,\mathcal{L}_\eta a(t) = \mathcal{L}_\eta\,\big(\partial_t + \mathcal{L}_{u(t)}\big)\,a(t) = 0\,. \qquad (5)$$

Examples of symmetry vector fields

Noether quantity	*Defining equation*	*Symmetry vector field*
Vorticity	$\boldsymbol{\omega} = \rho^{-1}\mathrm{curl}\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\right)$	$\eta = \rho^{-1}\mathrm{curl}\boldsymbol{\Psi}\cdot\nabla$
Helicity density	$\lambda_H = \left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\right)\cdot\mathrm{curl}\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\right)$	$\eta_H = \rho^{-1}\mathrm{curl}\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\right)\cdot\nabla$
Potential Vorticity	$q = \rho^{-1}\mathrm{curl}\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\right)\cdot\nabla T$	$\eta_{PV} = \rho^{-1}\left(\nabla\phi\times\nabla T\right)\cdot\nabla$
Cross helicity density	$\lambda_{CH} = \rho^{-1}\boldsymbol{B}\cdot\frac{\delta l}{\delta \boldsymbol{u}}$	$\eta_{CH} = \rho^{-1}\boldsymbol{B}\cdot\nabla$

Table: Vector fields of relabelling symmetries for ideal fluids and MHD. The vector $\mathbf{v} = \frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}$ is the circulation velocity in Kelvin's Theorem.

Lie symmetries: Case 1 (mass advection)

Solving the symmetry relations

The solutions of the symmetry relations

$$\delta u = \dot{\eta} + [u, \eta] = 0 \quad \text{and} \quad \delta a = -\mathcal{L}_\eta a = 0\,,$$

depend on the number and type of advected quantities, $a(t)$.

Case 1. If the only advected quantity is the mass density $a = \rho\, dV$, the symmetry condition is (by Cartan's formula)

$$\mathcal{L}_\eta(\rho\, dV) = d\,(\eta \,\lrcorner\, \rho\, dV) = 0.$$

In a simply connected domain, $d^2 = 0$ then implies that

$$\eta \,\lrcorner\, \rho\, dV = d(\mathbf{\Psi} \cdot d\boldsymbol{x}) = \mathrm{curl}\mathbf{\Psi} \cdot d\boldsymbol{S} \quad \Longrightarrow \quad \eta = \rho^{-1}\mathrm{curl}\mathbf{\Psi} \cdot \nabla$$

Case 1 (cont): Conservation of vorticity

Case 1 (cont): One advected quantity, $\rho\, dV$.

Substituting the solution $\eta = \rho^{-1}\mathrm{curl}\boldsymbol{\Psi}\cdot\nabla$ into Noether's theorem and using Corollary (4) yields

$$\begin{aligned} 0 &= \frac{d}{dt}\left\langle \frac{\delta l}{\delta u}, \eta \right\rangle = \frac{d}{dt}\int_\Omega \eta \,\lrcorner\, \frac{\delta l}{\delta u} \\ &= \frac{d}{dt}\int_\Omega \frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\cdot d\boldsymbol{x}\wedge(\eta\,\lrcorner\,\rho\, dV) = \frac{d}{dt}\int_\Omega \frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\cdot d\boldsymbol{x}\wedge d(\boldsymbol{\Psi}\cdot d\boldsymbol{x}) \\ &= -\int_\Omega \left(\frac{\partial}{\partial t} + \mathcal{L}_{u(t)}\right) d\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\cdot d\boldsymbol{x}\right)\wedge(\boldsymbol{\Psi}\cdot d\boldsymbol{x})\,. \end{aligned}$$

This is the weak form of conservation of the vorticity 2-form,

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_u\right)\left(\mathrm{curl}\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\cdot d\boldsymbol{S}\right) = 0\,.$$

Case 1 (re-cont): Conservation of helicity

Case 1 (re-cont).
Upon choosing the *arbitrary* vector $\boldsymbol{\Psi}$ to be

$$\eta \,\lrcorner\, \rho\,\mathrm{d}V = \mathrm{d}(\boldsymbol{\Psi}\cdot\mathrm{d}\boldsymbol{x}) = \mathrm{d}\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\cdot\mathrm{d}\boldsymbol{x}\right) \,,$$

Noether's theorem for the case of one advected quantity becomes

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\cdot\mathrm{d}\boldsymbol{x}\wedge\mathrm{d}(\boldsymbol{\Psi}\cdot\mathrm{d}\boldsymbol{x}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\cdot\mathrm{d}\boldsymbol{x}\wedge\mathrm{d}\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\cdot\mathrm{d}\boldsymbol{x}\right) . \end{aligned}$$

This is conservation of helicity.

Ertel's theorem in usual hydrodynamic notation

Identify the evolutionary operator with the Lagrangian time derivative

$$\partial_t + \mathcal{L}_{u(t)} = \frac{\mathrm{D}}{\mathrm{D}t}\,,$$

and define

$$\eta_H = \rho^{-1}\mathrm{curl}\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\right)\cdot\nabla =: \rho^{-1}\boldsymbol{\omega}\cdot\nabla \quad \text{with} \quad \boldsymbol{\omega} := \mathrm{curl}\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\right).$$

Write the symmetry relation (4) as

$$\frac{\mathrm{D}}{\mathrm{D}t}(\rho^{-1}\boldsymbol{\omega}\cdot\nabla)a(t) = (\rho^{-1}\boldsymbol{\omega}\cdot\nabla)\frac{\mathrm{D}}{\mathrm{D}t}a(t)\,. \tag{6}$$

This is the usual form of the classical Ertel theorem.

For a scalar advected function, $a \in \Lambda^0$, (6) yields another scalar conservation law, for $q = (\rho^{-1}\boldsymbol{\omega}\cdot\nabla)a$.

Lie symmetries: Case 2 (two advection laws)

Case 2. Two advected quantities: a density and a scalar

Theorem

With two advected quantities $a_1 = \rho\, dV \in \Lambda^3$, $a_2 = T \in \Lambda^0$, *the simultaneous solution of* $\mathcal{L}_\eta(\rho\, dV) = 0$ *and* $\mathcal{L}_\eta T = 0$ *is for any* $\phi \in \Lambda^0$,

$$\eta \,\lrcorner\, \rho\, dV = d(\phi\, dT)\,. \tag{7}$$

Proof.

The proof is simple, since $a_1 = \rho\, dV$ is a top form and $a_2 = T$ is a bottom form. Hence, $dT \wedge \eta \,\lrcorner\, \rho\, dV = (\nabla T \cdot \eta)\, \rho\, dV = 0$. Thus, the advected quantities $a_1 = \rho\, dV \in \Lambda^3$, $a_2 = T$ satisfy

$$0 = (\nabla T \cdot \eta)\, \rho\, dV = dT \wedge (\eta \,\lrcorner\, \rho\, dV) = dT \wedge d(\boldsymbol{\Psi} \cdot d\boldsymbol{x}) = dT \wedge d(\phi\, dT)\,.$$

□

Case 2 (cont): Conservation of potential vorticity

Substitute $\eta \lrcorner \rho \, \mathrm{d}V = \mathrm{d}(\phi \, \mathrm{d}T)$ into our previous Noether theorem calculation and recompute, finding this time that:

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \frac{\delta l}{\delta \boldsymbol{u}}, \boldsymbol{\eta} \right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}} \cdot \mathrm{d}\boldsymbol{x} \right) \wedge (\eta \lrcorner \rho \, \mathrm{d}V) \\
\text{By (7)} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}} \cdot \mathrm{d}\boldsymbol{x} \right) \wedge \mathrm{d}(\phi \, \mathrm{d}T) \\
&= -\int_\Omega \left(\left(\frac{\partial}{\partial t} + \mathcal{L}_{u(t)} \right) \left(\mathrm{d}\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}} \cdot \mathrm{d}\boldsymbol{x} \right) \wedge \mathrm{d}T \right) \right) \phi \, \mathrm{d}\, V .
\end{aligned}
$$

This is the weak form of potential vorticity (PV) conservation,

$$
\left(\frac{\partial}{\partial t} + \mathcal{L}_u \right) (q \, \rho \, \mathrm{d}V) = 0 \,, \tag{8}
$$

$$
(q \, \rho \, \mathrm{d}V) := \mathrm{d}\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}} \cdot \mathrm{d}\boldsymbol{x} \right) \wedge \mathrm{d}T = \mathrm{curl}\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}} \right) \cdot \nabla T \, dV \,. \tag{9}
$$

Lie symmetries: Case 3

Case 3. Two advected quantities: a density and a 2-form

Theorem

For the case that $a_1 = \rho\,\mathrm{d}V \in \Lambda^3$ *and* $a_2 = \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = \mathrm{d}(\mathbf{A}\cdot\mathrm{d}\boldsymbol{x}) \in \Lambda^2$*, the only simultaneous solution of* $\mathcal{L}_\eta \rho\,\mathrm{d}V = 0$ *and* $\mathcal{L}_\eta \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = 0$ *is*

$$\eta \,\lrcorner\, \rho\,\mathrm{d}V = \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = \rho^{-1}B \,\lrcorner\, \rho\,\mathrm{d}V\,. \qquad (10)$$

Proof.

Recall $\eta \,\lrcorner\, \rho\,\mathrm{d}V = \mathrm{d}(\boldsymbol{\Psi}\cdot\mathrm{d}\boldsymbol{x})$ and identify $\mathrm{d}(\boldsymbol{\Psi}\cdot\mathrm{d}\boldsymbol{x}) = \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}\,.$ □

Case 3 (cont): Conservation of cross vorticity

In this case, Noether's theorem implies the conserved quantity

$$
\begin{aligned}
0 &= \frac{d}{dt}\left\langle \frac{\delta l}{\delta \boldsymbol{u}}, \boldsymbol{\eta} \right\rangle = \frac{d}{dt}\int_\Omega \left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}} \cdot d\boldsymbol{x} \right) \wedge (\eta \;\lrcorner\; \rho\, dV) \\
\text{By (10)} &= \frac{d}{dt}\int_\Omega \frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}} \cdot d\boldsymbol{x} \wedge \boldsymbol{B}\cdot d\boldsymbol{S} \\
&= \frac{d}{dt}\int_\Omega \left(\boldsymbol{B}\cdot \frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}} \right) dV\,.
\end{aligned}
$$

This is the cross helicity, which is known to be conserved, in particular, for ideal magnetohydrodynamics (MHD)

Flows of Lie symmetries for Euler's equations

Our solutions of the symmetry relations in Noether's Theorem

$$\delta u = \dot{\eta} + [u, \eta] = 0 \quad \text{and} \quad \delta a = -\mathcal{L}_\eta a = 0\,,$$

has yielded the following conserved quantities and symmetry vector fields

Noether quantity	*Defining equation*	*Symmetry vector field*
Vorticity	$\boldsymbol{\omega} = \rho^{-1}\mathrm{curl}\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\right)$	$\eta = \rho^{-1}\mathrm{curl}\boldsymbol{\Psi}\cdot\nabla$
Helicity density	$\lambda_H = \left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\right)\cdot\mathrm{curl}\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\right)$	$\eta_H = \rho^{-1}\mathrm{curl}\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\right)\cdot\nabla$
Potential Vorticity	$q = \rho^{-1}\mathrm{curl}\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\right)\cdot\nabla T$	$\eta_{PV} = \rho^{-1}\left(\nabla\phi\times\nabla T\right)\cdot\nabla$
Cross helicity density	$\lambda_{CH} = \rho^{-1}\boldsymbol{B}\cdot\frac{\delta l}{\delta \boldsymbol{u}}$	$\eta_{CH} = \rho^{-1}\boldsymbol{B}\cdot\nabla$

Table: Vector fields of relabelling symmetries for ideal fluids and MHD.

The flows generated by the symmetry vector fields

Here are the flows associated with our Noether symmetries:

- The symmetry vector field $\eta = \rho^{-1}\mathrm{curl}\boldsymbol{\Psi}\cdot\nabla$ generates a flow along a simple closed curve $C(t)$ that is transported by the fluid velocity u, according to the symmetry relation $\left(\frac{\partial}{\partial t} + \mathcal{L}_{u(t)}\right)\eta = \dot{\eta} + [u, \eta] = \delta u = 0.$

  This is Kelvin's circulation theorem.
- $\eta_H = \rho^{-1}\mathrm{curl}\left(\frac{1}{\rho}\frac{\delta l}{\delta \boldsymbol{u}}\right)\cdot\nabla \Longrightarrow$ flow along vortex lines.
- $\eta_{PV} = \rho^{-1}\left(\nabla\phi \times \nabla T\right)\cdot\nabla \Longrightarrow$ flow along level sets of T.
- $\eta_{CH} = \rho^{-1}\boldsymbol{B}\cdot\nabla \Longrightarrow$ flow along magnetic field lines.

Each of these flows may be regarded as a particle relabelling symmetry, but Lagrangian fluid particles were *not* invoked!

Thanks for listening!

The key points of the lecture were:

Point #1:

Ideal fluid equations follow from Hamilton's principle

$$\delta S = 0 \quad \text{with} \quad S = \int \ell(u, a)dt.$$

Point #2:

The geometric approach reveals the symmetry vector fields responsible for the conservation laws for ideal fluids.

Point #3:

The result is Noether's theorem for ideal fluids in Eulerian form.

References for background reading

1. *Geometric Mechanics and Symmetry: From Finite to Infinite Dimensions*,
DD Holm,T Schmah and C Stoica.
Oxford University Press, (2009).
ISBN 978-0-19-921290-3
2. *Geometric Mechanics I: Dynamics and Symmetry*,
DD Holm.
World Scientific: Imperial College Press, Singapore, (2008).
ISBN 978-1-84816-195-5
3. *Geometric Mechanics II: Rotating, Translating and Rolling*
DD Holm
World Scientific: Imperial College Press, Singapore, (2008).
ISBN 978-1-84816-155-9
4. *The Euler–Poincaré equations and semidirect products with applications to continuum theories*,
DD Holm, JE Marsden and TS Ratiu,
Adv. in Math., **137** (1998) 1-81,
http://xxx.lanl.gov/abs/chao-dyn/9801015.