The 3D Navier-Stokes Problem

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Abstract

It is not known whether the three-dimensional (3D) incompressible Navier-Stokes equations possess unique smooth (continuously differentiable) solutions at high Reynolds numbers. This problem is quite important for basic science, practical applications, and numerical computations. This review presents a selective survey of the current state of the mathematical theory, focusing on the technical source of difficulties encountered with the construction of smooth solutions. It also highlights physical phenomena behind the mathematical challenges.

1. INTRODUCTION

The three-dimensional (3D) Navier-Stokes equations for a single-component, incompressible Newtonian fluid in three dimensions compose a system of four partial differential equations relating the three components of a velocity vector field $\vec{u} = \hat{i}u + \hat{j}v + \hat{k}w$ (adopting conventional vector notation) and a pressure field p, each of which is a function of the spatial variable $\vec{x} = \hat{i}x + \hat{j}y + \hat{k}z$ and the time t:

$$\dot{\vec{u}} + \vec{u} \cdot \vec{\nabla}\vec{u} + \vec{\nabla} p = \nu \Delta \vec{u} \qquad (1)$$

and

$$\vec{\nabla} \cdot \vec{u} = 0, \qquad (2)$$

where the fluid's kinematic viscosity ν is the single material parameter in the problem. The momentum equation (Equation 1) is Newton's second law of motion, balancing the acceleration of a fluid element ($\dot{\vec{u}} + \vec{u} \cdot \vec{\nabla}\vec{u}$) and the total force per unit mass density ($-\vec{\nabla} p + \nu \Delta \vec{u}$) imposed on it by neighboring elements. The continuity equation (Equation 2) expresses conservation of mass for the constant density fluid; the pressure p is the physical pressure divided by this constant density. Equation 1 should be solved forward in time, starting from an initial divergence-free velocity field $\vec{u}_0(\vec{x})$ in a subset of $\mathbf{R}^3$ with appropriate boundary conditions, with the pressure evolving in time to maintain the incompressibility constraint (Equation 2). That is, the pressure is determined instantaneously by the velocity field as a solution of the Poisson equation $-\Delta p = \nabla \cdot (\vec{u} \cdot \nabla \vec{u})$.

This mathematical description is presumed to apply to any simple isotropic fluid with an instantaneous linear relation between stress- and strain-rate describing motions at low Mach number (i.e., for relative flow speeds far below the speed of sound in the fluid). We also presume that these equations pertain to both laminar and fully developed turbulent flows, and they are widely applied throughout engineering and the natural sciences. With the addition of other effects (e.g., external forces and rotation, coupling to temperature and composition fields, and interactions with gravitational and electromagnetic fields or moving boundaries), these equations—or extended or reduced systems based on them—are routinely used to model systems across a tremendous range of length and time scales from microfluidics and biophysics to meteorology, oceanography, and astrophysics.

Given the fundamental nature of the Navier-Stokes equations and their importance for applications, the necessity of a firm mathematical foundation is evident. For models to be predictive and physically robust starting from reasonable initial conditions in reasonable domains, (*a*) the equations of motion should possess solutions, (*b*) the solutions should be unique, and (*c*) the solutions should depend continuously, in some appropriate sense, on the initial data. For analytically intractable solutions to be computationally accessible, they must display sufficient regularity (continuous differentiability) that reasonable numerical schemes can converge. We also desire solution regularity to ensure that the Navier-Stokes equations are physically self-consistent as a hydrodynamic model. Because they are derived to describe motions averaged over volumes far larger than the fluid's atomic constituents, irregularities such as singularities in derivatives signal a violation of that separation of scales, indicating predictions for arbitrarily small (in the hydrodynamic sense) regimes in which physical effects that are not present in the Navier-Stokes model come into play.

The situation now, in the first decade of the twenty-first century, is far from satisfactory. Although there has been considerable progress since this subject was last reviewed in this journal (Ladyzhenskaya 1975), a number of fundamental questions about solutions of the Navier-Stokes equations remain, and several constitute grand challenges for the mathematics community.

Indeed, a one-million-dollar prize is offered for the resolution of some of the most basic problems (Fefferman 2000).

This review intends to provide the mathematically minded applied science and engineering community with an informal and relatively self-contained description of the current state of knowledge and a selective survey of mathematical problems regarding the existence, uniqueness, and regularity of solutions. Although every attempt has been made to make the discussion accessible to a broad audience, the subject is by its very nature technical, so some analysis is necessary. In the end, however, many of the essential problems can be explicated with basic ideas from the theory of ordinary differential equations, Fourier series, rudimentary functional analysis, and elementary analytical estimates such as the Hölder and Cauchy-Schwarz inequalities (Doering & Gibbon 1995). Complete presentations of the modern mathematical theory can be found in a number of monographs (Constantin & Foias 1988, Galdi 2000, Temam 1995).

For definiteness, we focus on the free-decay problem for the incompressible Navier-Stokes equations (Equations 1 and 2) on a cubic periodic domain, $\vec{x} \in \Omega = [0, L]^3$. This eliminates complications that could be introduced by boundary layers at rigid interfaces, by external forces, or by events or actions at large distances. Then, besides the viscosity ν and the domain scale L, the only other input is the initial velocity field $\vec{u}_0(\vec{x})$.

In this periodic setting, the velocity vector fields have the simple Fourier representation

$$\vec{u}(\vec{x}, t) = \sum_{\vec{k}} \hat{u}_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}}, \tag{3}$$

where the wave numbers are $\vec{k} = \frac{2\pi}{L}(\hat{i}l + \hat{j}n + \hat{k}m)$ for integers l, n, and m and

$$\hat{u}_{\vec{k}}(t) = \frac{1}{L^3} \int_{\Omega} \vec{u}(\vec{x}, t) e^{-i\vec{k}\cdot\vec{x}}\, d^3x. \tag{4}$$

The Navier-Stokes equations (Equations 1 and 2) then compose an infinite set of coupled ordinary differential equations for the Fourier coefficients $\hat{u}_{\vec{k}}(t)$: For $\vec{k} \neq 0$,

$$\frac{d}{dt}\hat{u}_{\vec{k}}(t) = -\nu|\vec{k}|^2 \hat{u}_{\vec{k}}(t) - i\left(1 - \frac{\vec{k}\otimes\vec{k}}{|\vec{k}|^2}\right) \cdot \sum_{\vec{k}'+\vec{k}''=\vec{k}} \hat{u}_{\vec{k}'}(t) \cdot \vec{k}'' \hat{u}_{\vec{k}''}(t) \tag{5}$$

and

$$\vec{k} \cdot \hat{u}_{\vec{k}}(t) = 0. \tag{6}$$

The zero-mode Fourier coefficient $\hat{u}_{\vec{0}}$, the mean flow in the cell, is constant in time. This is just a statement of the bulk conservation of momentum; without loss of generality we take $\hat{u}_{\vec{0}}(0) = \hat{u}_{\vec{0}}(t) = 0$. The system of ordinary differential equations in Equation 5 should be solved forward in time starting from $\hat{u}_{\vec{k}}(0)$, the Fourier coefficients of the initial data $\vec{u}_0(\vec{x})$. The absence of the pressure in this formulation reflects that it is an auxiliary local variable introduced to maintain the nonlocal incompressibility constraint.

One desired feature of solutions, namely conservation of energy, is suggested by both formulations via formal calculation. Taking the dot product of $\vec{u}(\vec{x}, t)$ with Equation 1, integrating over the volume, and integrating by parts noting that there are no boundary contributions—assuming that the functions are smooth enough to carry out these operations—we find

$$\frac{d}{dt}\frac{1}{2}\|\vec{u}(\cdot, t)\|_2^2 = -\nu\|\vec{\nabla}\vec{u}(\cdot, t)\|_2^2. \tag{7}$$

Here we introduce the $L^2(\Omega)$ norm

$$\|f(\cdot)\|_2 = \sqrt{\int_\Omega |f(\vec{x})|^2\, d^3x} = \sqrt{L^3 \sum_{\vec{k}} \left|\hat{f}_{\vec{k}}\right|^2} \tag{8}$$

and use the notation $|\vec{\nabla}\vec{u}| = |\vec{\nabla} \otimes \vec{u}| = \sqrt{u_{i,j}u_{i,j}}$ (in Einstein's convention) for the local tensor norm. The analogous calculation proceeds in the Fourier representation by summing the dot products of the complex conjugate of $\hat{u}_{\vec{k}}(t)$ with Equation 5 and noting the collapse of the sum of cubic terms—assuming that the sums converge in the first place. Then

$$\frac{d}{dt}\frac{1}{2}\sum_{\vec{k}} \left|\hat{u}_{\vec{k}}(t)^2\right| = -\nu \sum_{\vec{k}} |\vec{k}|^2 |\hat{u}_{\vec{k}}(t)^2|. \tag{9}$$

The $L^2(\Omega)$ norm squared of the velocity field is just twice the bulk kinetic energy per unit mass density, so Equations 7 and 9 simply state that the initial budget of kinetic energy is continually and irreversibly dissipated into heat by viscosity at a rate proportional to the instantaneous value of the $L^2(\Omega)$ norm squared of velocity gradients.

Although these calculations and their implications appear elementary and obvious, they are not known to be valid for arbitrary solutions of the 3D Navier-Stokes equations. This predicament illustrates the significance of the shortcomings in the current mathematical state of affairs.

In Section 2, we consider the notion of weak solutions, which are generally available, and ponder their deficiencies. In Section 3, we discuss strong solutions that possess essentially all the mathematical properties that could be desired on physical grounds. Starting from sufficiently smooth initial data, strong solutions are known to exist at least for a while. The problem is that unless the flow is sufficiently weak (i.e., unless appropriate Reynolds numbers are sufficiently small), strong solutions are not known to exist for all times, even starting from smooth initial data and in the absence of any external forcing. It remains unknown whether solutions of the Navier-Stokes equations can develop singularities of some sort after just a finite time at high Reynolds numbers. Turbulence is ubiquitous at high Reynolds number, and not unexpectedly some of these mathematical issues are relevant for the analysis of turbulent flows. Finally, in Section 4 we consider some of the challenges that the ultimate resolution of the regularity question presents and discuss some aspects of high–Reynolds number (including turbulent) flows that can be studied fruitfully within the existing mathematical framework.

2. WEAK SOLUTIONS

Weak solutions satisfy the Navier-Stokes equations in the weak sense. That is, for square integrable (finite–kinetic energy) initial data $\vec{u}_0(\vec{x})$, weak solutions are functions $\vec{u}(\vec{x}, t)$ such that for any infinitely differentiable test function $q(\vec{x}, t)$ and divergence-free test field $\vec{v}(\vec{x}, t)$ with compact support in $t \in [0, \infty)$,

$$\int_0^\infty dt \int_\Omega d^3x(\vec{u} \cdot \dot{\vec{v}} + \vec{u} \cdot (\vec{\nabla}\vec{v}) \cdot \vec{u} + \nu\vec{u} \cdot \Delta\vec{v}) + \int_\Omega \vec{u}_0(\vec{x}) \cdot \vec{v}(\vec{x}, 0)\, d^3x = 0 \tag{10}$$

and

$$\int_0^\infty dt \int_\Omega d^3x \left(\vec{u} \cdot \vec{\nabla}q\right) = 0. \tag{11}$$

These are integrated versions of Equations 1 and 2 that make sense for velocity fields $\vec{u}(\vec{x}, t)$ that may not be smooth enough to differentiate as required to satisfy the partial differential equations pointwise in space and time. Leray (1934) introduced weak solutions three-quarters of a century ago. The basic idea behind the construction of weak solutions is to start with an appropriately regularized approximation to the Navier-Stokes equations to which solutions may be shown to exist for all times and then consider the limits of these approximate solutions as the regularization is removed. Leray originally modified Equation 1 by smearing part of the nonlinear term, but here we consider a regularization that is familiar from spectral methods in numerical analysis, a Galerkin approximation to the equations.

For positive K, we define the projection operators P_K acting on square integrable functions $f(\vec{x})$ according to

$$(P_K f)(\vec{x}) = \sum_{|\vec{k}|<K} \hat{f}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}. \tag{12}$$

P_K simply truncates the Fourier series for f to modes with wave numbers below K. It is a bounded linear, self-adjoint [i.e., $\int_\Omega f(P_K g) d^3x = \int_\Omega (P_K f) g d^3x$] operator satisfying $(P_K)^2 = P_K$ and $\|P_K f\|_2 \le \|f\|_2$. We note that $P_K(\vec{\nabla} f) = \vec{\nabla}(P_K f)$, at least in the sense of distributions if not in $L^2(\Omega)$. Moreover, $P_K \to I$, the identity operator, as $K \to \infty$ in a weak operator sense: For any $L^2(\Omega)$ function $f(\vec{x})$, $\|P_K f - f\|_2 \to 0$ as $K \to \infty$.

The Galerkin approximation $\vec{u}^K(\vec{x}, t)$ is defined as the solution of the regularized Navier-Stokes equations

$$\dot{\vec{u}}^K + P_K\left(\vec{u}^K \cdot \vec{\nabla}\vec{u}^K\right) + \vec{\nabla} p^K = \nu \Delta \vec{u}^K \tag{13}$$

and

$$\vec{\nabla} \cdot \vec{u}^K = 0 \tag{14}$$

starting from the initial data $\vec{u}^K(\vec{x}, 0) = (P_K \vec{u}_0)(\vec{x})$. Unique, bounded, smooth solutions to Equations 13 and 14 starting from $P_K \vec{u}_0$ exist for all times because in the Fourier representation the infinite system Equation 5 is approximated by a finite system of nonlinear ordinary differential equations for the $N \approx 4\pi L^3 K^3/3$ complex-valued Fourier coefficients of $\vec{u}^K$:

$$\frac{d}{dt}\hat{u}^K_{\vec{k}}(t) = -\nu|\vec{k}|^2 \hat{u}^K_{\vec{k}}(t) - i\left(1 - \frac{\vec{k}\otimes\vec{k}}{|\vec{k}|^2}\right) \cdot \sum_{\vec{k}'+\vec{k}''=\vec{k}} \hat{u}^K_{\vec{k}'}(t) \cdot \vec{k}\hat{u}^K_{\vec{k}}(t) \tag{15}$$

for $|\vec{k}| < K$ with $\vec{k} \cdot \hat{u}^K_{\vec{k}}(t) = 0$ and $\hat{u}^K_{\vec{k}}(t) \equiv 0$ for $|\vec{k}| \ge K$. The right-hand side of Equation 15 is a locally Lipshitz continuous function of the variables in the system phase space C^N, so local-in-time solutions can be constructed by classical Picard iteration. We can extend these local solutions to global-in-time solutions because of the a priori bound,

$$\sum_{\vec{k}} \left|\hat{u}^K_{\vec{k}}(t)\right|^2 \le \sum_{\vec{k}} \left|\hat{u}_{\vec{k}}(0)\right|^2 = \frac{1}{L^3}\|\vec{u}_0\|_2^2. \tag{16}$$

This estimate follows from the energy equation for the Galerkin system,

$$\frac{d}{dt}\frac{1}{2}\sum_{\vec{k}} |\hat{u}^K_{\vec{k}}(t)|^2 = -\nu \sum_{\vec{k}} |\vec{k}|^2 \left|\hat{u}^K_{\vec{k}}(t)\right|^2, \tag{17}$$

that shows that the $L^2(\Omega)$ norm of the solution is always decreasing and hence bounded by its initial value, which is in turn bounded by the $L^2(\Omega)$ norm of $\vec{u}_0$ uniformly in the approximation parameter K. The uniqueness of solutions to the finite system for the Fourier coefficients then follows as usual from the classical theory of ordinary differential equations.

Unlike the same exercise applied to the unregularized system, there is nothing delicate about deriving Equation 17 from Equation 15 because all the sums are finite there. Similarly, there is no subtlety in deriving the energy equation

$$\frac{d}{dt}\frac{1}{2}\left\|\vec{u}^K(\cdot,t)\right\|_2^2 = -\nu\left\|\vec{\nabla}\vec{u}^K(\cdot,t)\right\|_2^2 \tag{18}$$

from the regularized partial differential equations in Equations 13 and 14 because the Galerkin approximation $\vec{u}^K(\vec{x},t)$ is infinitely differentiable in all its variables. The energy equation satisfied by the regularized system ultimately informs us that the approximate solutions exist and are unique and perfectly regular for all times.

At this stage, the strategy is to send $K\to\infty$ and hope for convergence of the approximations $\vec{u}^K(\vec{x},t)$ to a limit $\vec{u}(\vec{x},t)$ that solves the unregularized Navier-Stokes equations. This is where the difficulties begin to appear. To establish convergence (in an appropriate sense), we must know that (suitable) norms of differences between approximations, $\|\vec{u}^K-\vec{u}^{K'}\|$, vanish as $K,K'\to\infty$. To deduce this information from the regularized Navier-Stokes equations, we must know that the approximations are uniformly regular in an appropriate sense as $K,K'\to\infty$. Below we present an example that illustrates the kind of calculations involved (without claim of optimal analysis) and demonstrates the conundrum.

Suppose we try to rigorously establish convergence of the sequence of Galerkin approximations $\{\vec{u}^K, K>0\}$ to a continuous function from $[0,T]$ to $L^2(\Omega)$ uniformly for some positive time $T<\infty$. We could accomplish this by showing that the approximations constitute a Cauchy sequence in the space of continuous functions from the time interval $[0,T]$ to $L^2(\Omega)$, i.e., by proving that

$$\lim_{K,K'\to\infty}\sup_{t\in[0,T]}\|\vec{u}^K(\cdot,t)-\vec{u}^{K'}(\cdot,t)\|_2 = 0. \tag{19}$$

For definiteness, we let $K'>K$ and define the difference $\delta\vec{u}=\vec{u}^{K'}-\vec{u}^K$, whose time evolution of the $L^2(\Omega)$ norm is computed from the difference of the regularized equations for $\vec{u}^{K'}$ and $\vec{u}^K$. Elementary integrations and estimates (the Hölder and Cauchy-Schwarz inequalities) yield

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|\delta\vec{u}(\cdot,t)\|_2^2 &= -\nu\|\vec{\nabla}\delta\vec{u}\|_2^2 - \int_\Omega \vec{u}^K\cdot\vec{\nabla}\vec{u}^K\cdot(P_{K'}-P_K)\,\delta\vec{u}d^3x - \int_\Omega \delta\vec{u}\cdot\vec{\nabla}\vec{u}^K\cdot\delta\vec{u}\,d^3x\\ &\le -\nu\|\vec{\nabla}\delta\vec{u}\|_2^2 + \|(P_{K'}-P_K)\,\delta\vec{u}\|_2\|\vec{\nabla}\vec{u}^K\|_\infty\|\vec{u}^K\|_2 + \|\delta\vec{u}\|_2^2\|\vec{\nabla}\vec{u}^K\|_\infty,\end{aligned} \tag{20}$$

which introduces the $L^\infty(\Omega)$ norm $\|f\|_\infty = \sup_{\vec{x}\in\Omega}|f(\vec{x})|$. Now we note

$$\|(P_{K'}-P_K)\,\delta\vec{u}\|_2^2 = \sum_{K<|\vec{k}|\le K'}\left|\delta\hat{u}_{\vec{k}}\right|^2 \le \sum_{K<|\vec{k}|<\infty}\frac{|\vec{k}|^2}{K^2}\left|\delta\hat{u}_{\vec{k}}\right|^2 \le \frac{1}{K^2}\|\vec{\nabla}\delta\vec{u}\|_2^2, \tag{21}$$

so the first two terms on the second line of Equation 20 may be bounded (using $2ab\le a^2+b^2$) as

$$\begin{aligned}-\nu\|\vec{\nabla}\delta\vec{u}\|_2^2 + \|(P_{K'}-P_K)\,\delta\vec{u}\|_2\|\vec{\nabla}\vec{u}^K\|_\infty\|\vec{u}^K\|_2 &\le -\nu\|\vec{\nabla}\delta\vec{u}\|_2^2 + \frac{1}{K}\|\vec{\nabla}\delta\vec{u}\|_2\|\vec{\nabla}\vec{u}^K\|_\infty\|\vec{u}^K\|_2\\ &\le -\nu\|\vec{\nabla}\delta\vec{u}(\cdot,t)\|_2^2 + \nu\|\vec{\nabla}\delta\vec{u}(\cdot,t)\|_2^2 + \frac{1}{4\nu K^2}\|\vec{\nabla}\vec{u}^K(\cdot,t)\|_\infty^2\|\vec{u}^K(\cdot,t)\|_2^2\\ &= \frac{1}{4\nu K^2}\|\vec{\nabla}\vec{u}^K(\cdot,t)\|_\infty^2\|\vec{u}^K(\cdot,t)\|_2^2.\end{aligned} \tag{22}$$

The $L^2(\Omega)$ norms of all $\vec{u}^K$ are bounded by the $L^2(\Omega)$ norm of $\vec{u}_0$, so

$$\frac{d}{dt}\frac{1}{2}\|\delta\vec{u}(\cdot,t)\|_2^2 \le \|\vec{\nabla}\vec{u}^K(\cdot,t)\|_\infty\|\delta\vec{u}(\cdot,t)\|_2^2 + \frac{1}{4\nu K^2}\|\vec{\nabla}\vec{u}^K(\cdot,t)\|_\infty^2\|\vec{u}_0\|_2^2. \tag{23}$$

Gronwall's inequality[1] then implies

$$\|\delta\vec{u}(\cdot,t)\|_2^2 \le \|\delta\vec{u}(\cdot,0)\|_2^2\, e^{2\int_0^t \|\vec{\nabla}\vec{u}^K(\cdot,s)\|_\infty ds} + \frac{\|\vec{u}_0\|_2^2}{2\nu K^2}\int_0^t \|\vec{\nabla}\vec{u}^K(\cdot,s)\|_\infty^2 e^{2\int_s^t \|\vec{\nabla}\vec{u}^K(\cdot,r)\|_\infty dr} ds. \tag{24}$$

Now if we knew that there were a finite number R so that

$$\int_0^T \|\vec{\nabla}\vec{u}^K(\cdot,s)\|_\infty^2 ds < R < \infty \tag{25}$$

uniformly in K (regardless of R's dependence on T, L, ν, and $\|\vec{u}_0\|_2$), then we could conclude that

$$\sup_{t\in[0,T]} \|\delta\vec{u}(\cdot,t)\|_2^2 \le \|\delta\vec{u}(\cdot,0)\|_2^2\, e^{2\sqrt{RT}} + \frac{\|\vec{u}_0\|_2^2}{2\nu K^2} Re^{2\sqrt{RT}}. \tag{26}$$

Moreover, because

$$\|\delta\vec{u}(\cdot,0)\|_2^2 = \sum_{K<|\vec{k}|\le K'} \left|\hat{u}_{\vec{k}}(0)\right|^2 \to 0 \quad \text{as} \quad K\to\infty, \tag{27}$$

we could then assert that$\sup_{t\in[0,T]} \|\delta\vec{u}(\cdot,t)\|_2^2 \to 0$ as K(and $K' > K$) $\to \infty$.

The key to establishing convergence in this particular approach is the existence of a uniform (in K) bound on the time integral of the $L^\infty(\Omega)$ norm squared of the Galerkin approximations' gradients in Equation 25. If such an a priori estimate were known, then one could proceed to (*a*) ask if the limit of the approximations satisfied the unregularized Navier-Stokes equations, (*b*) enquire as to further regularity of the limit, and (*c*) investigate the uniqueness of the solution so obtained. Indeed, if a unique smooth solution exists, then the Galerkin approximations converge to it, and in principle for particular cases this can be rigorously confirmed computationally (Chernyshenko et al. 2007).

However, no such uniform bound is known. In fact, we know relatively little along these lines. From the energy (Equation 18) for the Galerkin approximations, we can only deduce that for any positive times $0 \le t_0 < t < \infty$,

$$\frac{1}{2}\left\|\vec{u}^K(\cdot,t)\right\|_2^2 + \nu\int_{t_0}^t \left\|\vec{\nabla}\vec{u}^K(\cdot,s)\right\|_2^2 ds = \frac{1}{2}\left\|\vec{u}^K(\cdot,t_0)\right\|_2^2 \le \frac{1}{2}\|\vec{u}_0\|_2^2. \tag{28}$$

This confirms both that the energy in the approximations is bounded uniformly in K (by the initial energy) at every instant and that time integrals of the $L^2(\Omega)$ norm squared of the approximations' gradients are bounded uniformly in K by the initial energy divided by the viscosity, e.g.,

$$\int_0^T \left\|\vec{\nabla}\vec{u}^K(\cdot,s)\right\|_2^2 ds \le \frac{\|\vec{u}_0\|_2^2}{2\nu}\langle\infty. \tag{29}$$

However, this is far weaker control than that required by Equation 25 for the convergence argument above.

These uniform energy bounds are insufficient to establish strong (norm) convergence, but they are sufficient to establish the convergence of subsequences of the Galerkin approximations to weak solutions; this was Leray's fundamental insight (Hopf 1951, Leray 1934, Serrin 1963). The energy estimate and other a priori uniform bounds establish that the Galerkin approximations are

[1]Gronwall's inequality states that the differential inequality $df(t)/dt \le g(t)f(t) + h(t)$ implies a pointwise-in-time bound $f(t) \le f(0)e^{\int_0^t g(s)ds} + \int_0^t h(s)e^{\int_s^t g(r)dr} ds$. The proof follows by multiplying the differential inequality by a positive integrating factor to deduce $\frac{d}{dt}(f(t)e^{-\int_0^t g(s)ds}) \le h(t)e^{-\int_0^t g(s)ds}$ and integrating from 0 to $t \ge 0$.

contained in a compact (weak topology) set assuring the weak convergence[2] of a subsequence to a vector field that satisfies the weak form of the Navier-Stokes equations in Equations 10 and 11.

There may be more than one weakly convergent subsequence, and among these one can find weak solutions that also satisfy the energy inequality

$$\frac{1}{2}\,\|\vec{u}(\cdot,t)\|_2^2 + \nu\int_{t_0}^{t}\left\|\vec{\nabla}\vec{u}(\cdot,s)\right\|_2^2 ds \;\leq\; \frac{1}{2}\,\|\vec{u}(\cdot,t_0)\|_2^2 \tag{30}$$

for all t and almost all t_0 with $0 \leq t_0 < t < \infty$. Additionally, with some modification of the weak solution on a set of (time) measure zero, one can ensure that there are fields $\vec{u}(\vec{x},t)$ that are weakly continuous functions of time with values in $L^2(\Omega)$ that satisfy a slightly stronger form of Equation 10; namely, for any time $t \in [0,\infty)$ and any infinitely differentiable divergence-free $\vec{v}(\vec{x},t)$,

$$\begin{aligned}\int_\Omega \vec{u}(\vec{x},t)\cdot\vec{v}(\vec{x},t)d^3x = \int_0^t dt\int_\Omega d^3x(\vec{u}\cdot\cdot\vec{v} + \vec{u}\cdot(\vec{\nabla}\vec{v})\cdot\vec{u} + \nu\vec{u}\cdot\Delta\vec{v})\\ + \int_\Omega \vec{u}_0(\vec{x})\cdot\vec{v}(\vec{x},0)d^3x.\end{aligned} \tag{31}$$

Velocity vector fields satisfying Equations 11, 30, and 31 are called Leray-Hopf solutions of the Navier-Stokes equations.

The energy inequality in Equation 30 allows the possibility of singularities in the weak solutions in which energy is lost by means other than viscous dissipation. This situation implies a concentration of significant energy at arbitrarily high wave numbers. If such events arise in the course of refining the Galerkin approximations, then the norm of the weak limit of the approximations could be less than the limit of the norm. This leaking of energy to infinitesimal length scales is not necessarily unexpected mathematically,[3] but it is clearly a problem for the physics.

There is no assurance that weak solutions are unique, even with the same initial condition, and the question of solutions' uniqueness is ultimately one of regularity as well. To illustrate the issue, let us assume that we can formally manipulate the Navier-Stokes equations. Then the $L^2(\Omega)$ norm of the difference $\delta\vec{u} = \vec{u}' - \vec{u}$ between two solutions satisfies

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|\delta\vec{u}\|_2^2 &= -\nu\|\vec{\nabla}\delta\vec{u}\|_2^2 - \int_\Omega \delta\vec{u}\cdot\vec{\nabla}\vec{u}\cdot\delta\vec{u}d^3x = -\nu\|\vec{\nabla}\delta\vec{u}\|_2^2 + \int_\Omega \delta\vec{u}\cdot\vec{\nabla}\delta\vec{u}\cdot\vec{u}\; d^3x\\ &\leq -\nu\|\vec{\nabla}\delta\vec{u}\|_2^2 + \|\vec{\nabla}\delta\vec{u}\|_2\|\delta\vec{u}\|_2\|\vec{u}\|_\infty \leq \frac{\|\vec{u}\|_\infty^2}{4\nu}\|\delta\vec{u}\|_2^2.\end{aligned} \tag{32}$$

Now if $\int_0^T \|\vec{u}(\cdot,t)\|_\infty^2\, dt = \kappa < \infty$, Gronwall's inequality ensures that

$$\sup_{t\in[0,T]} \|\delta\vec{u}(\cdot,t)\|_2^2 \leq \|\delta\vec{u}(\cdot,0)\|_2^2\, e^{\kappa/2\nu} \tag{33}$$

so that $\vec{u}' = \vec{u}$ at $t = 0$ implies that $\vec{u}' = \vec{u}$ for all $t \in [0,T]$. However, it is not known that arbitrary time integrals of the square of the L^∞ norm of weak solutions are bounded—or that arbitrary time integrals of the square of the L^∞ norm of the Galerkin approximations are bounded uniformly in K—so the uniqueness of weak solutions remains an open question. The problem with uniqueness is that there is no canonical way to continue through a singularity, should one be encountered.

[2]The weak limit (if it exists) of a sequence of functions $f_n \in L^2(\Omega)$, written $w - \lim_{n\to\infty} f_n$, is the function f such that for every function $g \in L^2(\Omega)$, $\lim_{n\to\infty}\int_\Omega (f(\vec{x}) - f_n(\vec{x}))\,g(\vec{x})d^3x = 0$.

[3]The canonical example of this phenomenon is the following: Let us consider the function space $L^2\,([0,2\pi))$ and the sequence of functions $f_n(x) = (2\pi)^{-1/2}\,e^{inx}$ for $n = 1,2,\ldots$, each having norm 1. The Fourier coefficients of any $g(x) \in L^2\,([0,2\pi))$ are square summable, so $\lim_{n\to\infty}\hat{g}_n = \int_0^{2\pi} e^{-inx}g(x)dx = 0$. This proves that $w - \lim_{n\to\infty} f_n = 0$. Thus $\|w - \lim_{n\to\infty} f_n\|_2 = 0$ even though $\lim_{n\to\infty}\|f_n\|_2 = 1$. That is, the norm of the weak limit of a sequence may be less than the limit (inferior) of the sequence of norms.

One approach to these problems is to try to show that the weak solutions possess more regularity, enough to rule out finite-time singularities. This line of study has focused on deriving specific regularity criteria that would ensure regularity and uniqueness with the hope that eventually all weak solutions may be shown to satisfy them. The current state of affairs (Escauriaza et al. 2003, Giga 1986, Prodi 1959, Serrin 1962, von Wahl 1986) is that if a weak solution (or the Galerkin approximations, uniformly in K) satisfies

$$\left(\int_0^T \left(\int_\Omega |\vec{u}|^r \, d^3x\right)^{s/r}\right)^{1/s} < \infty \tag{34}$$

for $2/s + 3/r = 1$ with $3 \leq r \leq \infty$, then the solution is regular up to time T. In other words, if a singularity develops at time T, then these integrals have to diverge for that solution. One limiting case for these conditions is $r = \infty$ and $s = 2$,

$$\int_0^T \|\vec{u}(\cdot, t)\|_\infty^2 \, dt < \infty, \tag{35}$$

a criterion that could help establish uniqueness, as discussed above. The other limiting case is $r = 3$ and $s = \infty$,

$$\sup_{t\in[0,T]} \left(\int_\Omega |\vec{u}(\cdot, t)|^3 \, d^3x\right)^{1/3} = \sup_{t\in[0,T]} \|\vec{u}(\cdot, t)\|_3 < \infty, \tag{36}$$

plainly displaying the gap in our knowledge: The $L^2(\Omega)$ norms of weak solutions are known to be finite at all times, but we need to know that the $L^3(\Omega)$ norms remain finite to assure regularity.

Other regularity criteria may be formulated in terms of norms of the velocity gradients (Beirao da Viega 1995). For example, if

$$\left(\int_0^T \left(\int_\Omega \left|\vec{\nabla}\vec{u}\right|^{r'} d^3x\right)^{s'/r'} dt\right)^{1/s'} < \infty \tag{37}$$

with $2/s' + 3/r' = 2$ and $3/2 < r' \leq \infty$, then solutions would be regular up to time T. The boundary case $r = \infty$ and $s = 1$,

$$\int_0^T \|\vec{\nabla}\vec{u}(\cdot, t)\|_\infty dt < \infty, \tag{38}$$

is weaker than that considered in Equation 25, although with only a little more work, we can modify that argument to require only that a bound on $\int_0^T \|\vec{\nabla}\vec{u}^K(\cdot, t)\|_\infty^{1+\alpha} dt$ uniform in K for any $0 < \alpha \leq 1$ is sufficient to prove strong convergence of the Galerkin approximates. (Motivated readers can start at Equation 21 and insert a lower power of $|\vec{k}|/K$ into the sum.)

Let us consider the case $r' = 2$ and $s' = 4$, which is the condition

$$\int_0^T \|\vec{\nabla}\vec{u}(\cdot, t)\|_2^4 dt < \infty. \tag{39}$$

Here again the gap in our control is apparent because the energy inequality for weak solutions only ensures that

$$\int_0^T \|\vec{\nabla}\vec{u}(\cdot, t)\|_2^2 dt < \infty. \tag{40}$$

This criterion is of particular interest physically because $\varepsilon(t) = \nu\|\vec{\nabla}\vec{u}(\cdot, t)\|_2^2/L^3$ is the instantaneous, bulk viscous energy-dissipation rate per unit mass in the fluid. Although $\varepsilon(t)$ for weak solutions is not known to be limited instantaneously, the energy inequality for weak solutions assures us that its average $\bar{\varepsilon}$ over any nonzero time interval is bounded. To guarantee regularity, however, Equation 39 requires that the bulk dissipation-rate variance $\overline{(\varepsilon - \bar{\varepsilon})^2}$ is finite, as well. However, this is also not known to be true for general weak solutions. [In Section 3, we see explicitly how the integral condition (Equation 39) controls instantaneous norms of solutions.]

The control of a single velocity component may be sufficient to ensure regularity (Neistupa & Penel 1999). The most recent result in this direction (Kukavica & Zaine 2007) is that

$$\left(\int_0^T \left(\int_\Omega \left| \frac{\partial u}{dz} \right|^{r''} d^3x \right)^{s''/r''} dt \right)^{1/s''} < \infty \tag{41}$$

for $2/s'' + 3/r'' = 2$ with $9/4 < r'' \leq 3$ ensures regularity. These are the same scalings as for the full gradient but with more restrictions on the exponents. Of course, there is nothing special about the z derivative of the u component, and the appropriate regularity of any derivative of any component works as well.

In any event, none of these criteria presently makes contact with the known properties of weak solutions, so for the most part they serve as interesting, necessary conditions for the breakdown of smooth solutions. An alternative direction of investigation concerns the nature of the set of points in space and time upon which weak solutions could be singular (Caffarelli et al. 1982, Lin 1998, Scheffer 1976). The goal of this approach is to characterize some geometric properties and measure theoretic properties of such sets. This approach has established an estimate on a certain fractal dimension of the singular set that rules out singularities on 1D lines [e.g., of the form $\vec{X}(t)$] in space and time.

3. STRONG SOLUTIONS

If the initial flow field is sufficiently regular, then unique smooth solutions exist at least for a positive time, and if the initial data are sufficiently small, then the unique smooth solution may persist forever. Therefore, another approach to the Navier-Stokes equations is to study the sufficiency bounds to determine the longest time a smooth solution might persist with the aim of extending the lifetime estimates. We reiterate that it is not known whether such attempts might be futile (i.e., whether there are smooth initial data that can develop singularities within a finite time). At the very least, however, further understanding may be gained by investigating analytical barriers to these goals.

With regard to this approach, the strong regularity result (Foias & Temam 1989) demonstrates that if the gradient of the initial data is square integrable, i.e., if

$$\|\vec{\nabla}\vec{u}_0\|_2^2 = L^3 \sum_{\vec{k}} |\vec{k}|^2 |\hat{u}_{\vec{k}}(0)^2| < \infty, \tag{42}$$

then for any $\alpha > 0$ there exists a time $T > 0$, depending on α, ν, L, and$\|\vec{\nabla}\vec{u}_0\|_2$, such that

$$\sum_{\vec{k}} |\vec{k}|^2 e^{\alpha t |\vec{k}|} |\hat{u}_{\vec{k}}(t)|^2 \leq B < \infty \tag{43}$$

for $0 \leq t \leq T$. During this interval, the Fourier coefficients of the solutions vanish exponentially as $|\vec{k}| \to \infty$ so that the solution $\vec{u}(\vec{x}, t)$ and all its derivatives are bounded. We can prove this result by establishing an estimate such as Equation 43 for the Galerkin approximations $\vec{u}^K(\vec{x}, t)$ with bound $B < \infty$ and time $T > 0$ that are uniform in K. Such uniform smoothness is sufficient to establish the strong convergence of the Galerkin approximations to a limit that inherits the same regularity. This, in turn, is sufficient to establish uniqueness up to (and even slightly beyond) time T.

These ideas have been developed into explicit regularity statements about strong solutions in terms of bulk properties of the flow (Doering & Titi 1995). We suppose that for some time $T \leq \infty$,

$$\varepsilon_\infty = \sup_{0\leq t\leq T} \varepsilon(t) = \sup_{0\leq t\leq T} \frac{\nu\|\vec{\nabla}\vec{u}(\cdot, t)\|_2^2}{L^3} = \sup_{0\leq t\leq T} \nu \sum_{\vec{k}} |\vec{k}|^2 |\hat{u}_{\vec{k}}(t)|^2 < \infty. \tag{44}$$

This quantity is the extreme value of the energy-dissipation rate per unit mass over the interval. We define the transient time

$$t_* = C\frac{\nu^3}{\|\vec{\nabla}\vec{u}_0\|_2^4}, \tag{45}$$

where C is an absolute (dimensionless, computable) constant that can be chosen so that $t_* < T$. Then it may be shown that, for times $t_* \leq t \leq T$, the Fourier coefficients of the solutions are bounded according to

$$\left|\hat{u}_{\vec{k}}(t)\right|^2 \leq C' \frac{L^6 \varepsilon_\infty}{\nu|\vec{k}|^2} e^{-\ell|\vec{k}|}, \tag{46}$$

where C' is another absolute constant and the exponential decay length scale is

$$\ell = C'' \frac{\nu^3}{\varepsilon_\infty L^3}, \tag{47}$$

where C is another absolute constant. Therefore, as long as solutions are regular (i.e., as long as the bulk energy-dissipation rate is bounded), they are extremely regular.

The extreme value of the bulk energy-dissipation rate ε_∞ plays a feature role in this formulation, but it is worthwhile to remember that ε_∞ is finite already if some other norms are bounded on the time interval, although this fact has not helped to resolve matters. The important point in these considerations is that this additional initial regularity leads to strong regularity and uniqueness, lasting as long as some specific norms such as $\|\vec{\nabla}\vec{u}(\cdot, t)\|_2$ are finite. [It is worthwhile to contrast the length scale in Equation 47 with a Kolmogorov-like scale $\lambda_K \sim (\frac{\nu^3}{\varepsilon_\infty})^{1/4}$ based on ε_∞. It is typically much smaller: $\ell \frac{\lambda_K^4}{L^3}$.]

It is not too difficult to see how initial regularity at the level of $\|\vec{\nabla}\vec{u}\|_2$ persists at least for a while, so we now outline the analysis that goes into such an argument. This reckoning leads to an explicit estimate (lower bound) for the existence time of the smooth solution and produces sufficient smallness conditions on the initial data that guarantee the existence of global (in time) smooth solutions. These calculations can be carried out for the Galerkin approximations, but neither the individual steps nor the results depend on the approximation index K, so we do not carry it.

The energy equation for smooth solutions and the Galerkin approximations is

$$\frac{d}{dt}\frac{1}{2}\|\vec{u}\|_2^2 = -\nu\|\vec{\nabla}\vec{u}\|_2^2. \tag{48}$$

To derive an evolution equation for $\|\vec{\nabla}\vec{u}(\cdot, t)\|_2^2$, we multiply the momentum equation by $-\Delta\vec{u}$, integrate over the volume, and integrate by parts to find

$$\frac{d}{dt}\frac{1}{2}\|\vec{\nabla}\vec{u}\|_2^2 = -\nu\,\|\Delta\vec{u}\|_2^2 + \int_\Omega \vec{u}\cdot\vec{\nabla}\vec{u}\cdot\Delta\vec{u}\, d^3x. \tag{49}$$

The right-hand side of Equation 49 may be bounded as follows. First,

$$\int_\Omega \vec{u}\cdot\vec{\nabla}\vec{u}\cdot\Delta\vec{u}\, d^3x \leq \|\Delta\vec{u}\|_\infty \left\|\vec{\nabla}\vec{u}\right\|_2 \|\Delta\vec{u}\|_2 \leq c\left\|\vec{\nabla}\vec{u}\right\|_2^{3/2} \|\Delta\vec{u}\|_2^{3/2}. \tag{50}$$

Here we use the Hölder and Cauchy-Schwarz inequalities followed by the fact that, for mean-zero functions in three dimensions, $\|f\|_\infty \leq c\|\vec{\nabla} f\|_2^{1/2}\|\Delta f\|_2^{3/2}$, where c is an absolute constant, an inequality that may be proven by elementary means (Doering & Gibbon 1995). Thus

$$\frac{d}{dt}\frac{1}{2}\|\vec{\nabla}\vec{u}\|_2^2 \leq -\nu\|\Delta\vec{u}\|_2^2 + c\|\vec{\nabla}\vec{u}\|_2^{3/2}\|\Delta\vec{u}\|_2^{3/2} \leq -\frac{1}{2}\nu\|\Delta\vec{u}\|_2^2 + \frac{c'}{2\nu^3}\|\vec{\nabla}\vec{u}\|_2^6, \tag{51}$$

where c' is another (computable) absolute constant. The last step employs the inequality $ab \leq a^p/p + b^q/q$ when $1/p + 1/q = 1$ (everything positive) using $p = 4$ and $q = 4/3$. Finally, integration by parts and the Cauchy-Schwarz inequality imply

$$\|\vec{\nabla}\vec{u}\|_2^2 = -\int_\Omega \vec{u}\cdot\Delta\vec{u}\, d^3x \leq \|\vec{u}\|_2\|\Delta\vec{u}\|_2 \tag{52}$$

so that

$$\frac{d}{dt}\|\vec{\nabla}\vec{u}\|_2^2 \leq -\nu\frac{\|\vec{\nabla}\vec{u}\|_2^4}{\|\vec{u}\|_2^2} + \frac{c'}{\nu^3}\|\vec{\nabla}\vec{u}\|_2^6. \tag{53}$$

Together, Equations 48 and 53 form a system of differential inequalities. In Equation 49, the second power of $\|\vec{\nabla}\vec{u}\|_2^2$ with a negative sign on the right-hand side tends to keep $\|\vec{\nabla}\vec{u}\|_2^2$ from growing, and from Equation 48, we see that $\|\vec{u}\|_2^2$ decreases monotonically with time, so this effect only increases with time. However, the higher (third) power of $\|\vec{\nabla}\vec{u}\|_2^2$ with the positive sign overwhelms the negative term if $\|\vec{\nabla}\vec{u}\|_2^2$ is too large and might allow for a finite-time blowup. Once the right-hand side of Equation 53 becomes nonpositive, however, $\|\vec{\nabla}\vec{u}\|_2^2$ can only decrease from that time on, and the right-hand side stays nonpositive. That is, if the initial data are such that the right-hand side is nonpositive, then the solution stays regular forever. An elementary dynamical systems analysis shows that we can make a slightly stronger statement: If the initial condition satisfies

$$\|\vec{u}_0\|_2^2\|\vec{\nabla}\vec{u}_0\|_2^2 < \frac{3\nu^4}{c'}, \tag{54}$$

then the ensuing solution maintains finite $\|\vec{\nabla}\vec{u}\|_2^2$ for all times, and the Navier-Stokes equations have a unique smooth solution.

Beyond providing a sufficient condition for the existence of a global unique smooth solution to the Navier-Stokes equations, this result also assures us that weak solutions for free-decay problems eventually become—and thereafter stay—smooth solutions. If $\|\vec{u}(\cdot,t)\|_2^2\|\vec{\nabla}\vec{u}(\cdot,t)\|_2^2 < 3\nu^4/c'$ at any instant of time, it subsequently remains $<3\nu^4/c'$, and the flow becomes and stays smooth. The energy inequality (Equation 30) for weak solutions implies not only that $\|\vec{u}(\cdot,t)\|_2^2 \leq \|\vec{u}_0\|_2^2$, but also that

$$\nu\int_0^t \|\vec{\nabla}\vec{u}(\cdot,s)\|_2^2 ds \leq \frac{1}{2}\|\vec{u}_0\|_2^2. \tag{55}$$

For arguments' sake, assume that $\|\vec{u}(\cdot,t)\|_2^2\|\vec{\nabla}\vec{u}(\cdot,t)\|_2^2 \geq 3\nu^4/c'$ from $t = 0$ onward. Then

$$\frac{1}{2}\|\vec{u}_0\|_2^2 \geq \frac{3\nu^5}{c'}\int_0^t \|\vec{u}(\cdot,s)\|_2^{-2}\, ds \geq \frac{3\nu^5 t}{c'}\|\vec{u}_0\|_2^{-2}, \tag{56}$$

producing a contradiction at large times. We therefore conclude that weak solutions are actually strong solutions for

$$t > \frac{c'}{6\nu^5}\|\vec{u}_0\|_2^4. \tag{57}$$

If initial data with finite $\|\vec{\nabla}\vec{u}_0\|_2^2$ initially violate Equation 54, then there is still a nonempty time interval in which $\|\vec{\nabla}\vec{u}(\cdot,t)\|_2^2$ remains bounded so that the Navier-Stokes equations have a unique smooth solution. To estimate this existence time, we note that Equation 53 implies

$$\frac{d}{dt}\|\vec{\nabla}\vec{u}\|_2^2 \leq \frac{c'}{\nu^3}\|\vec{\nabla}\vec{u}\|_2^6, \tag{58}$$

yielding

$$\|\vec{\nabla}\vec{u}(\cdot,t)\|_2^2 \leq \frac{\|\vec{\nabla}\vec{u}_0\|_2^2}{\sqrt{1-2c'\|\vec{\nabla}\vec{u}_0\|_2^4 t/\nu^3}}, \tag{59}$$

while the right-hand side is finite and real. Hence the solution is smooth and unique (at least) for

$$0 < t < \frac{\nu^3}{2c'\|\vec{\nabla}\vec{u}_0\|_2^4}. \tag{60}$$

This lower bound on time of the regularity and uniqueness depends in the same manner on the viscosity and the initial data as the transient time in Equation 45. This is because the analysis there also reduces to a differential inequality of the form of Equation 58. Of course, we can quantitatively improve this estimate for the time of existence of the unique smooth solution by utilizing all the terms in Equation 53, but it still cannot rule out a loss of regularity after just a finite time for large initial data.

At this point, we also demonstrate an assertion made above that a finite a priori bound on the variance of the energy-dissipation rate ensures regularity. Indeed, let us suppose that

$$V(t) = \int_0^t \|\vec{\nabla}\vec{u}(\cdot,s)\|_2^4 ds < \infty \tag{61}$$

for some period of time. Then, multiplying Equation 58 by a positive integrating factor, we see that

$$\frac{d}{dt}\left(\|\vec{\nabla}\vec{u}(\cdot,t)\|_2^2 \exp\left\{-\frac{c'}{\nu^3}\int_0^t \|\vec{\nabla}\vec{u}(\cdot,s)\|_2^4 ds\right\}\right) \leq 0. \tag{62}$$

The means that the quantity in braces decreases from its initial value, so

$$\|\vec{\nabla}\vec{u}(\cdot,t)\|_2^2 \leq \|\vec{\nabla}\vec{u}_0\|_2^2 \exp\left\{\frac{c'V(t)}{\nu^3}\right\} \tag{63}$$

pointwise in time as long as $V(t)$ remains finite. This is characteristic of results in 3D Navier-Stokes analysis: A variety of sufficient conditions for regularity can be derived, but none has produced an a priori verifiable criterion.

Finally, Lu (2006) recently discovered that the analysis leading from Equation 49 to Equation 58 is sharp in the sense that there is a positive constant c so that for large values of $\|\vec{\nabla}\vec{u}\|_2^2$ there exist smooth divergence-free vector fields that satisfy

$$-\nu\,\|\Delta\vec{u}\|_2^2 + \int_\Omega \vec{u}\cdot\vec{\nabla}\vec{u}\cdot\Delta\vec{u}\; d^3x = \frac{c''}{\nu^3}\|\vec{\nabla}\vec{u}\|_2^6. \tag{64}$$

This means that the sequence of estimates leading to it cannot be improved substantially, but it does not mean that a finite-time singularity develops necessarily. The relation in Equation 64 is an instantaneous one, and the evolution of the velocity field by the Navier-Stokes equations may (and in Lu's example does) drive the flow away from saturation. Rather, this result suggests that a substantially new approach to the analysis involving the dynamics in a deeper manner is required to improve on the current state of affairs.

4. DISCUSSION

The physical process responsible for the mathematical difficulties in the 3D Navier-Stokes equations is vortex stretching. The vorticity $\vec{\omega} = \vec{\nabla} \times \vec{u}$ is twice the local angular velocity of a fluid element. Its bulk mean square, the enstrophy of the flow, is (for sufficiently smooth flows in periodic domains and many others)

$$\|\vec{\omega}(\cdot, t)\|_2^2 = \|\vec{\nabla}\vec{u}(\cdot, t)\|_2^2, \tag{65}$$

one of the key quantities determining the regularity and uniqueness of solutions. If the vorticity is sufficiently regular, then all aspects of the flow are controlled. The vorticity evolves according to

$$\dot{\vec{\omega}} + \vec{u} \cdot \vec{\nabla}\vec{\omega} = \nu \Delta \vec{\omega} + \vec{\omega} \cdot \vec{\nabla}\vec{u}. \tag{66}$$

The content of this equation is simply that the angular acceleration of a fluid element (proportional to $\dot{\vec{\omega}} + \vec{u} \cdot \vec{\nabla}\vec{\omega}$) results from diffusive exchange of angular momentum with neighboring elements, i.e., the viscous torques applied by neighboring elements ($\nu \Delta \vec{\omega}$), and vortex stretching ($\vec{\omega} \cdot \vec{\nabla}\vec{u}$).

Vortex stretching can contribute to the amplification of enstrophy. At locations in the flow where $\vec{\omega} \cdot \vec{\nabla}\vec{u} \cdot \vec{\omega} > 0$, the angular velocity is predominantly aligned in a direction with a positive eigenvalue of the rate of strain matrix, the symmetric part of $\vec{\nabla}\vec{u}$. For an incompressible fluid, this means the fluid element is stretched in the direction of $\vec{\omega}$ and compressed from at least one other direction, and a relevant moment of inertia of the element is decreasing. By conservation of angular momentum, the speed of rotation tends to increase. Of course, this amplification process competes with viscous diffusion, and the mechanism can work in reverse to suppress vorticity at other locations as well, but there is no global conservation principle that relates growth to suppression in three dimensions. Vortex stretching is absent in the 2D Navier-Stokes equations (where $w = 0$ and $\partial_z = 0$), and in that case there are no such fundamental open questions concerning the existence of unique smooth solutions.

The enstrophy evolution equation obtained by integrating $\vec{\omega}$ dotted into Equation 66 is

$$\frac{d}{dt}\frac{1}{2}\|\vec{\omega}(\cdot, t)\|_2^2 = -\nu\|\vec{\nabla}\vec{\omega}\|_2^2 + \int_\Omega \vec{\omega} \cdot \vec{\nabla}\vec{u} \cdot \vec{\omega}\, d^3x, \tag{67}$$

which is the same as the evolution equation for $\|\vec{\nabla}\vec{u}(\cdot, t)\|_2^2$. The indefinite cubic term that prevents global (in time) control of solutions—or uniform in K global control of the Galerkin approximations—is precisely the vortex-stretching contribution. This suggests that the detailed study of vortex dynamics, specifically local vorticity-stretching mechanisms, could be a fruitful avenue of investigation. Indeed, one can express conditions for the avoidance of a singularity in terms of local vorticity alignments (Constantin & Fefferman 1993). However, it remains unknown whether these conditions are always realized.

Similar to the situation with the energy-dissipation rate per unit mass $\varepsilon(t)$, time averages $\overline{\|\vec{\omega}\|_2^2}$ of the enstrophy over arbitrary nonzero intervals are finite for weak solutions, but we would need to know more to confirm that solutions are regular over the interval [e.g., that the variance of the enstrophy $\overline{(\|\vec{\omega}\|_2^2 - \overline{\|\vec{\omega}\|_2^2})^2}$ is finite]. This observation highlights the importance of intermittency in Navier-Stokes dynamics (Frisch 1995, Vassilicos 2000). Intermittency refers to the phenomena of rare but large fluctuations usually associated with turbulence. If enstrophy fluctuations are bounded at the level of the time average of the second moment, then solutions are regular. However, infinite variance given finite mean, which we cannot generally rule out, necessitates the presence of large rare (in both space and time) fluctuations in vorticity that even spatial averaging cannot mollify. Whether the intermittency is contemplated in terms of the time series for spatially integrated quantities or for fluctuations in space at a specific time [e.g., the possibility of an infinite $L^3(\Omega)$

norm for $\vec{u}$ at some instant even when we know that its $L^2(\Omega)$ norm remains finite at each instant of time], our inability to determine whether such events can occur is at the root of the problem.

Many rigorous analytical results, such as sufficient conditions for the existence of unique smooth solutions to the 3D Navier-Stokes equations, can be expressed in terms of familiar quantities such as Reynolds numbers. For the decay problem, for example, natural Reynolds numbers $Re = UL/\nu$ and $Re_\lambda = U\lambda/\nu$ may be defined by the root-mean-square speed $U = \|\vec{u}\|_2/L^{3/2}$, the domain scale L, the Taylor microscale $\lambda = \|\vec{u}\|_2/\|\vec{\nabla}\vec{u}\|_2$ (with $\lambda \leq L/2\pi$ for the spatially mean-zero flows considered here), and the viscosity ν. The condition in Equation 54 that confirms the smoothness and uniqueness of solution is that the Reynolds numbers satisfy

$$Re^3 < \sqrt{\frac{3}{c'}} Re_\lambda. \tag{68}$$

However, $Re_\lambda < Re/2\pi$ so this requires that both Reynolds numbers be $O(1)$ and $\lambda > \sqrt{3/c'}\mathrm{Re}^2 L = O(L)$. If the relation in Equation 68 is violated but the initial Reynolds number $Re_0 < \infty$ and the initial Taylor microscale $\lambda_0 > 0$ are nonzero, then solutions are unique and smooth, at least while

$$0 < t < \frac{1}{2c'} \times \frac{L^2}{\nu} \times \left(\frac{\lambda_0}{L}\right)^4 \times Re_0^{-4}. \tag{69}$$

During decay from any finite-energy initial condition, the solutions eventually become and thereafter stay smooth, but according to Equation 57, we may have to wait until

$$t > \frac{c' L^2}{6\nu} Re_0^4 \tag{70}$$

for Equation 68 to be satisfied.

These observations make it clear that solutions are known to be smooth and unique only for very low Reynolds numbers, emphatically excluding regimes in which turbulence is possible. The rigorous analysis of turbulent flows, therefore, and in particular the analysis of solutions to the Navier-Stokes equations for Re_λ, $Re \gg 1$, and/or $\lambda \ll L$ generally require the consideration of weak solutions. Then the uniqueness of solutions is not known, and energy may not be conserved. Nevertheless, rigorous estimates for an assortment of physically relevant quantities are accessible, results that apply to strong, as well as weak, solutions and to fully developed turbulent, as well as laminar, flows (Foias et al. 2001).

Steady-state or statistically steady-state flows require energy sources, and flows can be driven in a variety of ways. Often a fluid is forced at or by a rigid surface, which requires formulating appropriate domain boundaries and inhomogeneous boundary conditions for the Navier-Stokes equations. For theoretical and computational studies (especially for high Reynolds numbers and turbulent flows), it is convenient to stay in a periodic domain, simply add a body force to the Navier-Stokes equations, and consider solutions to the system

$$\dot{\vec{u}} + \vec{u} \cdot \vec{\nabla}\vec{u} + \vec{\nabla} p = \nu \Delta \vec{u} + \vec{f} \tag{71}$$

and

$$\vec{\nabla} \cdot \vec{u} = 0, \tag{72}$$

where $\vec{f}(\vec{x}, t)$ is an externally applied force (per unit mass density) that we may assume (without loss of generality) is spatially mean zero and divergence-free at all times.

For finite-energy initial data and sufficiently regular forcing, for example,

$$\sup_{t\geq 0} \sum_{\vec{k}\neq 0} \frac{|\hat{f}_{\vec{k}}(t)|^2}{|\vec{k}|^2} < \infty, \tag{73}$$

there are global weak solutions satisfying an energy inequality of the form

$$\frac{1}{2}\|\vec{u}(\cdot,t)\|_2^2 + \nu \int_{t_0}^{t} \|\vec{\nabla}\vec{u}(\cdot,s)\|_2^2 ds \le \frac{1}{2}\|\vec{u}(\cdot,t_0)\|_2^2 + \int_{t_0}^{t}\int_{\Omega} \vec{u}(\cdot,s)\cdot\vec{f}(x,s)d^3x\,ds \tag{74}$$

for all t and almost all t_0 with $0 \le t_0 < t < \infty$. These weak solutions for forced flows also have finite kinetic energy at all times and finite time-averaged energy-dissipation rates (and enstrophies). Similarly, if the initial data and the forcing are sufficiently smooth, then there are unique strong solutions satisfying conservation of energy in the differential form

$$\frac{d}{dt}\frac{1}{2}\|\vec{u}(\cdot,t)\|_2^2 = -\nu\|\vec{\nabla}\vec{u}(\cdot,t)\|_2^2 + \int_{\Omega}\vec{u}(\cdot,t)\cdot\vec{f}(\cdot,t)\,d^3x, \tag{75}$$

at least for a finite time after which we cannot rule out the appearance of singularities. Moreover, if the initial data and the forcing are both sufficiently smooth and sufficiently small, then there are unique smooth solutions for all times. Not unexpectedly, rigorous requirements ensuring global regularity are generally not satisfied in situations capable of producing or sustaining turbulence.

To illustrate the kind of result one can obtain for weak solutions of the body-forced Navier-Stokes equations at high or low Reynolds numbers (Doering & Foias 2002), we consider a steady square-integrable body force, $\vec{f}(\vec{x}) \in L^2(\Omega)$, with the largest scale $\ell \le L$ (with integer L/ℓ), i.e.,

$$\vec{f}(\vec{x}) = \sum_{|\vec{k}|\ge 2\pi/\ell} \hat{f}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} = F\vec{\Phi}(\vec{x}/\ell), \tag{76}$$

where F is the amplitude and divergence-free $\vec{\Phi}(\vec{y}) \in L^2([0,1]^3)$ is the shape of the force. The amplitude F is uniquely defined, for example, if we normalize $\vec{\Phi}$ by $\|\vec{\Phi}\|_{L^2([0,1]^3)} = 1$. Then, there are global weak solutions, possibly nonunique, starting from finite-energy initial data. Let us suppose a weak solution has long-time mean square velocity $\bar{U}$ defined by

$$\bar{U} = \sqrt{\lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{\|\vec{u}(\cdot,t)\|_2^2}{L^3}dt}, \tag{77}$$

assuming for simplicity that the $T \to \infty$ limit exists. Then the long time-averaged power energy-dissipation rate per unit mass (again assuming the existence of the $T \to \infty$ limit),

$$\bar{\varepsilon} = \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{\nu\,\|\vec{u}(\cdot,t)\|_2^2}{L^3}dt, \tag{78}$$

is bounded from above by

$$\frac{\bar{\varepsilon}\ell}{\bar{U}^3} \le C_1 + \frac{C_2}{Re}, \tag{79}$$

where $Re = \bar{U}\ell/\nu$ and the coefficients C_1 and C_2 do not depend on the initial data or any of the parameters ν, F, L, or ℓ but only on finite homogeneous ratios of the body-force shape function $\vec{\Phi}$. (Such prefactor dependence on details of $\vec{\Phi}$ is inevitable given arbitrariness in the precise definition of the length scale employed in the nondimensionalization.) Both $\bar{\varepsilon}$ and $\bar{U}$ are emergent quantities [i.e., not directly set by the control parameters defining the problem (ν, F, L, ℓ, and $\vec{\Phi}$)], so Equation 79 is a conditional result in the sense that if a particular value of $\bar{U}$ (or $\bar{\varepsilon}$) is observed, then $\bar{\varepsilon}$ (or $\bar{U}$) must be bounded from above (or below) as indicated.

The rigorous result in Equation 79 is remarkable because this is precisely the way $\bar{\varepsilon}\ell/\bar{U}^3$ is expected to behave as a function of the Reynolds number. At low Reynolds numbers, $\bar{\varepsilon} \sim \nu\bar{U}^2/\ell^2$ as for Stokes flows. At high Reynolds numbers, the development of a turbulent energy cascade and the zeroth law of turbulence suggest that a residual dissipation $\bar{\varepsilon} \sim \bar{U}^3/\ell$ persists in the inviscid ($\nu \to 0$) limit of statistically steady-state turbulence (Frisch 1995). The estimate in Equation 79 is thus in accord with experience, suggesting that turbulent velocity fields tend to saturate the functional estimates utilized in its derivation. Similar energy-dissipation rate bounds have also been derived for time-dependent body forces (Petrov & Doering 2005), for forced flows in other domains (Doering et al. 2003, Petrov et al. 2005), and for boundary-driven flows (Busse 1978, Doering & Constantin 1992, Doering et al. 2000, Howard 1972, Plasting & Kerswell 2003, Wang 1997).

The parameter dependences in the estimate for $\bar{\varepsilon}$ in Equation 79 apply for square integrable body forces whose Fourier coefficients are square summable, but weak solutions exist even for rougher force fields in which case the coefficients C_1 and C_2 in the theorem above diverge. This does not mean that $\bar{\varepsilon}$ is unbounded, but rather that the Reynolds number scaling may change for flows driven by force fields that stir more directly at small scales. The analysis has been generalized recently (Cheskidov et al. 2007) to handle force shape functions $\vec{\Phi}(\vec{y}) \notin L^2([0,1]^3)$ but with

$$\sum_{\vec{\xi}\neq 0} \frac{|\hat{\Phi}_{\vec{\xi}}|^2}{|\vec{\xi}|^{2\gamma}} = 1, \tag{80}$$

where $0 < \gamma \leq 1$. In this situation, there are finite numbers $\tilde{C}_1$ and $\tilde{C}_2$ depending only on homogeneous ratios of finite norms of $\vec{\Phi}$, and not on any of the parameters ν, F, L, or ℓ, so that

$$\frac{\bar{\varepsilon}\ell}{\bar{U}^3} \leq \left(\tilde{C}_1 + \frac{\tilde{C}_2}{Re}\right)^{2/(2-\gamma)} Re^{\gamma/(2-\gamma)} \tag{81}$$

for all $0 < Re < \infty$. Interestingly, this kind of estimate is consistent with some modern theoretical and computational studies of power consumption and dissipation in flows driven by fractal forces (Biferale et al. 2004, Mazzi & Vassilicos 2004).

One particular physical effect may regularize solutions when added to the Navier-Stokes equations: rotation. The Taylor-Proudman theorem suggests that flows in a rapidly rotating frame lose their dependence on the direction along the axis of rotation. This effective two-dimensionalization has a profound effect on the behavior of solutions. Because 2D flows are known to be better behaved with regard to regularity than 3D flows, rotation can have a mollifying effect on potential problems, and given sufficiently strong rotation, correspondingly stronger rigorous results can be derived. That rotation may aid in the analysis of the 3D Navier-Stokes equations is thus not unexpected, but mathematical developments along these lines are only relatively recent (Babin et al. 1997). Rotation is a central feature of geophysical fluid dynamics (Gill 1982, Pedlosky 1987), and the rigorous mathematical theory is an active area of investigation (Chemin et al. 2006). The rigorous mathematical analysis of geophysically motivated reduced models based on the 3D Navier-Stokes equations is also a topic of great current interest (Cao & Titi 2007).

Compressible flows or inviscid fluid dynamics are not discussed above. Compressible flows generically display shock waves and solutions with discontinuities, even in one dimension (Smoller 1983). Many mathematical regularity questions of concern for the incompressible Navier-Stokes equations are thus irrelevant, but these kinds of singularities are not inconsistent with the physics that the equations are intended to describe. Inviscid incompressible fluid motion described by the Euler equations (i.e., Equations 1 and 2 without the $\nu\Delta\vec{u}$ term) is another issue altogether (Bertozzi & Majda 2002). Two-dimensional global (in time) strictly weak solutions are known to exist, most simply in the form of point-vortex solutions. For sufficiently smooth initial data, local (in time) smooth solutions exist in three dimensions, but without viscous dissipation they cannot be expected to be smoother than the initial data. However, it is not known whether any smooth (e.g., infinitely continuously differentiable) finite-energy initial data can produce a solution that displays a finite-time singularity (Gibbon 2008). Although this question for the 3D incompressible Euler equations does not have a one-million-dollar prize attached, it is as fundamental to mathematics, physics, and fluid dynamics as the 3D Navier-Stokes problem. The resolution of one of these puzzles likely will shed considerable light on the other.

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