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Notation mainly:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \nu\nabla^2\mathbf{u},$$

$$\nabla\cdot\mathbf{u} = 0,$$

where

$$\begin{aligned}\mathbf{u}(\mathbf{x},t) &= \hat{i}u + \hat{j}v + \hat{k}w,\\ &= \hat{e}_i u_i.\end{aligned}$$

We have initial data $\mathbf{u}_0(x)$ - without loss of generality we have mean zero. We also have pressure where

$$\begin{aligned}p(x,t) &= -\Delta^{-1}\left(\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})\right),\\ &= -\Delta^{-1}\left[(\partial_i u_j)(\partial_j u_i)\right].\end{aligned}$$

Fourier representation:

$$\mathbf{u}(\mathbf{x},t) = \sum_{\mathbf{k}\neq 0}\hat{u}(\mathbf{k},t)e^{i\mathbf{k}\mathbf{x}},$$

where

$$\hat{u}(\mathbf{k},t) = \frac{1}{L^3}\int_{T^3} e^{-i\mathbf{k}\mathbf{x}}\mathbf{u}(\mathbf{x},t)\ d^3\mathbf{x},$$

with $\mathbf{k} = \frac{2\pi}{L}\hat{e}_i u_i$.

$$\nabla\cdot\mathbf{u} = 0 \quad\leftrightarrow\quad \mathbf{k}\cdot\hat{u}(\mathbf{k},t) = 0.$$

$$\begin{aligned}\|\nabla\mathbf{u}(\cdot,t)\|_2^2 &= \int_{T^3}|\mathbf{u}(\mathbf{x},t)|^2\ d^3\mathbf{x},\\ &= L^3\sum_k|\hat{u}(\mathbf{k},t)|^2.\end{aligned}$$

$$\begin{aligned}\|\nabla\mathbf{u}\|_2^2 &= \int|\nabla\mathbf{u}|^2\ d^3\mathbf{x},\\ &= \int(\partial_i u_j)(\partial_j u_i)\ d^3\mathbf{x},\\ &= L^3\sum_{\mathbf{k}}|\mathbf{k}|^2|\hat{u}(\mathbf{k},t)|.\end{aligned}$$

Galerkin:

$$P^N(f)(\mathbf{x},t) = \sum_{|\mathbf{k}|\leq N} e^{i\mathbf{k}\ \mathbf{x}}f(\mathbf{k},t).$$

Then the Galerkin method applied to the Navier Stokes equation would be

$$\frac{d\mathbf{u}^N}{dt} + P^N(\mathbf{u}^N \cdot \nabla \mathbf{u}^N) + \nabla p^N = \nu \nabla^2 \mathbf{u}^N,$$

$$\nabla \cdot \mathbf{u}^N = 0.$$

where $P^N(\mathbf{u}^N) = \mathbf{u}^N$.
Note that

$$(P^N)^2 = P^N, \quad [P^N, \nabla] = 0, \quad P^N = (P^N)^+.$$

This is a significant simplification. If we now use the Galerkin approximation and dot $\mathbf{u}^N$ with the Navier Stokes equation we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^N\|_2^2 + \int \mathbf{u}^N \cdot P^N(\mathbf{u}^N \cdot \nabla \mathbf{u}^N)\ d^3\mathbf{x} = -\nu \|\nabla \mathbf{u}^N\|_2^2.$$

We have global in time, smooth solutions.

Fact A: Given $\mathbf{u}_0(x)$, $\|\nabla \mathbf{u}_0\|_2^2 = E_0 < \infty$. This says that given a velocity, if the enstrophy is finite then $\exists T > 0$ and $E(T) < \infty$ s.t $\forall t \in (0, T)$ $\|\nabla \mathbf{u}^N(\cdot, t)\|_2^2 < E(T)$ uniformly in N.

Fact B: Given $\mathbf{u}_0$ with finite enstrophy ($\|\nabla \mathbf{u}_0\|_2^2 = E_0 < \infty$) and any $\alpha > 0$, then $\exists\, T(\alpha) > 0$ and $g(T(\alpha)) < \infty$ so that for $t \in (0, T(\alpha))$ $\|e^{\alpha|\nabla|t}\nabla \mathbf{u}^N(\cdot, t)\|_2^2 < G(T(\alpha))$ uniformly in N.

$$(|\nabla|\mathbf{u})(\mathbf{x}) = \sum_{\mathbf{k}} |\mathbf{k}|\hat{u}(\mathbf{k})e^{i\mathbf{kx}}$$

then

$$e^{\alpha|\nabla|t}(|\nabla|\mathbf{u})(\mathbf{x}) = \sum_{\mathbf{k}} |\mathbf{k}|\hat{u}(\mathbf{k})e^{i\mathbf{kx}}e^{\alpha|\nabla|t}$$

and

$$\|e^{\alpha|\nabla|t}\mathbf{u}(\cdot, t)\|_2^2 = \sum_{\mathbf{k}} |\mathbf{k}|^2 e^{2\alpha|\mathbf{k}|t}|\hat{u}(\mathbf{k}, t)|^2.$$

This kind of regularity is called a **Gevrey regularity**.

Fact C: (follows from fact A).
$\exists c > 0$ s.t given $\|\mathbf{u}_0\|_2 \times \|\nabla \mathbf{u}_0\|_2 < C\nu^2$ then $\mathbf{u}^k(\cdot, t) \in C^\infty$ with uniform in N norms $\forall t > 0$.

Fact C': If $\||\nabla|^{\frac{1}{2}}\mathbf{u}_0\|_2 = \|\mathbf{u}\|_{H^{\frac{1}{2}}} < C'\nu$, then the enstrophy and all other norms are bounded uniformly in N $\forall t > 0$.

Fact D: Analysis that leads to fact A 'cannot be improved', i.e. all the estimates that go into proving fact A are saturated.

Here we will define

$$K^N(t) = \frac{1}{2}\|\mathbf{u}^N(\cdot,t)\|_2^2$$

as the natural kinetic energy and the enstrophy is defined as

$$E(t) = \|\nabla \mathbf{u}^N\|_2^2 = \|\omega^N\|_2^2.$$

Poincare:

$$E^N(t) = L^3 \sum_{\mathbf{k}\neq 0} |\mathbf{k}|^2 |\hat{u}(\mathbf{k},t)|^2 \geq \frac{4\pi^2}{L^2} K^N(t).$$

Energy equation:

$$\frac{d}{dt} K^N = -\nu E^N.$$

Enstrophy equation:

$$\begin{aligned} \frac{d}{dt} E^N &= -2\nu\|\nabla\omega^N\|_2^2 + 2\int \omega^N \cdot (\nabla \mathbf{u}^N)\cdot\omega^N, \\ &= -2\nu\|\Delta \mathbf{u}^N\|_2^2 + 2\int \mathbf{u}^N \cdot (\nabla \mathbf{u}^N)\cdot \Delta\mathbf{u}^N. \end{aligned}$$

Now we will curl the Navier Stokes on the Galerkin approximation, to get

$$\frac{d\omega^N}{dt} + P^N(\mathbf{u}^N \cdot \nabla\omega^N) = \nu\nabla^2\omega^N \cdot \nabla\mathbf{u}^N.$$

This is the vorticity equation generated by the Galerkin approximation. Also if we take the Navier Stokes and dot it with $-\nabla^2\mathbf{u}^N$ then it will give us terms in the enstrophy equation shown previously. From here we will drop the 'N'. So the enstrophy equation becomes

$$\frac{dE}{dt} = -2\nu\|\nabla\mathbf{u}\|_2^2 + 2\int \mathbf{u}\cdot\mathbf{u}\cdot\Delta\mathbf{u}.$$

Note: Regarding the term $\omega^N \cdot \nabla\mathbf{u} \cdot \omega^N$ - if we consider the symmetric part of this then we know that the trace of the matrix must be zero as $\nabla\cdot\mathbf{u} = 0$ (divergence-free). Thus the real part of the eigenvalues will be zero, some will have $\lambda < 0$ and some $\lambda > 0$, which will tell us whether the vortex is stretching or contracting.

Now we want to consider the nonlinear term, i.e.

$$\left|\int \mathbf{u}\cdot\nabla\mathbf{u}\cdot\nabla\mathbf{u}^2\right| \leq \|\mathbf{u}\|^{\infty}\|\nabla\mathbf{u}\|_2\|\nabla\mathbf{u}\|_2$$

this is due to Holder inequality.

Fact: $\|\mathbf{u}\|_\infty \leq \|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\nabla^2\mathbf{u}\|_2^{\frac{1}{2}}$ for three dimensions.

To show this we say

$$
\begin{aligned}
|\mathbf{u}(\mathbf{x})| &= \left|\sum_{\mathbf{k}\neq 0} e^{i\mathbf{kx}}\hat{u}(\mathbf{k})\right|, \\
&\leq \sum_{\mathbf{k}\neq 0} |\hat{u}(\mathbf{k})|, \\
&= \sum_{0\neq\mathbf{k}\leq\Lambda} \frac{|\mathbf{k}|}{|\mathbf{k}|}|\hat{u}(\mathbf{k}| + \sum_{\mathbf{k}>\Lambda} \frac{|\mathbf{k}|^2}{|\mathbf{k}|^2}\hat{u}(\mathbf{k}), \\
&\leq \left(\sum_{|\mathbf{k}|<\Lambda} \frac{1}{|\mathbf{k}|^2}\right)^{\frac{1}{2}} \left(\sum_{|\mathbf{k}\leq\Lambda} |\mathbf{k}|^2|\hat{u}(\mathbf{k})|^2\right)^{\frac{1}{2}} + \left(\sum_{|\mathbf{k}|>\Lambda} \frac{1}{|\mathbf{k}|^2}\right)^{\frac{1}{2}} \left(\sum_{|\mathbf{k}|>\Lambda} |\mathbf{k}^4|\hat{u}(\mathbf{k})|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

$$
\begin{aligned}
\sum_{0<|\mathbf{k}|<\Lambda} \frac{1}{|\mathbf{k}|^2}\left(\frac{2\pi}{L}\right)^3\left(\frac{L}{2\pi}\right)^3 &= \left(\frac{L}{2\pi}\right)^3 \int_{\frac{2\pi}{L}}^{\Lambda} \frac{4\pi k^2}{k^2}\, dk, \\
&\leq \left(\frac{L}{2\pi}\right)^3 4\pi\Lambda.
\end{aligned}
$$

$$
\begin{aligned}
\sum_{|\mathbf{k}|>\Lambda} \frac{1}{|\mathbf{k}|^4}\left(\frac{2\pi}{L}\right)^3\left(\frac{L}{2\pi}\right)^3 &= \left(\frac{L}{2\pi}\right)^3 4\pi \int_{\Lambda}^{\infty} \frac{k^2}{k^4}\, dk, \\
&\leq \left(\frac{L}{2\pi}\right)^3 \frac{4\pi}{\Lambda}.
\end{aligned}
$$

This calculation tells us that at each point:

$$
|\mathbf{u}(\mathbf{x})| \leq \left(\frac{L}{2\pi}\right)^{\frac{3}{2}} (4\pi)^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\|\nabla\mathbf{u}\|_2^2 + \left(\frac{L}{2pi}\right)^{\frac{3}{2}} (4\pi)^{\frac{1}{2}}\frac{1}{\Lambda^{\frac{1}{2}}}\|\nabla^2\mathbf{u}\|_2^2.
$$

Now we choose

$$
\begin{aligned}
\Lambda^{\frac{1}{2}} &= \frac{\|\nabla^2\mathbf{u}\|_2^{\frac{1}{2}}}{\|\nabla\mathbf{u}\|_2^{\frac{1}{2}}}, \\
&\leq 2\left(\frac{4\pi}{(2\pi)^3}\right)^{\frac{1}{2}} \|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\nabla^2\mathbf{u}\|_2^{\frac{1}{2}}, \\
&= \left(\frac{2}{\pi^2}\right)^{\frac{1}{2}} \|\nabla\mathbf{u}\|_2^{\frac{1}{2}}\|\nabla^2\mathbf{u}\|_2^{\frac{1}{2}}.
\end{aligned}
$$

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Regarding fact C':

$$
\left(\sum_{\mathbf{k}\neq 0} |\mathbf{k}|\hat{u}(\mathbf{k})|^2\right)^{\frac{1}{2}} \leq E^{\frac{1}{4}}(2K)^{\frac{1}{4}}.
$$

Now we dot the Navier Stokes equation with $|\nabla|\mathbf{u}$ to get

$$\frac{d}{dt}\frac{1}{2}\||\nabla|^{\frac{1}{2}}\mathbf{u}_0\|_2^2 + \int (|\nabla|\mathbf{u})\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) = -\nu\||\nabla|^{\frac{3}{2}}\mathbf{u}\|_2^2,$$

$$\int (\mathbf{u}\cdot\nabla\mathbf{u})\cdot(|\nabla|\mathbf{u}) \leq \|\mathbf{u}\|_{L^3}\|\nabla\mathbf{u}\|_{L^3}^2.$$

Sobolev in 3D says that

$$\|f\|_{L^3} \leq c\||\nabla|^{\frac{1}{2}}f\|_{L^2}$$

then we can use this to obtain

$$\int (\mathbf{u}\cdot\nabla\mathbf{u})\cdot(|\nabla|\mathbf{u}) \leq \||\nabla|^{\frac{1}{2}}\mathbf{u}\|_2\||\nabla|^{\frac{3}{2}}\mathbf{u}\|_2^2$$

then

$$\frac{d}{dt}\frac{1}{2}\||\nabla|^{\frac{1}{2}}\mathbf{u}\|_2^2 + \left(\nu - c\||\nabla|^{\frac{1}{2}}\mathbf{u}\|_2\right)\||\nabla|^{\frac{3}{2}}\mathbf{u}\|_2^2 \leq 0.$$

Again, we will write out the enstrophy equation

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|\nabla\mathbf{u}\|_2^2 &= -\nu\|\nabla^2\mathbf{u}\|_2^2 + \int \mathbf{u}\cdot(\nabla\mathbf{u})\cdot\nabla^2\mathbf{u}, \\ &\leq -\left(\nu - \tilde{c}\||\nabla|^{\frac{1}{2}}\mathbf{u}\|_2\right)\|\Delta\mathbf{u}\|_2^2.\end{aligned}$$

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We take the Navier Stokes equations

$$\frac{d\mathbf{u}}{dt} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \nu\nabla^2\mathbf{u} + \mathbf{f}(\mathbf{x}),$$

$$\nabla\cdot\mathbf{u} = 0,$$

where $\mathbf{f}$ is some forcing term, in only terms of space. We now consider periodic cells of length scale l but in whole the length scale is L. Note that we should consider velocity length scales that are not necessarily periodic. We have $L/l = \alpha$ where α is an integer. Now let $\mathbf{f}(\mathbf{x}) = F\Phi(l^{-1}x)$ where $|\Phi\|_{L^2}([0,1]^3) = 1$. If $\mathbf{f}(\mathbf{x})$ is square integrable then the kinetic energy must remain bounded.

If we take the average of the Navier Stokes equation we obtain

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{u}\|_2^2 = -\nu\|\nabla\mathbf{u}\|_2^2 + \int \mathbf{u}\cdot\mathbf{f}. \tag{1}$$

Suppose $\mathbf{f} \in L_2$, then by Poincare (periodic function):

$$\int_{(0,L)^3}\mathbf{f} = 0, \quad \int_{(0,L)^3}\mathbf{u}_0 = 0, \quad \int_{(0,L)^3}\mathbf{u} = 0.$$

From using this and the Cauchy-Schwartz inequality we would get

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{u}\|_2^2 \leq -\frac{4\pi^2}{L^2}\nu\|\mathbf{u}\|_2^2 + \|\mathbf{u}\|_2\|\mathbf{f}\|_2,$$

$$\|\mathbf{u}\|_2\frac{d}{dt}\|\mathbf{u}\|_2 \leq -\frac{4\pi^2}{L^2}\nu\|\mathbf{u}\|_2^2 + \|\mathbf{u}\|_2\|\mathbf{f}\|_2,$$

which will simplify to become

$$\frac{d}{dt}\|\mathbf{u}\|_2 \leq -\frac{4\pi^2}{L^2}\nu\|\mathbf{u}\|_2 + \|\mathbf{f}\|_2.$$

By Gronwalls inequality:

$$\|\mathbf{u}(\cdot,t)\|_2 \leq \|\mathbf{u}_0\|_2 e^{-\frac{4\pi^2\nu}{L^2}t} + \|\mathbf{f}\|_2\frac{(1-e^{-\frac{4\pi^2\nu}{L^2}})}{\frac{4\pi\nu^2}{L^2}}.$$

Going back to equation (1), we can now take the time average. We get

$$\frac{1}{T}\int_0^T dt\left(\frac{d}{dt}\frac{1}{2}\|\mathbf{u}\|_2^2\right) = \nu\frac{1}{T}\int_0^T dt\|\nabla\mathbf{u}(\cdot,t)\|_2^2 + \frac{1}{t}\int_0^T dt\mathbf{f}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x},t).$$

Note that by the fundamental theorem of calculus

$$\frac{1}{T}\int_0^T dt\left(\frac{d}{dt}\frac{1}{2}\|\mathbf{u}\|_2^2\right) = \frac{1}{T}\left(\frac{1}{2}\|\mathbf{u}(\cdot,T)\|_2^2 - \frac{1}{2}\|\mathbf{u}_0\|_2^2\right),$$

but as $T \to \infty$, the term above will go to zero. This is because the energy is conserved.

Given

$$< F >_T = \frac{1}{T}\int_0^T dt\frac{1}{L^3}\int_{(0,L)^3} d^3x\ F(x,t).$$

This equation says that

$$< \nu|\nabla\mathbf{u}|^2 >_T = < \mathbf{f}\cdot\mathbf{u} >_T + O\left(\frac{1}{T}\right).$$

$$< F > = \lim_{T\to\infty} < F >_T,$$

(we could always say it goes to the supremum limit). Now we define

$$\epsilon = < \nu|\nabla\mathbf{u}|^2 >$$

which is the rate at which the force is converted to heat.

$$\begin{aligned}\epsilon &= < \nu|\nabla\mathbf{u}|^2 >,\\ &= < \mathbf{f}\cdot\mathbf{u} >,\\ &\leq < |\mathbf{f}|^2 >^{\frac{1}{2}} < |\mathbf{u}|^2 >^{\frac{1}{2}}.\end{aligned}$$

Now let $U =< |\mathbf{u}|^2 >^{\frac{1}{2}}$ which gives

$$< |\mathbf{f}|^2 >^{\frac{1}{2}} < |\mathbf{u}|^2 >^{\frac{1}{2}} = F < |\Phi|^2 >^{\frac{1}{2}} U.$$

We want to get rid of the 'F'. If we go back to the Navier Stokes equation and dot it with $\mathbf{f}$, where $\nabla \cdot \mathbf{f} = 0$, then the time average of the time derivate will vanish. But we should then multiply the Navier Stokes by $(-\Delta^{-1}\mathbf{f})$, where

$$-\Delta^{-2}\mathbf{f} = \sum_{\mathbf{k}\neq 0} e^{i\mathbf{k}\ \mathbf{x}} \frac{\hat{f}(\mathbf{k})}{|\mathbf{k}|^4}.$$

But we need to do a precalculation:

$$\mathbf{f}(\mathbf{x}) = F\Phi(l^{-1}x) \qquad \Rightarrow \qquad < |\mathbf{f}|^2 >= F^2 < |\Phi|^2 > .$$

Then for scaling purposes we have

$$\nabla^a \mathbf{f} = \frac{F^2}{l^{2a}} < |\nabla'^a \Phi|^2 > \tag{2}$$

If we now dot the Navier Stokes with $\Delta^{-2}\mathbf{f}$ and take the average, we get

$$\left\langle (\Delta^{-2}\mathbf{f}) \cdot \frac{d\mathbf{u}}{dt} \right\rangle + \left\langle (\Delta^{-2}\mathbf{f}) \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \right\rangle = -\nu \left\langle (\Delta^{-2}\mathbf{f}) \cdot (\Delta \mathbf{u}) \right\rangle + \left\langle (\Delta^{-2}\mathbf{f}) \cdot \mathbf{f} \right\rangle .$$

From rearrangement and some manipulation we obtain

$$\left\langle |\Delta^{-1}\mathbf{f}|^2 \right\rangle = \nu \left\langle (\Delta^{-1}\mathbf{f}) \cdot \mathbf{u} \right\rangle - \left\langle \nabla \Delta^{-2}\mathbf{f} : (\mathbf{u}\mathbf{u}) \right\rangle$$

From equation (2) by by the Cauchy-Schwartz inequality we get:

$$\begin{aligned} F^2 l^4 \left\langle \Delta^{-1}\Phi|^2 \right\rangle &\leq \nu \left\langle |\Delta^{-1}\mathbf{f}|^2 \right\rangle^{\frac{1}{2}} + \sup_{\mathbf{x}} |\nabla \Delta^{-2}\mathbf{f}| U^2, \\ &\leq \nu F l^2 \left\langle |\Delta^{-1'}\Phi|^2 \right\rangle^{\frac{1}{2}} U + \|\Delta^{-1}\mathbf{f}\|^{\frac{1}{2}}_{L^2(0,l)^3} \|\nabla^{-1}\mathbf{f}\|^{\frac{1}{2}}_{L^2(0,l)^3} U^2, \\ &= \nu F l^2 \left\langle |\Delta^{-1'}\Phi^2| \right\rangle^{\frac{1}{2}} U + l^3 F \left\langle |\Delta^{-1'}\Phi|^2 \right\rangle^{\frac{1}{4}} \left\langle |\nabla^{-1'}\Phi|^2 \right\rangle^{\frac{1}{4}} U^2. \end{aligned}$$

Note that the amplitude of the force is bounded from below, i.e.

$$F \leq \frac{\nu}{l^2} \frac{\left\langle |\Delta^{-1}\Phi|^2 \right\rangle^{\frac{1}{2}}}{\left\langle |\Delta^{-1}\Phi|^2 \right\rangle} + \frac{U^2}{l} \frac{\left\langle |\Delta^{-1}\Phi|^2 \right\rangle^{\frac{1}{4}} \left\langle \nabla \Phi|^2 \right\rangle^{\frac{1}{4}}}{\left\langle |\Delta^{-1}\Phi|^2 \right\rangle}.$$

Remember that

$$\begin{aligned} \epsilon &\leq FU \left\langle |\Phi|^2 \right\rangle^{\frac{1}{2}}, \\ &\leq a\frac{\nu U^2}{l} + \frac{bU^3}{l}. \end{aligned}$$

$$\begin{aligned} \frac{\epsilon l}{U^3} &\leq \frac{a\nu}{Ul} + b, \\ &= \frac{a}{\mathrm{Re}} + b, \end{aligned}$$

where a and b are purely shape functions in the form of

$$
\begin{aligned}
a &= \frac{\left\langle|\Phi|^2\right\rangle^{\frac{1}{2}}}{\left\langle|\Delta^{-1}\Phi|^2\right\rangle^{\frac{1}{2}}}, \\
b &= \frac{\left\langle\nabla^{-1}\Phi|^2\right\rangle^{\frac{1}{4}}\left\langle|\Phi|^2\right\rangle^{\frac{1}{2}}}{\left\langle|\Delta^{-1}\Phi|^2\right\rangle^{\frac{3}{4}}}.
\end{aligned}
$$