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Lattice methods and the pressure field for solutions of the Navier–Stokes equations

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Abstract. As an alternative to removing the pressure field in regularity arguments for strong solutions of the 3D periodic Navier–Stokes equations, we show that if the pressure field $\mathcal{P}$ is assumed to be uniformly bounded for all t in $L^{15/8+\epsilon}$ ($\epsilon > 0$), then the Navier–Stokes equations are regular. The method of proof uses a so-called 'lattice theorem' which gives a set of differential inequalities for the quantities $H_{N,m} \equiv \|D^N u\|_{2m}^{2m}$ ($m \geqslant 1$). As a parallel result, this theorem also gives Serrin's $L^{3+\epsilon}$ regularity result for the velocity field.

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1. Introduction

The classical texts on the Navier–Stokes equations [1–7] have discussed the problem of 3D global regularity in great detail. Ladyzhenskaya's result [1] concerning the regularity of strong solutions; namely that the velocity field u must be assumed to be bounded in L^4 to obtain regularity†, followed by Serrin's reduction [3] of this assumption to $\|u\|_{3+\varepsilon}$, remind us how small yet how large the gap is between what must be assumed and what can be proved. In this type of work on strong solutions, it is conventional to remove the pressure field $\mathcal{P}$, usually using a nonlocal projection operator (see [5, 6]), which necessarily forces any assumptions that must be made onto the velocity field. For the incompressible Navier–Stokes equations, on periodic boundary conditions, on the domain $\Omega \equiv [0, 1]^d$ with ν as viscosity and a zero momentum condition $\int_\Omega u \,\mathrm{d}x = 0$

$$u_t + (u\cdot\nabla)u = \nu\Delta u - \nabla\mathcal{P} + f \qquad \operatorname{div} u = 0 \tag{1.1}$$

the removal of the pressure field $\mathcal{P}$ from the problem tends to be preferred because u is a dynamical variable. Because of the Boussinesq approximation, $\mathcal{P}$ obeys no independent PDE of its own. Instead, $\mathcal{P}$ is slaved to u via a Poisson equation which can be obtained simply by taking the divergence‡ of (1.1)

$$\Delta\mathcal{P} = -\sum_{i,j} u_{i,j}u_{j,i}. \tag{1.2}$$

† We use the notation $\int_\Omega |u|^s \,\mathrm{d}x \equiv \|u\|_s^s$ for the norm in L^s.

‡ In (1.1) we take f to be a divergence-free C^∞ forcing function.

Indeed, it is the solution of this equation for $\mathcal{P}$ which adds to the already great difficulties which are encountered in 3D computations of the Navier–Stokes equations. The pressure field $\mathcal{P}$ does play a role, however, in partial regularity arguments for weak solutions. Papers by Foias, Guillope and Temam [8], Struwe [9] and Caffarelli, Kohn and Nirenberg [10] have considered this.

This paper, however, is not concerned with regularity results for weak solutions [8–10] but is concerned with how assumptions on the pressure field instead of the velocity field can be used as an alternative in providing estimates for strong solutions. Removing the pressure field and loading all assumptions onto the velocity field has no great merit in itself because Serrin's $\|u\|_{3+\epsilon}$ assumption has no obvious physical interpretation. Purely as a matter of taste, it is equally possible to reverse the process by making no assumptions at all on u and then one can set about investigating the assumptions that need to be made on $\mathcal{P}$ which would give 3D regularity. This exercise has some value if the assumption which needs to be made on $\mathcal{P}$ turns out not to be too severe: for instance, if it could be reduced to $\|\mathcal{P}\|_s$ needing to be uniformly bounded for all time with s not too high. From physical arguments (e.g. Bernouilli's equation), one might expect $s = 1$ to be the sharp result. The main theorem of this paper, which will be proved in section 4, will show that if $\|\mathcal{P}\|_s$ is assumed to be uniformly bounded for all t for $s > 15/8$, then the 3D incompressible Navier–Stokes equations (1.1) are regular. While not quite sharp, this theorem means that we are close to the sharp result and that values of s can be chosen for which $s \leqslant 2$. This can be achieved through the use of the expression for $\Delta\mathcal{P}$ from (1.2) in a Galiardo–Nirenberg inequality for the pressure which re-introduces the Navier–Stokes velocity field back into the problem at that point.

To prove this theorem, which is the main result of the paper, it is necessary to prove a subsidiary result, called a lattice theorem, which is proved in section 3. This generalizes the idea of what the authors have called a ladder theorem [11], first introduced in [12, 13] for the complex Ginzburg–Landau equation. For clarity, this is explained briefly in the next section. Both the ladder and lattice theorems also give the $\|u\|_{3+\epsilon}$ regularity result, as they should if they are sharp.

2. Ladders and lattices

2.1. A summary of the ladder structure

In [11] it was shown how a 'ladder' could be constructed which generalizes the bootstrapping idea [4, 5] through which the velocity field in one Sobolev space can be controlled using the bounds on lower spaces. In this subsection we give a quick summary. Define a set of quantities in d dimensions ($d = 2, 3$)

$$H_N = \sum_{i=1}^{d} \sum_{|n|=N} \int |D^n u_i|^2 \,\mathrm{d}x \equiv \|D^N u\|_2^2. \tag{2.1}$$

D^n is the usual notation for all derivatives of order n in d dimensions where n is a multi-index such that $n_1 + n_2 + \cdots + n_d = |n| = N$. Because $\operatorname{div} u = 0$, it is also possible to write (2.1) as

$$H_N = \int |\operatorname{curl}^N u|^2 \,\mathrm{d}x \tag{2.2}$$

where $\mathrm{curl}^N u$ means taking the curl of u, N times. In [11] the following pair of differential inequalities were found

$$\tfrac{1}{2}\dot{H}_N \leqslant -\nu H_{N+1} + cH_N \|Du\|_\infty + H_N^{1/2}\mathcal{F}_N^{1/2} \tag{2.3}$$

$$\tfrac{1}{2}\dot{H}_N \leqslant -\nu H_{N+1} + cH_N^{1/2}H_{N+1}^{1/2}\|u\|_\infty + H_N^{1/2}\mathcal{F}_N^{1/2} \tag{2.4}$$

where $\mathcal{F}_N = \sum_i \|D^N f_i\|_2^2$. This result is not unlike that of Beale, Kato and Majda [14] for the Euler equations and (2.3) reduces to their result when $\nu = 0$. To get control over the H_{N+1} we now need one further step to get a differential inequality where we can obtain a set of absorbing balls.

Theorem 1. For each d, $N \geqslant 1$ and $1 \leqslant s \leqslant N$

$$\frac{1}{2}\dot{H}_N \leqslant -\nu \frac{H_N^{1+1/s}}{H_{N-s}^{1/s}} + cH_N \|Du\|_\infty + H_N^{1/2}\mathcal{F}_N^{1/2} \tag{2.5}$$

$$\frac{1}{2}\dot{H}_N \leqslant -\frac{\nu}{2} \frac{H_N^{1+1/s}}{H_{N-s}^{1/s}} + c\frac{H_N \|u\|_\infty^2}{\nu} + H_N^{1/2}\mathcal{F}_N^{1/2}. \tag{2.6}$$

Proof of theorem 1. We use (2.3) and (2.4) together with the following lemma:

Lemma 1. If, on periodic boundary conditions, we define $H_N = \sum \int |D^n u_i|^2$ then $\forall r, s, N \in \mathbb{N}$ with $s \leqslant N$,

$$H_N \leqslant H_{N+r}^{s/r+s} H_{N-s}^{r/r+s}. \tag{2.7}$$

Proof of Lemma. Step 1. Firstly, let $\tilde{H}_M = \sum \int |D^m \phi_i|^2$. Now we show that

$$\tilde{H}_M \leqslant \tilde{H}_{M+1}^{1/2}\tilde{H}_{M-1}^{1/2} \tag{2.8}$$

$$\tilde{H}_M = -\sum_{i,M} \int (D^{m+1}\phi_i)(D^{m-1}\phi_i) \leqslant \left[\sum \int (D^{m+1}\phi_i)^2\right]^{1/2} \left[\sum \int (D^{m-1}\phi_i)^2\right]^{1/2} \tag{2.9}$$

using the Cauchy–Schwarz inequality.

Step 2. Secondly, we show that $\forall M \in \mathbb{N}$,

$$\tilde{H}_M \leqslant \tilde{H}_{M+1}^{M/(M+1)}\tilde{H}_0^{1/(M+1)} \tag{2.10}$$

To achieve this, we know from (2.8) that (2.10) holds for $M = 1$. Assume (2.10) holds for M. Then

$$\tilde{H}_{M+1} \leqslant \tilde{H}_{M+2}^{1/2}\tilde{H}_M^{1/2} \leqslant \tilde{H}_{M+2}^{1/2}\tilde{H}_{M+1}^{M/2(M+1)}\tilde{H}_0^{1/2(M+1)} \tag{2.11}$$

so

$$\tilde{H}_{M+1} \leqslant \tilde{H}_{M+2}^{(M+1)/(M+2)}\tilde{H}_0^{1/(M+2)}. \tag{2.12}$$

Hence (2.10) is true $\forall M \in \mathbb{N}$ by induction.

Step 3. Thirdly, we show that $\forall M, r \in \mathbb{N}$

$$\tilde{H}_M \leqslant \tilde{H}_{M+r}^{M/(M+r)} \tilde{H}_0^{r/(M+r)}. \tag{2.13}$$

We know from (2.10) that (2.13) holds for $r = 1$. Assume (2.13) holds for r. Then

$$\tilde{H}_M \leqslant \tilde{H}_{M+r}^{M/(M+r)} \tilde{H}_0^{r/(M+r)} \leqslant \tilde{H}_{M+r+1}^{M/(M+r+1)} \tilde{H}_0^{(r+1)/(M+r+1)} \tag{2.14}$$

where we have used (2.10). Hence (2.13) is true $\forall M, r \in \mathbb{N}$, by induction.

Step 4. We know (2.13) holds with

$$H_M = \sum \int |D^m \phi_i|^2. \tag{2.15}$$

Now suppose that originally $\phi_i = D^{N-M} u_i$, then

$$\tilde{H}_M = H_N \qquad \text{and} \qquad \tilde{H}_0 = H_{N-M}. \tag{2.16}$$

Then (2.13) becomes,

$$\Psi_{M+p} \leqslant \Psi_{M+p+K}^{M/(M+K)} \Psi_p^{K/(M+K)}$$

and so with $M = s$ we have

$$H_N \leqslant H_{N+r}^{s/s+r} H_{N-s}^{r/s+r}.$$

Thus we have proved theorem 1. □

2.2. *Consequence of the ladder structure*

To find absorbing balls for the H_N we must find some control over either the $\|Du\|_\infty$ or $\|u\|_\infty$ terms and the next lowest rung of the ladder. To achieve this, the $\|Du\|_\infty$ term or the $\|u\|_\infty$ term can be bounded above using a Gagliardo–Nirenberg inequality [15].

$$\|Du\|_\infty^2 \leqslant c H_{N+1}^a H_0^{(1-a)} \tag{2.18}$$

where $a = (d+2)/[2(N+1)]$ which implies that $2N > d$. Alternatively

$$\|u\|_\infty^2 \leqslant c H_N^b H_0^{(1-b)} \tag{2.19}$$

where $b = d/2N$ and and so again $2N > d$. In the first case, going back a step to (2.3), we can peel off the H_{N+1} from the central term using a Young's inequality to combine with the $-\nu H_{N+1}$ term and then appeal to (2.7). Indeed, this is the slightly sharper of the two alternatives:

$$\frac{1}{2}\dot{H}_N \leqslant -\frac{\nu}{2}(2-a)\frac{H_N^2}{H_{N-1}} + c_N[\nu^{-a} H_N^2]^{1/(2-a)} + H_N^{1/2} \mathcal{F}_N^{1/2} \tag{2.20}$$

where we have absorbed a term in H_0 into the constant as it is bounded above. Because we are restricted by $2N > d$ to ensure $a < 1$ then, for $d = 2$ or 3, we must choose $N \geqslant 2$.

Consequently, absorbing balls exist for all H_N provided one has control over H_1 for large times. Returning to (2.2), we can see that the quantity H_1 is

$$H_1 = \int |\operatorname{curl} u|^2 \, \mathrm{d}x = \int |\omega|^2 \, \mathrm{d}x \tag{2.21}$$

which is the enstrophy. It is well known that when $d = 2$ this can be bounded above using a maximum principle. No such bound is known when $d = 3$. Hence we have an attractor [16] made up from C^∞ functions for $d = 2$ but nothing more than an L^2 bound on the velocity field when $d = 3$.

It is here where the ladder structure shows that control over H_1 is a sufficient condition for a ball for all the H_N but, as we have said earlier, this is by no means the weakest result. One of the classical results of Navier–Stokes analysis (see references in [3–5]) is that the assumption that the velocity field is uniformly bounded in $L^{3+\epsilon}$ is sufficient to show regularity. To achieve this from the ladder requires the following procedure. We not only use the ladder where we step along in gradients but also step up in L^p to form a 'lattice' which is, in effect, a $W^{N,p}$-space although we keep only highest derivatives. Consequently, we define

$$H_{N,m} = \sum_{i=1}^{d} \sum_{|n|=N} \int |D^n u_i|^{2m} \, \mathrm{d}x \equiv \|D^N u\|_{2m}^{2m} \tag{2.22}$$

where $m \geqslant 1$, for which we can prove the following theorem.

Theorem 2. For $N > 1 + 3/2m$ and $K > N + \frac{3}{2}(1 - 1/m)$ we have

$$\overline{\lim_{t\to\infty}} \|Du\|_\infty \leqslant c \left[\overline{\lim_{t\to\infty}} H_{0,m} \right]^{(1-AB)/[m(2-ABK)]} \tag{2.23}$$

where $A = 2mN/[2mK - 3(m-1)]$ and $B = (2m+3)/2mN$ with $ABK < 2$.

Remark. The latter condition, $ABK < 2$, means that

$$m > \frac{3K-6}{2K-6} > \frac{3}{2}. \tag{2.24}$$

Consequently, the object which controls $\|Du\|_\infty$ and all the H_N is $H_{0,3/2+\epsilon}$ $(\epsilon > 0)$, which is Serrin's result.

Proof. Firstly we use a Gagliardo–Nirenberg inequality to obtain $(m \geqslant 1)$

$$H_{N,m} \leqslant c H_{K,1}^{Am} H_{0,m}^{1-A} \tag{2.25}$$

where $A = 2mN/[2mK - 3(m-1)]$ and $K > N + \frac{3}{2}(1 - 1/m)$. Secondly, iterate the ladder (2.5) K times to get

$$\overline{\lim_{t\to\infty}} H_{K,1} \leqslant c \left[\nu^{-1} \overline{\lim_{t\to\infty}} \|Du\|_\infty \right]^K \overline{\lim_{t\to\infty}} H_{0,1}. \tag{2.26}$$

In (2.26) we are taking the leading order term only: the forcing produces terms of lower order. Thirdly, using another Gagliardo–Nirenberg inequality

$$\|Du\|_\infty^{2m} \leqslant c H_{N,m}^{B} H_{0,m}^{1-B} \tag{2.27}$$

where $B = (2m+3)/2mN$ so $N > 1+3/2m$. Putting (2.25), (2.26) and (2.27) together, we can see that to get $\|Du\|_\infty$ controlled by $H_{0,m}$ for large t, we must have $ABK < 2$ which, in turn, means that $m > (3K-6)/(2K-6)$, and so (2.23) holds only for m satisfying (2.24). □

3. The lattice structure

We can fully generalize the ladder theorem in the following way and at the same time obtain control over $\overline{\lim}_{t\to\infty} H_{1,2}$ (a special case of the lattice theorem outlined) which is essential for proving the consequent theorem for the pressure. This new generalized structure also produces the $\|u\|_{3+\epsilon}$ result quite naturally (as we should expect).

3.1. The lattice theorem

Theorem 3. For $d = 3$, $N \geqslant 1$ and $1 \leqslant m \leqslant 2$, the $\dot{H}_{N,m}$ satisfy the following pair of differential inequalities,

$$\frac{1}{2m}\dot{H}_{N,m} \leqslant -\nu c_1 \frac{H_{N,m}^{1+1/m}}{H_{N-1,m}^{1/m}} + c_{N,m}\nu^{-3/5} H_{N,m}\|Du\|_4^{8/5} \tag{3.1}$$

$$\frac{1}{2m}\dot{H}_{N,m} \leqslant -\nu c_1 \frac{H_{N,m}^{1+1/m}}{H_{N-1,m}^{1/m}} + c_{N,m}\nu^{\frac{-p}{(1-p)}} H_{N,m}^{1+\frac{1}{2mq}} \|u\|_4^{(N-1)/q} \tag{3.2}$$

where $p = 3[m(N-1)+2]/8mN$ and $q = N(1-p)$.

Proof of theorem 3. From the incompressible Navier–Stokes equations and our definition for $H_{N,m}$,

$$\begin{aligned} \frac{1}{2m}\dot{H}_{N,m} &= \sum_{i=1}^{d}\sum_{|n|=N}\int (D^n u_i)^{2m-1} D^n\left[-(u\cdot\nabla)u_i + \nu\Delta u_i - \mathcal{P}_{,i} + f_i\right] \\ &= T_{\rm NL} + T_{\rm L} + T_{\rm P} + T_{\rm F}. \end{aligned} \tag{3.3}$$

3.1.1. The Laplacian term T_L.
We see that using parts gives

$$T_{\rm L} = \nu\sum_i\sum_N \int (D^n u_i)^{2m-1} D^n \Delta u_i = -\nu\frac{(2m-1)}{m^2}\sum_{i,k}\sum_N\int\left[\partial_k\left((D^n u_i)^m\right)\right]^2. \tag{3.4}$$

Consequently, if we define

$$B_{i,n}^{(m)} = B_m = (D^n u_i)^m \tag{3.5}$$

so

$$\|B_m\|_2^2 = H_{N,m} \tag{3.6}$$

then we find that

$$T_{\rm L} = -\nu\frac{(2m-1)}{m^2}\|DB_m\|_2^2. \tag{3.7}$$

Now, note that if we perform integration by parts on $H_{N,m}$ and then use Cauchy's inequality,

$$H_{N,m}^2 \leqslant (2m-1)^2\|DB_m\|_2^2\sum_i\sum_N\int |D^n u_i|^{2(m-1)}|D^{n-1}u_i|^2. \tag{3.8}$$

Using Hölder's inequality, it turns out that

$$H_{N,m}^2 \leqslant (2m-1)^2\|DB_m\|_2^2 H_{N,m}^{1-1/m} H_{N-1,m}^{1/m}. \tag{3.9}$$

Consequently,

$$-\|DB_m\|_2^2 \leqslant -\frac{1}{(2m-1)^2}\frac{H_{N,m}^{1+1/m}}{H_{N-1,m}^{1/m}} \tag{3.10}$$

which gives us the expression for the Laplacian term in theorem 3.

3.1.2. The pressure term T_P.

$$|T_P| = \left|\sum_i \sum_N \int (D^n u_i)^{2m-1} D^n(\mathcal{P}_{,i})\right| \leqslant H_{N,2m-1}^{1/2} T_S^{1/2} \tag{3.11}$$

where

$$T_S = \sum_i \sum_N \int (D^n \mathcal{P}_{,i})^2 = \sum_N \int (D^n \nabla \mathcal{P})^2 = \sum_{N-1} \int (D^{(n-1)} \Delta \mathcal{P})^2. \tag{3.12}$$

Now we prove,

Lemma 2.

(i) $$\Delta \mathcal{P} = -\sum_{i,j} u_{i,j} u_{j,i} \tag{3.13}$$

(ii) $$\|\Delta \mathcal{P}\|_r \leqslant c H_{1,r}^{1/r} \qquad \forall r \geqslant 1. \tag{3.14}$$

Proof of lemma. (i) Taking the divergence of the Navier–Stokes equations gives

$$\Delta \mathcal{P} = -\nabla \cdot (u \cdot \nabla u) = -\partial_i (u_j u_{i,j}) = -u_{i,j} u_{j,i}. \tag{3.15}$$

(ii) Now simply take the rth-norm of both sides, and apply Cauchy's inequality,

$$\|\Delta \mathcal{P}\|_r^r \leqslant \sum_{i,j} \int |u_{i,j}|^r |u_{j,i}|^r \leqslant H_{1,r} \tag{3.16}$$

and hence the result. □

Remark. Note that we have assumed a divergence-free forcing function for clarity of argument only—we do *not* need to make this assumption; theorem 3 would still be correct (we simply get lower order terms in $H_{N,m}$).

We can deal with the T_S-term as follows:

$$T_S = \sum_{N-1} \int \left|\sum_{i,j} D^{n-1}(u_{i,j} u_{j,i})\right|^2. \tag{3.17}$$

Using the Schwarz inequality and Leibnitz's theorem:

$$T_S \leqslant d^2 \sum_{i,j} \sum_{N-1} \int \left|\sum_\ell C_\ell^{n-1} D^\ell(u_{i,j}) D^{n-1-\ell}(u_{j,i})\right|^2. \tag{3.18}$$

Next, we define

$$A_{i,j}^{(\ell)} = \sum_{N-1} \int |D^{\ell+1_j} u_i|^2 |D^{(n-1)-\ell+1_i} u_j|^2 \tag{3.19}$$

where $i, j = 1, \ldots, d$; $n = (n_1, n_2, \ldots, n_d)$, $\ell = (\ell_1, \ell_2, \ldots, \ell_d)$ are multi-indices. $(n-1)$ is also a multi-index (given in this form for notational purposes only) such that $|(n-1)| = (n-1)_1 + \cdots + (n-1)_d = N-1$. From the Leibnitz operation we must have $\ell_i \leqslant (n-1)_i$, $\forall i$.

Consequently,

$$T_S \leqslant d^2 \sum_{i,j} \sum_\ell C_\ell^N A_{i,j}^{(\ell)}. \tag{3.20}$$

A Hölder inequality gives

$$A_{i,j}^{(\ell)} \leqslant \|D^{\ell+1_j} u_i\|_p^2 \|D^{(n-1)-\ell+1_i} u_j\|_q^2 \tag{3.21}$$

where $1/p + 1/q = \frac{1}{2}$. There are two paths we can now follow:

(i) The $\|Du\|_4$ lattice. Consider the following set of Gagliardo–Nirenberg inequalities,

$$\|D^{\ell+1}u_i\|_p \leqslant c\|D^n u_i\|_r^a \|Du_i\|_s^{1-a} \tag{3.22}$$

$$\|D^{(n-1)-\ell+1}u_j\|_q \leqslant c\|D^n u_j\|_r^b \|Du_j\|_s^{1-b} \tag{3.23}$$

where $1/p + 1/q = \frac{1}{2}$, and we also require

$$\frac{1}{p} = \frac{L}{d} + a\left(\frac{1}{r} - \frac{N-1}{d}\right) + \frac{1-a}{s} \qquad 0 \leqslant \frac{L}{N-1} \leqslant a < 1 \tag{3.24}$$

$$\frac{1}{q} = \frac{N-L-1}{d} + b\left(\frac{1}{r} - \frac{N-1}{d}\right) + \frac{1-b}{s} \qquad 0 \leqslant \frac{N-L-1}{N-1} \leqslant b < 1. \tag{3.25}$$

Choose

$$\frac{1}{ap} = \frac{1}{r} + \frac{1-a}{as} \Rightarrow a = \frac{L}{N-1} \qquad \forall d \tag{3.26}$$

$$\frac{1}{bq} = \frac{1}{r} + \frac{1-b}{bs} \Rightarrow b = \frac{N-L-1}{N-1} \qquad \forall d. \tag{3.27}$$

Hence

$$0 \leqslant a < 1 \iff 0 \leqslant L < N-1 \tag{3.28}$$

$$0 \leqslant b < 1 \iff 0 \leqslant N-L-1 < N-1 \tag{3.29}$$

where in fact equality also holds above by inspection. Since $a+b=1$ and $1/p+1/q=\frac{1}{2}$ we see that we must have

$$\frac{1}{r} + \frac{1}{s} = \frac{1}{2}. \tag{3.30}$$

Therefore,

$$A_{i,j}^{(\ell)} \leqslant c\|D^n u\|_r^2 \|Du\|_s^2 \tag{3.31}$$

and we get

$$T_{\mathrm{P}} \leqslant cH_{N,2m-1}^{1/2} H_{N,r/2}^{1/r} \|Du\|_s. \tag{3.32}$$

Since $1/r + 1/s = \frac{1}{2}$, a convenient, a natural choice here is $s = r = 4$, which gives

$$T_{\mathrm{P}} \leqslant cH_{N,2m-1}^{1/2} H_{N,2}^{1/4} \|Du\|_4. \tag{3.33}$$

Now, note that we can write

$$H_{N,2m-1}^{1/2} \equiv \|B_m\|_{\frac{2}{m}(2m-1)}^{(2m-1)/m} \qquad \text{and} \qquad H_{N,r/2}^{1/r} \equiv \|B_m\|_{r/m}^{1/m} \tag{3.34}$$

and we can perform the following two Gagliardo–Nirenberg inequalities:

$$\|B_m\|_{\frac{2}{m}(2m-1)} \leqslant c\|DB_m\|_2^{a_1} \|B_m\|_2^{1-a_1} \tag{3.35}$$

$$\|B_m\|_{r/m} \leqslant c\|DB_m\|_2^{a_2} \|B_m\|_2^{1-a_2} \tag{3.36}$$

where $a_1 = 3(m-1)/2(2m-1)$ and $a_2 = 3(r-2m)/2r$ and where we must restrict ourselves to $1 \leqslant m \leqslant 2$ when $r = 4$. Combining these two inequalities in our expression above for T_{P}, we find

$$T_{\mathrm{P}} \leqslant c\left[\|DB_m\|_2^2\right]^{3/8} H_{N,m}^{5/8} \|Du\|_4 \qquad 1 \leqslant m \leqslant 2. \tag{3.37}$$

If we now use a Young's inequality (multiply and divide by $\nu^{3/8}$) to peel off the $\|DB_m\|_2$ term in (3.37) and combine it with the Laplacian term, then we obtain the $\|Du\|_4$ lattice.

(ii) The $\|u\|_4$ lattice. Instead of the Gagliardo–Nirenberg inequalities employed above, we consider the following set

$$\|D^{\ell+1_j}u_i\|_p \leqslant c\|D^n u_i\|_r^a \|u_i\|_s^{1-a} \tag{3.38}$$

$$\|D^{(n-1)-\ell+1_i}u_j\|_q \leqslant c\|D^n u_j\|_r^b \|u_j\|_s^{1-b} \tag{3.39}$$

where $1/p + 1/q = \frac{1}{2}$ and $r, s \geqslant 1$, and we require

$$\frac{1}{p} = \frac{L+1}{d} + a\left(\frac{1}{r} - \frac{N}{d}\right) + \frac{1-a}{s} \qquad 0 \leqslant \frac{L+1}{N} \leqslant a < 1 \tag{3.40}$$

$$\frac{1}{q} = \frac{N-L}{d} + b\left(\frac{1}{r} - \frac{N}{d}\right) + \frac{1-b}{s} \qquad 0 \leqslant \frac{N-L}{N} \leqslant b < 1. \tag{3.41}$$

If we choose

$$\frac{1}{ap} = \frac{1}{r} + \frac{1-a}{as} \Rightarrow a = \frac{L+1}{N} \qquad \forall d \tag{3.42}$$

$$\frac{1}{bq} = \frac{1}{r} + \frac{1-b}{bs} \Rightarrow b = \frac{N-L}{N} \qquad \forall d \tag{3.43}$$

then,

$$0 \leqslant a < 1 \iff 0 \leqslant L+1 < N \tag{3.44}$$

$$0 \leqslant b < 1 \iff 0 \leqslant N - L < N \tag{3.45}$$

where equality also holds here, by inspection. Since we require $1/p + 1/q = \frac{1}{2}$, this means we must have

$$r = \frac{2s(N+1)}{sN - 2(N-1)}. \tag{3.46}$$

Hence, since $a + b = (N+1)/N$ we get

$$T_{\rm P} \leqslant dcH_{N,2m-1}^{1/2} H_{N,r/2}^{(N+1)/rN} \|u\|_s^{(N-1)/N}. \tag{3.47}$$

We can now see that when $N = 1$, then we must have $r = 4$, independent of s, and further, for general N, if we choose† $s = 4$ then r is again exactly equal to 4 (independent of N). With this choice,

$$T_{\rm P} \leqslant dcH_{N,2m-1}^{1/2} H_{N,2}^{(N+1)/4N} \|u\|_4^{(N-1)/N}. \tag{3.48}$$

Since from (3.34) we know that $H_{N,r/2} \equiv \|B_m\|_{r/m}^{r/m}$, we can use the inequalities (3.35) and (3.36) to get

$$T_{\rm P} \leqslant dc\left[\|DB_m\|_2^2\right]^{\tilde{p}} H_{N,m}^{1-\tilde{p}+\frac{1}{2mN}} \|u\|_4^{(N-1)/N} \tag{3.49}$$

where $\tilde{p} = 3[m(N-1)+2]/8mN$. Consequently, an application of Young's inequality gives us the $\|u\|_4$ lattice.

† Another choice would be $r = 2(2m-1)$ or $r = 2m$ and leave the parameter s free. In the latter case, convert (3.10) into an inequality purely in terms of $H_{N,m}$ and $\|u\|_s$ and get a condition on s for an absorbing ball. This condition turns out to be exactly $s > 3$, thereby reproducing the results of theorem 3.

3.1.3. The nonlinear and forcing terms T_{NL} and T_F. Although it is possible to bound the nonlinear term above as in [11], it is also possible to show that it has an upper bound proportional to the pressure term.

$$T_{\mathrm{NL}} = -\sum_{i,j}\sum_{N}\int |D^n u_i|^{2m-1}|D^n(u_j u_{i,j})| \qquad N \geqslant 1. \tag{3.50}$$

Consequently, we can use Leibnitz's theorem to obtain

$$T_{\mathrm{NL}} = -\sum_{i,j}\sum_{N}\int |D^n u_i|^{2m-1}\left|\sum_{\ell\neq 0} C_\ell^n D^\ell u_j D^{n-\ell}u_{i,j}\right|$$

where integration by parts reveals that the $\ell = 0$ term is zero. Cauchy's inequality implies

$$T_{\mathrm{NL}} \leqslant H_{N,2m-1}^{1/2}\left[\sum_{i,j}\sum_{N}\sum_{\ell\neq 0} C_\ell^n \int |D^\ell u_j|^2 |D^{n-\ell}u_{i,j}|^2\right]^{1/2}. \tag{3.51}$$

It is now easy to see that we can deal with the nonlinear term by following a procedure very similiar to that previously used for the pressure term. Note that $\ell \neq 0$ allows us to find the appropriate upper bound for the nonlinear term.

The forcing term can be bounded with a single application of Hölder's inequality, as follows:

$$T_{\mathrm{F}} \leqslant \sum_{i}\sum_{N}\int |D^n u_i|^{2m-1}|D^n f_i| \leqslant H_{N,m}^{1-1/2m}\|D^N f\|_{2m}. \tag{3.52}$$

Thus we have now proved both parts of theorem 3. □

3.2. The $\|u\|_{3+\epsilon}$ result from the lattice

Note that in (3.2) the bottom point of our lattice can be found by letting $N = 1$; the $\|u\|_4$ term vanishes and an absorbing ball occurs when $2q > 1$. This implies that when $N = 1$, we must have $m > 3/2$ and so $H_{0,3/2+\epsilon}$ is the bottom point. For general $N \geqslant 1$, we can find an absorbing ball via the following Gagliardo–Nirenberg inequality,

$$\|u\|_4 \leqslant c H_{N,m}^{a_2/2m} H_{0,m}^{(1-a_2)/2m} \tag{3.53}$$

where $a_2 = 3(2-m)/4Nm$, and we have the restriction $1 \leqslant m \leqslant 2$, so that we get for $N \geqslant 1$,

$$\overline{\lim_{t\to\infty}} H_{N,m} \leqslant c\,\overline{\lim_{t\to\infty}}\left[\nu^{-8N^2m^2} H_{N-1,m}^{N\alpha} H_{0,m}^{(\alpha-Nm)(N-1)}\right]^{1/\beta} \tag{3.54}$$

where $\alpha = [5Nm + 3m - 6]$ and $\beta = [\alpha(N-1) - 6N + 4Nm]$.

Alternatively, from (3.1), $\|Du\|_4$ can be controlled by

$$\|Du\|_4 \leqslant c H_{N,m}^{a_1/2m} H_{0,m}^{(1-a_1)/2m} \tag{3.55}$$

where $a_1 = (m+6)/4Nm$ and $1/N \leqslant a_1 < 1$ means that for $N = 2$ we must restrict ourselves to $1 \leqslant m \leqslant 2$, and

$$H_{N-1,m} \leqslant c H_{N,m}^{1-\frac{1}{N}} H_{0,m}^{1/N}. \tag{3.56}$$

If we substitute (3.55) and (3.56) into the lattice (3.1) and look for the absorbing ball, we find that

$$\overline{\lim_{t\to\infty}} H_{N,m} \leqslant c\,\overline{\lim_{t\to\infty}}\left[\nu^{-4m^2N} H_{0,m}^{[2m(N+1)-3]}\right]^{1/(2m-3)} \qquad N \geqslant 2,\ m > 3/2. \tag{3.57}$$

It is also transparent from (3.57) that $H_{0,3/2+\epsilon}$ is the bottom point of this lattice.

4. Boundedness of $\|\mathcal{P}\|_s (s > 15/8)$ and regularity

We now come to the main theorem of the paper, the proof of which depends strongly on theorem 3.

Theorem 4. If $\|\mathcal{P}\|_s$ is assumed to be uniformly bounded for all t for $s > 15/8$, then the 3D incompressible Navier–Stokes equations (1.1) are regular.

Proof of theorem 4. We begin with the following.

Lemma 3. Provided $m \geqslant 2$ and with $(m-1)/(2m+1) < \delta \leqslant m-1$,

$$\frac{1}{2m}\dot{H}_{0,m} \leqslant -\nu c_{1,m}\frac{H_{0,m}^{(2m+1)/3}}{H_{0,m-1}^{2m/3}} + c_{2,m}H_{0,m}^{\beta_m/\gamma_m}\left(\|\mathcal{P}\|_{2(1+\delta)}\right)^{2m/\gamma_m} \tag{4.1}$$

where

$$\beta_m = (1-a)(m-1) \qquad \gamma_m = m - a(m-1) \tag{4.2}$$

and

$$a = \frac{3}{2}\left[1 - \frac{m\delta}{(m-1)(\delta+1)}\right]. \tag{4.3}$$

Proof of lemma 3. Beginning with $H_{0,m} = \int |u|^{2m}$ and differentiating with respect to time gives

$$\frac{1}{2m}\dot{H}_{0,m} \leqslant -\nu(2m-1)\int |Du|^2|u|^{2(m-1)} + \left|\int (\nabla\mathcal{P})u^{2m-1}\right|. \tag{4.4}$$

Now we take the last term, integrate by parts and use a Hölder inequality

$$T_{\mathcal{P}} = \left|\int (\nabla\mathcal{P})u^{2m-1}\right| \leqslant (2m-1)\int |\mathcal{P}||Du||u|^{2(m-1)}$$
$$\leqslant (2m-1)\|\mathcal{P}\|_{2(1+\delta)}\|u\|_{2m\eta}^{m-1}\left[\int |Du|^2|u|^{2(m-1)}\right]^{1/2} \tag{4.5}$$

where $\delta/(1+\delta) = (m-1)/m\eta$. Now write

$$\|u\|_{2m\eta} = \|u^m\|_{2\eta}^{1/m} \leqslant c\left(\int |Du|^2|u|^{2(m-1)}\right)^{a/2m} H_{0,m}^{(1-a)/2m} \tag{4.6}$$

where we have used a Gagliardo–Nirenberg inequality in 3D with $a = 3[\eta-1]/2\eta$. Since $0 \leqslant a < 1$ we find that $1 \leqslant \eta < 3$ which, in turn, implies that δ must lie in the range

$$\frac{m-1}{2m+1} < \delta \leqslant m-1. \tag{4.7}$$

We see now that the pressure term becomes

$$T_{\mathcal{P}} \leqslant c(2m-1)\|\mathcal{P}\|_{2(1+\delta)}\left(\int |Du|^2|u|^{2(m-1)}\right)^{[m+a(m-1)]/2m} H_{0,m}^{(1-a)(m-1)/2m}. \tag{4.8}$$

We can peel off the $\int |Du|^2|u|^{2(m-1)}$ term using a Young's inequality and combine it with the same term from the Laplacian and then use interpolation on this term itself ($d = 3$)

$$-m^2\int |Du|^2\,|u|^{2(m-1)} = -\int |D(u^m)|^2 \leqslant -c\frac{H_{0,m}^{(2m+1)/3}}{H_{0,m-1}^{2m/3}} \tag{4.9}$$

We have now proved the lemma. □

Now consider the following four steps:

A. We know that if $\|u\|_4^4 \equiv H_{0,2}$ is controlled for all t then this is a sufficient condition for regularity [1]. This is also the conclusion that can be drawn from theorem 3 (3.2). Note that this is also the conclusion of theorem 2. We also know that $\|u\|_2^2 \equiv H_{0,1}$ is also controlled for all t. Next we appeal to lemma 3 using the value $m = 2$ to obtain

$$\overline{\lim}_{t\to\infty} H_{0,2} \leqslant \nu^{-3(7\delta+1)/4(5\delta+2)} \left(\overline{\lim}_{t\to\infty} H_{0,1}\right)^{(7\delta+1)/(5\delta+2)} \left(\overline{\lim}_{t\to\infty} \|\mathcal{P}\|_{2(1+\delta)}\right)^{[6(1+\delta)]/(5\delta+2)} \tag{4.10}$$

where δ lies in the range $1/5 < \delta \leqslant 1$. This is sufficient to give regularity if one assumes that $\|\mathcal{P}\|_{2(1+\delta)}$ is bounded for all t.

B. In fact, we can do better than this and weaken the condition on $\mathcal{P}$ even further. To do this we use (1.2) in a Gagliardo–Nirenberg inequality to obtain

$$\|\mathcal{P}\|_{2(1+\delta)} \leqslant c\|\Delta\mathcal{P}\|_2^b \|\mathcal{P}\|_s^{1-b} \leqslant cH_{1,2}^{b/2} \|\mathcal{P}\|_s^{1-b} \tag{4.11}$$

where $s \leqslant 2(1+\delta)$ and is to be determined. The exponent $b(s, \delta)$ is given by

$$b(s, \delta) = 3\frac{2(1+\delta) - s}{(1+\delta)(s+6)}. \tag{4.12}$$

C. To perform the next step, we need to control $H_{1,2}$ by $H_{0,2}$. This is conveniently furnished from theorem 3 (3.54) by choosing $N = 1$ and $m = 2$ to give

$$\overline{\lim}_{t\to\infty} H_{1,2} \leqslant c\nu^{-16} \left(\overline{\lim}_{t\to\infty} H_{0,2}\right)^5. \tag{4.13}$$

Note that both sides match dimensionally so the constant is dimensionless.

D. Using the results from B and C in A we easily find that $\overline{\lim}_{t\to\infty} H_{0,2}$ is controlled by $\overline{\lim}_{t\to\infty} H_{0,1}$ and $\|\mathcal{P}\|_s$ provided

$$1 > 12b(s, \delta)\left(\frac{1+\delta}{5\delta+2}\right) \tag{4.14}$$

which yields

$$s > \frac{78 + 60\delta}{47 + 5\delta}. \tag{4.15}$$

Since δ lies in the range $1/5 < \delta \leqslant 1$ we find that any choice of s which satisfies $s > 15/8$ will do. Hence the assumption that $\|\mathcal{P}\|_s$ is bounded for all t is enough to control $H_{0,2} \equiv \|u\|_4^4$ and hence give C^∞ regularity. □

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