Lecture notes

Ideal fluid mechanics

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1 Introduction

The derivation of the equations of motion for an *ideal fluid* by Euler in 1755, and then for a *viscous fluid* by Navier (1822) and Stokes (1845) were a tour-de-force of 18th and 19th century mathematics. These equations have been used to describe and explain so many physical phenomena around us in nature, that currently billions of dollars of research grants in mathematics, science and engineering now revolve around them. They can be used to model the coupled atmospheric and ocean flow used by the meteorological office for weather prediction down to any application in chemical engineering you can think of, say to development of the thrusters on NASA's Apollo programme rockets. The incompressible *Navier–Stokes equations* are given by

$$\begin{aligned}\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= \nu \nabla^2 \boldsymbol{u} - \nabla p + \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} &= 0,\end{aligned}$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ is a three dimensional incompressible fluid velocity (indicated by the last equation), $p = p(\boldsymbol{x}, t)$ is the pressure and $\boldsymbol{f}$ is an external force field. The frictional force due to stickiness of a fluid is represented by the term $\nu \nabla^2 \boldsymbol{u}$. Here we will consider ideal fluids only, and this corresponds to the case $\nu = 0$, when the equations above are known as the *Euler equations* for a homogeneous incompressible ideal fluid. We will derive the Euler equations and in the process learn about the subtleties of fluid mechanics and along the way see lots of interesting applications.

2 Fluid flow

2.1 Flow

A material exhibits *flow* if shear forces, however small, lead to a deformation which is unbounded—we could use this as definition of a *fluid*. A *solid* has a fixed shape, or at least a strong limitation on its deformation when force is applied to it. With the category of "fluids", we include liquids and gases. The main distinguishing feature between these two fluids is the notion of compressibility. Gases are usually compressible—as we know from everyday aerosols and air canisters. Liquids are generally incompressible—a feature essential to all modern car braking mechanisms.

Fluids can be further subcatergorized. Here we will only consider *ideal* fluids. This means that the *only* internal force present is pressure which acts so that fluid flows from a region of high pressure to one of low pressure. However fluids can exhibit internal frictional forces which model a "stickiness" property of the fluid which involves energy loss—such fluids are known as viscous fluids—we will not consider them here. Some fluids/material known as "non-Newtonian or complex fluids" exhibit even stranger behaviour, their reaction to deformation may depend on: (i) past history (earlier deformations), for example some paints; (ii) temperature, for example some polymers or glass; (iii) the size of the deformation, for example some plastics or silly putty.

The notion of an ideal fluid however, can be used to describe and predict a lot of phenomena observed in common working fluids, such as air, water, blood, and so forth, and have been applied to wing and aircraft design (as a limit of high Reynolds number flow).

2.2 **Continuum hypothesis**

For any *real* fluid there are three natural length scales:

1. $L_{\text{molecular}}$, the molecular scale characterized by the mean free path distance of molecules between collisions;
2. L_{fluid}, the medium scale of a fluid parcel, the fluid droplet in the pipe or ocean flow;
3. L_{macro}, the macro-scale which is the scale of the fluid geometry, the scale of the container the fluid is in, whether a beaker or an ocean.

And, of course we have the asymptotic inequalities:

$$L_{\text{molecular}} \ll L_{\text{fluid}} \ll L_{\text{macro}}.$$

We will assume that the properties of an elementary volume/parcel of fluid, however small, are the same as for the fluid as a whole—i.e. we suppose that the properties of the fluid at scale L_{fluid} propagate all the way down and through the molecular scale $L_{\text{molecular}}$. This is the *continuum assumption.* For everyday fluid mechanics engineering, this assumption is extremely accurate (Chorin and Marsden [3, Page 2]).

2.3 **Conservation principles**

Our derivation of the basic equations underlying the dynamics of ideal fluids is based on three basic principles (see Chorin and Marsden [3, Page 2]):

1. *Conservation of mass*, mass is neither created or destroyed;
2. *Newton's 2nd law/balance of momentum*, for a parcel of fluid the rate of change of momentum equals the force applied to it;
3. *Conservation of energy*, energy is neither created nor destroyed.

In turn these principles generate the:

1. *Continuity equation* which governs how the density of the fluid evolves locally and thus indicates compressibility properties of the fluid;
2. *Euler equations* of motion for a fluid which indicates how the fluid moves around from regions of high pressure to those of low pressure;
3. *Equation of state* which indicates the mechanism of energy exchange within the fluid.

3 Trajectories and streamlines

Suppose that our ideal fluid is contained with a region/domain $\mathcal{D} \subseteq \mathbb{R}^d$ where $d = 2$ or 3, and $\boldsymbol{x} = (x, y, z) \in \mathcal{D}$ is a position/point in $\mathcal{D}$. Imagine a small fluid particle or a speck of dust moving in a fluid flow field prescribed by the *velocity field* $\boldsymbol{u}(\boldsymbol{x}, t) = (u, v, w)$. Suppose the position of the particle at time t is recorded by the variables $\big(x(t), y(t), z(t)\big)$. The velocity of the particle at time t at position $\big(x(t), y(t), z(t)\big)$ is

$$\begin{aligned}
\dot{x}(t) &= u\big(x(t), y(t), z(t), t\big), \\
\dot{y}(t) &= v\big(x(t), y(t), z(t), t\big), \\
\dot{z}(t) &= w\big(x(t), y(t), z(t), t\big).
\end{aligned}$$

In shorter vector notation this is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t).$$

The *trajectory* or *particle path* of a fluid particle is the curve traced out by the particle as time progresses. It is the solution to the differential equation above (with suitable initial conditions).

Suppose now for a given fluid flow $\boldsymbol{u}(\boldsymbol{x}, t)$ we fix time t. A *streamline* is an integral curve of $\boldsymbol{u}(\boldsymbol{x}, t)$ for t fixed, i.e. it is a curve $\boldsymbol{x} = \boldsymbol{x}(s)$ parameterized by the variable s, that satisfies the system of equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{x}(s) = \boldsymbol{u}(\boldsymbol{x}(s), t),$$

with t held constant. If the velocity field $\boldsymbol{u}$ is time-independent, i.e. $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ only, or equivalently $\partial_t \boldsymbol{u} = \boldsymbol{0}$, then trajectories and streamlines coincide. Flows for which $\partial_t \boldsymbol{u} = \boldsymbol{0}$ are said to be *stationary*.

Example. Suppose a velocity field $\boldsymbol{u}(\boldsymbol{x}, t) = (u, v, w)$ is given for $t > -1$ by

$$u = \frac{x}{1+t}, \qquad v = \frac{y}{1+\frac{1}{2}t} \qquad \text{and} \qquad w = z.$$

To find the *particle paths or trajectories*, we must solve the system of equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = v \qquad \text{and} \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = w,$$

and then eliminate the time variable t between them. Hence for the particle paths we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x}{1+t}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y}{1+\frac{1}{2}t} \qquad \text{and} \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = z.$$

Using the method of separation of variables and integrating in time from t_0 to t, in each of the three equations, we get

$$\ln\left(\frac{x}{x_0}\right) = \ln\left(\frac{1+t}{1+t_0}\right), \qquad \ln\left(\frac{y}{y_0}\right) = 2\ln\left(\frac{1+\frac{1}{2}t}{1+\frac{1}{2}t_0}\right) \qquad \text{and} \qquad \ln\left(\frac{z}{z_0}\right) = \frac{t}{t_0},$$

where we have assumed that at time t_0 the particle is at position (x_0, y_0, z_0). Exponentiating the first two equations and solving the last one for t, we get

$$\frac{x}{x_0} = \frac{1+t}{1+t_0}, \qquad \frac{y}{y_0} = \frac{(1+\frac{1}{2}t)^2}{(1+\frac{1}{2}t_0)^2} \qquad \text{and} \qquad t = t_0 \ln(z/z_0).$$

We can use the last equation to eliminate t so the particle path/trajectory through (x_0, y_0, z_0) is the curve in three dimensional space given by

$$x = x_0 \cdot \frac{\big(1 + t_0 \ln(z/z_0)\big)}{(1+t_0)}, \qquad \text{and} \qquad y = y_0 \cdot \frac{\big(1+\frac{1}{2}t_0 \ln(z/z_0)\big)^2}{(1+\frac{1}{2}t_0)^2}.$$

To find the *streamlines*, we fix time t. We must then solve the system of equations

$$\frac{\mathrm{d}x}{\mathrm{d}s} = u, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = v \qquad \text{and} \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = w,$$

with t fixed, and then eliminate s between them. Hence for streamlines we have

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{x}{1+t}, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = \frac{y}{1+\frac{1}{2}t} \qquad \text{and} \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = z.$$

Assuming that we are interested in the streamline that passes through the point (x_0, y_0, z_0), we again use the method of separation of variables and integrate with respect to s from s_0 to s, for each of the three equations. This gives

$$\ln\left(\frac{x}{x_0}\right) = \frac{s-s_0}{1+t}, \qquad \ln\left(\frac{y}{y_0}\right) = \frac{s-s_0}{1+\frac{1}{2}t} \qquad \text{and} \qquad \ln\left(\frac{z}{z_0}\right) = s - s_0.$$

Using the last equation, we can substitute for $s - s_0$ into the first equations. If we then multiply the first equation by $1+t$ and the second by $1+\frac{1}{2}t$, and use the usual log law $\ln a^b = b \ln a$, then exponentiation reveals that

$$\left(\frac{x}{x_0}\right)^{1+t} = \left(\frac{y}{y_0}\right)^{1+\frac{1}{2}t} = \frac{z}{z_0},$$

which are the equations for the streamline through (x_0, y_0, z_0).

4 Conservation of mass

4.1 Continuity equation

Recall, we suppose our ideal fluid is contained with a region/domain $\mathcal{D} \subseteq \mathbb{R}^d$ (here we will assume $d = 3$, but everything we say is true for the collapsed two dimensional case $d = 2$). Hence $\boldsymbol{x} = (x, y, z) \in \mathcal{D}$ is a position/point in $\mathcal{D}$. At each time t we will suppose that the fluid has a well defined *mass density* $\rho(\boldsymbol{x}, t)$ at the point $\boldsymbol{x}$. Further, each fluid particle traces out a well defined path in the fluid, and its motion along that path is governed by the *velocity field* $\boldsymbol{u}(\boldsymbol{x}, t)$ at position $\boldsymbol{x}$ at time t. Consider an arbitrary subregion $\Omega \subseteq \mathcal{D}$. The total mass of fluid contained inside the region Ω at time t is

$$\int_\Omega \rho(\boldsymbol{x}, t)\,\mathrm{d}V.$$

where $\mathrm{d}V$ is the volume element in $\mathbb{R}^d$. Let us now consider the rate of change of mass inside Ω. By the principle of conservation of mass, the rate of increase of the mass in Ω is given by the mass of fluid entering/leaving the boundary $\partial\Omega$ of Ω.

To compute the total mass of fluid entering/leaving the boundary $\partial\Omega$, we consider a small area patch $\mathrm{d}S$ on the boundary of $\partial\Omega$, which has unit outward normal $\boldsymbol{n}$. The total mass of fluid flowing out of Ω through the area patch $\mathrm{d}S$ per unit time is

$$\text{mass density} \times \text{fluid volume leaving per unit time} = \rho(\boldsymbol{x}, t)\,\boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S,$$

where $\boldsymbol{x}$ is at the center of the area patch $\mathrm{d}S$ on $\partial\Omega$. Note that to estimate the fluid volume leaving per unit time we have decomposed the fluid velocity at $\boldsymbol{x} \in \partial\Omega$, time t, into velocity components normal $(\boldsymbol{u} \cdot \boldsymbol{n})$ and tangent to the surface $\partial\Omega$ at that point. The velocity component tangent to the surface pushes fluid across the surface—no fluid enters or leaves Ω via this component. Hence we only retain the normal component—see Fig. 2.

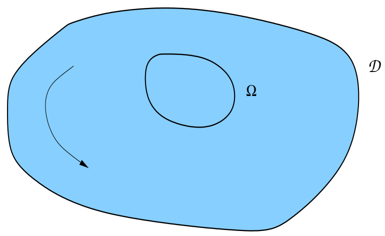

Fig. 1 The fluid of mass density $\rho(\boldsymbol{x},t)$ swirls around inside the container $\mathcal{D}$, while Ω is an imaginary subregion.

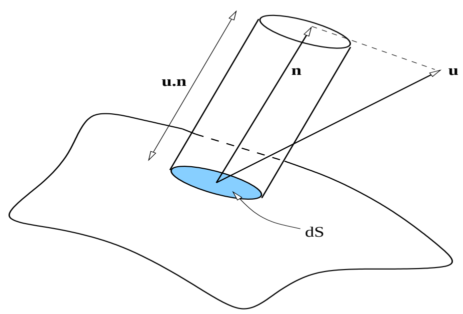

Fig. 2 The total mass of fluid moving through the patch $\mathrm{d}S$ on the surface $\partial\Omega$ per unit time, is given by the mass density $\rho(\boldsymbol{x},t)$ times the volume of the cylinder shown which is $\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S$.

Returning to the principle of conservation of mass, this is now equivalent to the *integral form of the law of conservation of mass*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\boldsymbol{x},t)\,\mathrm{d}V = -\int_{\partial\Omega}\rho\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S.$$

The divergence theorem and that the rate of change of the total mass inside Ω equals the total rate of change of mass density inside Ω imply, respectively,

$$\int_\Omega \nabla\cdot(\rho\boldsymbol{u})\,\mathrm{d}V = \int_{\partial\Omega}(\rho\boldsymbol{u})\cdot\boldsymbol{n}\,\mathrm{d}S \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\,\mathrm{d}V = \int_\Omega \frac{\partial\rho}{\partial t}\,\mathrm{d}V.$$

Using these two relations, the law of conservation of mass is equivalent to

$$\int_\Omega \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u})\,\mathrm{d}V = 0.$$

Now we use that Ω is arbitrary to deduce the *differential form of the law of conservation of mass* or *continuity equation* that applies pointwise:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0.$$

This is the first of our three conservation laws.

4.2 **Incompressible flow**

Having established the continuity equation we can now define a subclass of flows which are *incompressible*. The classic examples are water, and the brake fluid in your car whose incompressibility properties are vital to the effective transmission of pedal pressure to brakepad pressure.

Definition 1 (Incompressibility) A fluid with the property $\nabla \cdot \boldsymbol{u} = 0$ is *incompressible*.

The continuity equation and the identity, $\nabla \cdot (\rho\, \boldsymbol{u}) = \nabla\rho \cdot \boldsymbol{u} + \rho\, \nabla \cdot \boldsymbol{u}$, imply

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla\rho + \rho\, \nabla \cdot \boldsymbol{u} = 0.$$

Hence since $\rho > 0$, a flow is incompressible if and only if

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla\rho = 0.$$

If the fluid is *homogeneous* so that ρ is constant in space, then the flow is incompressible if and only if ρ is constant in time.

4.3 **Stream functions**

A *stream function* exists for a given flow $\boldsymbol{u} = (u, v, w)$ if the velocity field $\boldsymbol{u}$ is *solenoidal*, i.e. $\nabla \cdot \boldsymbol{u} = 0$, *and* we have an additional symmetry that allows us to eliminate one coordinate. For example, a two dimensional incompressible fluid flow $\boldsymbol{u} = \boldsymbol{u}(x, y, t)$ is solenoidal since $\nabla \cdot \boldsymbol{u} = 0$, and has the symmetry that it is uniform with respect to z. For such a flow we see that

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \Leftrightarrow \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

This equation is satisfied if and only if there exists a function $\psi(x, y, t)$ such that

$$\frac{\partial \psi}{\partial y} = u(x, y, t) \qquad \text{and} \qquad -\frac{\partial \psi}{\partial x} = v(x, y, t).$$

The function ψ is called *Lagrange's stream function*. A stream function is always only defined up to any arbitrary additive constant. Further note that for t fixed, *streamlines* are given by constant contour lines of ψ.

Note that if we use plane polar coordinates so $\boldsymbol{u} = \boldsymbol{u}(r, \theta, t)$ and the velocity components are $\boldsymbol{u} = (u_r, u_\theta)$ then

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \Leftrightarrow \qquad \frac{1}{r}\frac{\partial}{\partial r}(r\, u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial r} = 0.$$

This is satisfied if and only if there exists a function $\psi(r, \theta, t)$ such that

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = u_r(r, \theta, t) \qquad \text{and} \qquad -\frac{\partial \psi}{\partial r} = u_\theta(r, \theta, t).$$

Example Suppose that in Cartesian coordinates we have the two dimensional flow $\boldsymbol{u} = (u, v)$ given by

$$(u, v) = (k\, x, -k\, y),$$

for some constant k. Note that $\nabla \cdot \boldsymbol{u} = 0$ so there exists a stream function satisfying

$$\frac{\partial \psi}{\partial y} = k\,x \qquad \text{and} \qquad -\frac{\partial \psi}{\partial x} = -k\,y.$$

Consider the first partial differential equation. Integrating with respect to y we get

$$\psi = k\,xy + C(x)$$

where $C(x)$ is an arbitrary function of x. However we know that ψ must simultaneously satisfy the second partial differential equation above. Hence we substitute this last relation into the second partial differential equation above to get

$$-\frac{\partial \psi}{\partial x} = -k\,y \qquad \Leftrightarrow \qquad -k\,y + C'(x) = -k\,y.$$

We deduce $C'(x) = 0$ and therefore C is an arbitrary constant. Since a stream function is only defined up to an arbitrary constant we take $C = 0$ for simplicity and the stream function is given by

$$\psi = k\,xy.$$

Now suppose we used plane polar coordinates instead. The corresponding flow $\boldsymbol{u} = (u_r, u_\theta)$ is given by

$$(u_r, u_\theta) = (k\,r\cos 2\theta, -k\,r\sin 2\theta).$$

First note that $\nabla \cdot \boldsymbol{u} = 0$ using the polar coordinate form for $\nabla \cdot \boldsymbol{u}$ indicated above. Hence there exists a stream function $\psi = \psi(r, \theta)$ satisfying

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = k\,r\cos 2\theta \qquad \text{and} \qquad -\frac{\partial \psi}{\partial r} = -k\,r\sin 2\theta.$$

As above, consider the first partial differential equation shown, and integrate with respect to θ to get

$$\psi = \tfrac{1}{2}\,k\,r^2 \sin 2\theta + C(r).$$

Substituting this into the second equation above reveals that $C'(r) = 0$ so that C is a constant. We can for convenience set $C = 0$ so that

$$\psi = \tfrac{1}{2}\,k\,r^2 \sin 2\theta.$$

Comparing this form with its Cartesian equivalent above, reveals they are the same.

5 Balance of momentum

5.1 Differentiation following the fluid

Recall our image of a small fluid particle moving in a fluid flow field prescribed by the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$. The velocity of the particle at time t at position $\boldsymbol{x}(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t).$$

As the particle moves in the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$, say from position $\boldsymbol{x}(t)$ to a nearby position an instant in time later, two dynamical contributions change: (i) a small instant in time has elapsed and the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, which depends on time, will have changed a little; (ii) the position of the particle has changed in that short time as it moved slightly, and the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, which depends on position, will be slightly different at the new position.

Let us compute the *acceleration* of the particle to explicitly observe these two contributions. By using the chain rule we see that

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{x}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}\big(\boldsymbol{x}(t),t\big)\\ &= \frac{\partial \boldsymbol{u}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \boldsymbol{u}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \boldsymbol{u}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial \boldsymbol{u}}{\partial t}\\ &= \left(\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial}{\partial z}\right)\boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial t}\\ &= \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial t}.\end{aligned}$$

Indeed for any function $F(x,y,z,t)$, scalar or vector valued, the chain rule implies

$$\frac{\mathrm{d}}{\mathrm{d}t}F\big(x(t),y(t),z(t),t\big) = \frac{\partial F}{\partial t} + \boldsymbol{u}\cdot\nabla F.$$

Definition 2 (Material derivative) If the velocity field components are

$$\boldsymbol{u} = (u,v,w) \qquad \text{and} \qquad \boldsymbol{u}\cdot\nabla \equiv u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z},$$

then we define the *material derivative* following the fluid to be

$$\frac{\mathrm{D}}{\mathrm{D}t} := \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla.$$

5.2 **Internal fluid forces**

Let us consider the forces that act on a small parcel of fluid in a fluid flow. There are two types:

1. *external or body forces*, these may be due to gravity or external electromagnetic fields. They exert a force per unit volume on the continuum.
2. *surface or stress forces*, these are forces, molecular in origin, that are applied by the neighbouring fluid across the surface of the fluid parcel.

The force per unit area exerted across a surface (imaginary in the fluid) is called the *stress*. Let $\mathrm{d}S$ be a small imaginary surface in the fluid centered on the point $\boldsymbol{x}$—see Fig. 3. The force $\mathrm{d}\boldsymbol{F}$ on side (2) by side (1) of $\mathrm{d}S$ in the fluid/material is given by

$$\mathrm{d}\boldsymbol{F} = \boldsymbol{\Sigma}(\boldsymbol{n})\,\mathrm{d}S.$$

Here $\boldsymbol{\Sigma}$ is the stress at the point $\boldsymbol{x}$. It is a function of the normal direction $\boldsymbol{n}$ to the surface $\mathrm{d}S$, in fact it is given by:

$$\boldsymbol{\Sigma}(\boldsymbol{n}) = \sigma(\boldsymbol{x})\,\boldsymbol{n}.$$

Note $\sigma = [\sigma_{ij}]$ is a 3×3 matrix known as the *stress tensor*. The diagonal components of σ_{ij}, with $i=j$, generate *normal stresses*, while the off-diagonal components, with

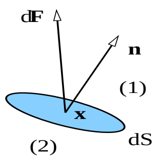

Fig. 3 The force $\mathrm{d}\boldsymbol{F}$ on side (2) by side (1) of $\mathrm{d}S$ is given by $\boldsymbol{\Sigma}(\boldsymbol{n})\,\mathrm{d}S$.

$i \neq j$, generate *tangential* or *shear stresses*. Here we will only consider normal stresses, in fact those for which there exists a scalar function p such that (here I is the 3×3 identity matrix)

$$\sigma(\boldsymbol{x}) = -p\,I \qquad \Rightarrow \qquad \boldsymbol{\Sigma}(\boldsymbol{n}) = -p\,\boldsymbol{n}.$$

Definition 3 (Ideal fluid) For any motion of an *ideal fluid* there is a function $p(\boldsymbol{x}, t)$ called the *pressure* such that the stress across a surface S, which has unit normal $\boldsymbol{n}$ at the point $\boldsymbol{x} \in S$ at time t, is given by $p(\boldsymbol{x}, t)\boldsymbol{n}$, i.e.

$$\text{the stress across } S \text{ at } \boldsymbol{x} \; = p(\boldsymbol{x}, t)\,\boldsymbol{n}.$$

Remark 1 Hence there are no tangential forces in an ideal fluid. This has some important consequences, quoting from Chorin and Marsden [3, Page 5]:

> ...there is no way for rotation to start in a fluid, nor, if there is any at the beginning, to stop... ... even here we can detect trouble for ideal fluids because of the abundance of rotation in real fluids (near the oars of a rowboat, in tornadoes, etc.).

5.3 **Equation of motion (Euler 1755)**

Consider again an arbitrary imaginary subregion Ω of $\mathcal{D}$ as in Fig. 1. At any instant in time t, the total force exerted on the fluid inside Ω through the stresses exerted across its boundary $\partial\Omega$ is given by

$$-\int_{\partial\Omega} p\,\boldsymbol{n}\,\mathrm{d}S \equiv -\int_{\Omega} \nabla p\,\mathrm{d}V.$$

If $\boldsymbol{f}(\boldsymbol{x}, t)$ is a body force (external force) per unit mass then the total body force on the fluid inside Ω is

$$\int_{\Omega} \rho\,\boldsymbol{f}\,\mathrm{d}V.$$

Thus on any parcel of fluid, the force per unit volume acting on it is

$$-\nabla p + \rho\,\boldsymbol{f}.$$

Hence using Newton's 2nd law (force = mass × acceleration) we can deduce the following relation—the differential form of the *balance of momentum*:

$$\rho\,\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \rho\,\boldsymbol{f}.$$

This is *Euler's equation of motion for an ideal fluid*. It represents the second of our three conservation laws.

6 Transport theorem

Suppose that the region within which the fluid is moving is $\mathcal{D}$. Suppose Ω is a subregion of $\mathcal{D}$ identified at time $t = 0$. As the fluid flow evolves the fluid particles that originally made up Ω will subsequently fill out a volume Ω_t at time t. We think of Ω_t as the volume *moving with the fluid*.

Theorem 1 (Transport theorem) *For any function $F = F(\boldsymbol{x}, t)$ and density function $\rho = \rho(\boldsymbol{x}, t)$ satisfying the continuity equation, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \rho\, F \,\mathrm{d}V = \int_{\Omega_t} \rho\, \frac{\mathrm{D}F}{\mathrm{D}t} \,\mathrm{d}V.$$

The transport theorem is useful because it allows us to deduce Euler's equation of motion from the *primitive* integral form of the balance of momentum. From Newton's second law this is

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \rho\, \boldsymbol{u} \,\mathrm{d}V = \int_{\Omega_t} -\nabla p + \rho\, \boldsymbol{f} \,\mathrm{d}V.$$

Hence, using the transport theorem with $F \equiv \boldsymbol{u}$ and that Ω and thus Ω_t are arbitrary, we see that Euler's equation of motion must hold pointwise at each $\boldsymbol{x} \in \mathcal{D}$.

Proof There are four steps; see Chorin and Marsden [3, Pages 6–11].

Step 1: Fluid flow map. For any $\boldsymbol{x} \in \mathcal{D}$ let $\varphi(\boldsymbol{x}, t)$ denote the trajectory followed by the particle that starts at position $\boldsymbol{x}$ at time $t = 0$. Let φ_t denote the map $\boldsymbol{x} \mapsto \varphi(\boldsymbol{x}, t)$, i.e. φ_t is the map that advances each particle at position $\boldsymbol{x}$ at time $t = 0$ to its position at time t later; it is the *fluid flow-map*. Hence, for example $\varphi_t(\Omega) = \Omega_t$. We assume φ_t is sufficiently smooth and invertible for all our subsequent manipulations.

Step 2: Change of variables. For any two functions $\rho = \rho(\boldsymbol{x}, t)$ and $F = F(\boldsymbol{x}, t)$ we can perform the change of variables—with $J(\boldsymbol{x}, t)$ as the Jacobian of φ_t:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \rho\, F \,\mathrm{d}V &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} (\rho\, F)(\varphi(\boldsymbol{x}, t), t)\, J(\boldsymbol{x}, t) \,\mathrm{d}V \\
&= \int_{\Omega} \frac{\partial}{\partial t} \big((\rho\, F)(\varphi(\boldsymbol{x}, t), t)\, J(\boldsymbol{x}, t) \big) \,\mathrm{d}V \\
&= \int_{\Omega} \frac{\partial}{\partial t} (\rho\, F)(\varphi(\boldsymbol{x}, t), t)\, J(\boldsymbol{x}, t) + (\rho\, F)(\varphi(\boldsymbol{x}, t), t) \frac{\partial}{\partial t} J(\boldsymbol{x}, t)\, r dV \\
&= \int_{\Omega} \left(\frac{\mathrm{D}}{\mathrm{D}t} (\rho\, F) \right) (\varphi(\boldsymbol{x}, t), t)\, J(\boldsymbol{x}, t) + (\rho\, F)(\varphi(\boldsymbol{x}, t), t) \frac{\partial}{\partial t} J(\boldsymbol{x}, t) \,\mathrm{d}V.
\end{aligned}$$

Step 3: Evolution of the Jacobian. By direct computation (see Appendix A)

$$\frac{\partial}{\partial t} J(\boldsymbol{x}, t) = \big(\nabla \cdot \boldsymbol{u}(\varphi(\boldsymbol{x}, t), t) \big)\, J(\boldsymbol{x}, t).$$

Step 4: Conservation of mass. We see that we thus have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \rho\, F \,\mathrm{d}V &= \int_{\Omega} \left(\frac{\mathrm{D}}{\mathrm{D}t} (\rho\, F) + (\rho\, F) \big(\nabla \cdot \boldsymbol{u} \big) \right) (\varphi(\boldsymbol{x}, t), t)\, J(\boldsymbol{x}, t) \,\mathrm{d}V \\
&= \int_{\Omega_t} \left(\frac{\mathrm{D}}{\mathrm{D}t} (\rho\, F) + \big(\rho\, \nabla \cdot \boldsymbol{u} \big)\, F \right) \mathrm{d}V \\
&= \int_{\Omega_t} \rho \frac{\mathrm{D}F}{\mathrm{D}t} \,\mathrm{d}V,
\end{aligned}$$

where in the last step we have used the conservation of mass equation. □

7 Conservation of energy

We have derived two conservation laws thusfar, first, conservation of mass,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0,$$

and second, balance of momentum,

$$\rho \, \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \rho \, \boldsymbol{f}.$$

If we assume we are in three dimensional space so $d = 3$, we have four equations, but five unknowns—namely $\boldsymbol{u}$, p and ρ. We cannot specify the fluid motion completely without specifying one more condition.

Definition 4 (Kinetic energy) The *kinetic energy* of the fluid in the region $\Omega \subseteq \mathcal{D}$ is

$$E := \tfrac{1}{2} \int_{\Omega} \rho \, |\boldsymbol{u}|^2 \, \mathrm{d}V.$$

The rate of change of the kinetic energy, using the transport theorem, is given by

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2} \int_{\Omega_t} \rho \, |\boldsymbol{u}|^2 \, \mathrm{d}V \right) \\
&= \tfrac{1}{2} \int_{\Omega_t} \rho \frac{\mathrm{D}|\boldsymbol{u}|^2}{\mathrm{D}t} \, \mathrm{d}V \\
&= \tfrac{1}{2} \int_{\Omega_t} \rho \frac{\mathrm{D}}{\mathrm{D}t} (\boldsymbol{u} \cdot \boldsymbol{u}) \, \mathrm{d}V \\
&= \int_{\Omega_t} \rho \boldsymbol{u} \cdot \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \, \mathrm{d}V \\
&= \int_{\Omega_t} \boldsymbol{u} \cdot \left(\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \right) \mathrm{d}V.
\end{aligned}$$

Here we assume that *all* the energy is kinetic. The principal of conservation of energy states (from Chorin and Marsden, page 13):

> *the rate of change of kinetic energy in a portion of fluid equals the rate at which the pressure and body forces do work.*

In other words we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int_{\partial \Omega_t} p \, \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S + \int_{\Omega_t} \rho \, \boldsymbol{u} \cdot \boldsymbol{f} \, \mathrm{d}V.$$

We compare this with our expression above for the rate of change of the kinetic energy. Equating the two expressions, using Euler's equation of motion, and noticing that the body force term immediately cancels, we get

$$\begin{aligned}
& & \int_{\partial \Omega_t} p \, \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S &= \int_{\Omega_t} \boldsymbol{u} \cdot \nabla p \, \mathrm{d}V \\
\Leftrightarrow & & \int_{\Omega_t} \nabla \cdot (\boldsymbol{u} p) \, \mathrm{d}V &= \int_{\Omega_t} \boldsymbol{u} \cdot \nabla p \, \mathrm{d}V \\
\Leftrightarrow & & \int_{\Omega_t} \nabla \cdot (\boldsymbol{u} p) \, \mathrm{d}V &= \int_{\Omega_t} \nabla \cdot (\boldsymbol{u} p) - (\nabla \cdot \boldsymbol{u}) \, p \, \mathrm{d}V \\
\Leftrightarrow & & \int_{\Omega_t} (\nabla \cdot \boldsymbol{u}) \, p \, \mathrm{d}V &= 0.
\end{aligned}$$

Since Ω and therefore Ω_t is arbitrary we see that the assumption that all the energy is kinetic implies

$$\nabla \cdot \boldsymbol{u} = 0.$$

Hence our third conservation law, conservation of energy has lead to the *equation of state*, $\nabla \cdot \boldsymbol{u} = 0$, i.e. that an ideal fluid is *incompressible*.

Hence the *Euler equations* for a homogeneous incompressible flow in $\mathcal{D}$ are

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \boldsymbol{f},$$
$$\nabla \cdot \boldsymbol{u} = 0,$$

together with the boundary condition on $\partial\mathcal{D}$ which is $\boldsymbol{u} \cdot \boldsymbol{n} = 0$.

Remark 2 Often Euler's equation is quoted with density ρ not present. Since ρ is constant $(\nabla p)/\rho \equiv \nabla(p/\rho)$, and we could re-label the term p/ρ to be p. Equivalently, we could simply normalize the uniform constant density ρ to unity.

Example (sink or bath drain) As the water (of uniform density ρ) flows out through a hole at the bottom of a bath the residual rotation is confined to a core of radius a, so that the water particles may be taken to move on horizontal circles with

$$u_\theta = \begin{cases} \Omega r, & r \leqslant a, \\ \frac{\Omega a^2}{r}, & r > a. \end{cases}$$

As we have all observed when water runs out of a bath or sink, the free surface of the water directly over the drain hole has a depression in it—see Fig. 4. The question is, what is the form/shape of this free surface depression?

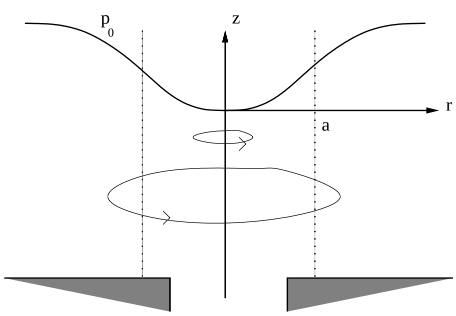

Fig. 4 Water draining from a bath.

We know that the pressure at the free surface is uniform, it is atmospheric pressure, say P_0. We need the Euler equations for a homogeneous incompressible fluid in *cylindrical coordinates* (r, θ, z) with the velocity field $\boldsymbol{u} = (u_r, u_\theta, u_z)$. These are

$$\frac{\partial u_r}{\partial t} + (\boldsymbol{u} \cdot \nabla)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + f_r,$$
$$\frac{\partial u_\theta}{\partial t} + (\boldsymbol{u} \cdot \nabla)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + f_\theta,$$
$$\frac{\partial u_z}{\partial t} + (\boldsymbol{u} \cdot \nabla)u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + f_z,$$

where $p = p(r, \theta, z, t)$ is the pressure, ρ is the uniform constant density and $\boldsymbol{f} = (f_r, f_\theta, f_z)$ is the body force per unit mass. Here we also have

$$\boldsymbol{u} \cdot \nabla = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}.$$

Further the incompressibility condition $\nabla \cdot \boldsymbol{u} = 0$ is given in cylindrical coordinates by

$$\frac{1}{r} \frac{\partial (r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0.$$

Now we look at the setting we are presented with for this problem. Note the flow is steady and $u_r = u_z = 0$, $f_r = f_\theta = 0$. The force due to gravity implies $f_z = -g$. The whole problem is also symmetric with respect to θ, so that all partial derivatives with respect to θ should be zero. Combining all these facts reduces Euler's equations above to

$$-\frac{u_\theta^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r}, \qquad 0 = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} \qquad \text{and} \qquad 0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g.$$

The incompressibility condition is satisfied trivially. The second equation above tells us the pressure p is independent of θ, as we might have already suspected. Hence we assume $p = p(r, z)$ and focus on the first and third equation above.

Assume $r \leqslant a$. Using that $u_\theta = \Omega r$ in the first equation we see that

$$\frac{\partial p}{\partial r} = \rho \Omega^2 r \qquad \Leftrightarrow \qquad p(r, z) = \tfrac{1}{2} \rho \Omega^2 r^2 + C(z),$$

where $C(z)$ is an arbitrary function of z. If we then substitute this into the third equation above we see that

$$\frac{1}{\rho} \frac{\partial p}{\partial z} = -g \qquad \Leftrightarrow \qquad C'(z) = -\rho g,$$

and hence $C(z) = -\rho g z + C_0$ where C_0 is an arbitrary constant. Thus we now deduce that the pressure function is given by

$$p(r, z) = \tfrac{1}{2} \rho \Omega^2 r^2 - \rho g z + C_0.$$

At the free surface of the water, the pressure is constant atmospheric pressure P_0 and so if we substitute this into this expression for the pressure we see that

$$P_0 = \tfrac{1}{2} \rho \Omega^2 r^2 - \rho g z + C_0 \qquad \Leftrightarrow \qquad z = (\Omega^2/2g)\, r^2 - (C_0 - P_0)/\rho g.$$

Hence the depression in the free surface for $r \leqslant a$ is a *parabolic surface* of revolution. Note that pressure is only ever globally defined up to an additive constant so we are at liberty to take $C_0 = 0$ or $C_0 = P_0$ if we like.

For $r > a$ a completely analogous argument using $u_\theta = \Omega a^2 / r$ shows that

$$p(r, z) = -\frac{\rho \Omega^2 a^4}{2\, r^2} - \rho g z + K_0,$$

where K_0 is an arbitrary constant. Since the pressure must be continuous at $r = a$, we substitute $r = a$ into the expression for the pressure here for $r > a$ and the expression for the pressure for $r \leqslant a$, and equate the two. This gives

$$-\tfrac{1}{2} \rho \Omega^2 a^2 - \rho g z + K_0 = \tfrac{1}{2} \rho \Omega^2 a^2 - \rho g z \qquad \Leftrightarrow \qquad K_0 = \rho \Omega^2 a^2.$$

Hence the pressure for $r > a$ is given by

$$p(r,z) = -\frac{\rho \Omega^2 a^4}{2\,r^2} - \rho g z + \rho \Omega^2\, a^2.$$

Using that the pressure at the free surface is $p(r,z) = P_0$, we see that for $r > a$ the free surface is given by

$$z = -\frac{\Omega^2 a^4}{g\,r^2} + \frac{\Omega^2\, a^2}{g}.$$

8 Bernoulli's Theorem

Theorem 2 (Bernoulli's Theorem) *Suppose we have a homogeneous incompressible stationary flow with a conservative body force $\boldsymbol{f} = -\nabla\phi$, where ϕ is the potential function. Then the quantity*

$$H := \tfrac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} + \phi$$

is constant along streamlines.

Proof We need the following identity that can be found in Appendix B:

$$\tfrac{1}{2}\nabla\big(|\boldsymbol{u}|^2\big) = \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\times(\nabla\times\boldsymbol{u}).$$

Since the flow is stationary, Euler's equation of motion imply

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla\Big(\frac{p}{\rho}\Big) - \nabla\phi.$$

Using the identity above we see that

$$\begin{aligned}
& \tfrac{1}{2}\nabla\big(|\boldsymbol{u}|^2\big) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) = -\nabla\Big(\frac{p}{\rho}\Big) - \nabla\phi \\
\Leftrightarrow\quad & \nabla\Big(\tfrac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} + \phi\Big) = \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) \\
\Leftrightarrow\quad & \nabla H = \boldsymbol{u}\times(\nabla\times\boldsymbol{u}),
\end{aligned}$$

using the definition for H given in the theorem. Now let $\boldsymbol{x}(s)$ be a streamline that satisfies $\boldsymbol{x}'(s) = \boldsymbol{u}\big(\boldsymbol{x}(s)\big)$. By the fundamental theorem of calculus, for any s_1 and s_2,

$$\begin{aligned}
H\big(\boldsymbol{x}(s_2)\big) - H\big(\boldsymbol{x}(s_1)\big) &= \int_{s_1}^{s_2} \mathrm{d}H\big(\boldsymbol{x}(s)\big) \\
&= \int_{\boldsymbol{x}(s_1)}^{\boldsymbol{x}(s_2)} \nabla H\cdot\boldsymbol{x}'(s)\,\mathrm{d}s \\
&= \int_{\boldsymbol{x}(s_1)}^{\boldsymbol{x}(s_2)} \big(\boldsymbol{u}\times(\nabla\times\boldsymbol{u})\big)\cdot\boldsymbol{u}\big(\boldsymbol{x}(s)\big)\,\mathrm{d}s \\
&= 0,
\end{aligned}$$

where we used that $(\boldsymbol{u}\times\boldsymbol{a})\cdot\boldsymbol{u} \equiv \boldsymbol{0}$ for any vector $\boldsymbol{a}$ (since $\boldsymbol{u}\times\boldsymbol{a}$ is orthogonal to $\boldsymbol{u}$). Since s_1 and s_2 are arbitrary we deduce that H does change along streamlines. □

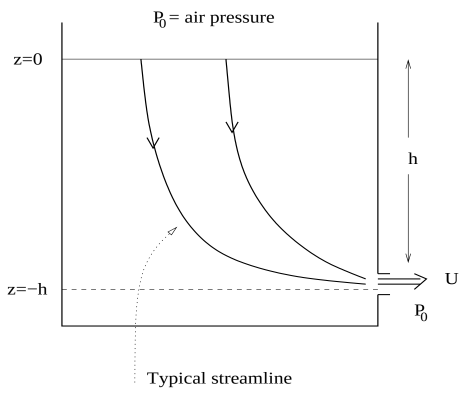

Fig. 5 *Torricelli problem*: the pressure at the top surface and outside the puncture hole is atmospheric pressure P_0. Suppose the height of water above the puncture is h. The goal is to determine how the velocity of water U out of the puncture hole varies with h.

Example (Torricelli 1643). Consider the problem of an oil drum full of water that has a small hole punctured into it near the bottom. The problem is to determine the velocity of the fluid jetting out of the hole at the bottom and how that varies with the amount of water left in the tank—the setup is shown in Fig 5. We shall assume the hole has a small cross-sectional area α. Suppose that the cross-sectional area of the drum, and therefore of the free surface (water surface) at $z = 0$, is A. We naturally assume $A \gg \alpha$. Since the rate at which the amount of water is dropping inside the drum must equal the rate at which water is leaving the drum through the punctured hole, we have

$$\left(-\frac{\mathrm{d}h}{\mathrm{d}t}\right) \cdot A = U \cdot \alpha.$$

We observe that if U is of order one, and $A \gg \alpha$ so that $\alpha/A \ll 1$, then

$$\left|\frac{\mathrm{d}h}{\mathrm{d}t}\right| = U\,(\alpha/A) \ll 1,$$

and hence the velocity of the free surface is asymptotically small.

Since the flow is quasi-stationary, incompressible as it's water, and there is conservative body force due to gravity, we apply Bernoulli's Theorem for one of the typical streamlines shown in Fig. 5. This implies that the quantity H is the same at the free surface and at the puncture hole outlet, hence

$$\tfrac{1}{2}\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 + \frac{P_0}{\rho} = \tfrac{1}{2}U^2 + \frac{P_0}{\rho} - gh.$$

If we ignore the term $(\mathrm{d}h/\mathrm{d}t)^2$, which is small, and cancel the P_0/ρ terms then we can deduce that

$$U = \sqrt{2gh}.$$

Remark 3 Note the pressure inside the container at the puncture hole level is $P_0 + \rho gh$. The difference between this and the atmospheric pressure P_0 outside, accelerates the water through the puncture hole.

9 Vorticity

The *vorticity field* of a flow with velocity field $\boldsymbol{u}$ is defined as

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}.$$

It encodes the magnitude of, and direction of the axis about which, the fluid rotates, locally.

Theorem 3 *Suppose $\boldsymbol{x} \in \mathbb{R}^3$ and $\boldsymbol{x} + \boldsymbol{h} \in \mathbb{R}^3$ is a nearby point. Then for any function $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ we can always write*

$$\boldsymbol{u}(\boldsymbol{x} + \boldsymbol{h}) = \boldsymbol{u}(\boldsymbol{x}) + D(\boldsymbol{x}) \cdot \boldsymbol{h} + \tfrac{1}{2}\boldsymbol{\omega} \times \boldsymbol{h} + \mathcal{O}(h^2),$$

where $D(\boldsymbol{x})$ is a 3×3 symmetric matrix.

Proof By Taylor expansion we have

$$\boldsymbol{u}(\boldsymbol{x} + \boldsymbol{h}) = \boldsymbol{u}(\boldsymbol{x}) + \big(\nabla \boldsymbol{u}(\boldsymbol{x})\big) \cdot \boldsymbol{h} + \mathcal{O}(h^2),$$

where $(\nabla \boldsymbol{u}) \cdot \boldsymbol{h}$ is simply matrix multiplication of the 3×3 matrix $\nabla \boldsymbol{u}$ by the column vector $\boldsymbol{h}$. We can always write

$$\nabla \boldsymbol{u} = \tfrac{1}{2}\big((\nabla \boldsymbol{u}) + (\nabla \boldsymbol{u})^{\mathrm{T}}\big) + \tfrac{1}{2}\big((\nabla \boldsymbol{u}) - (\nabla \boldsymbol{u})^{\mathrm{T}}\big).$$

Set

$$D := \tfrac{1}{2}\big((\nabla \boldsymbol{u}) + (\nabla \boldsymbol{u})^{\mathrm{T}}\big),$$
$$R := \tfrac{1}{2}\big((\nabla \boldsymbol{u}) - (\nabla \boldsymbol{u})^{\mathrm{T}}\big).$$

Note that D is symmetric. Direct computation reveals that

$$R = \tfrac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

where ω_1, ω_2 and ω_3 are the three components of $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$, i.e. $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$. Hence again by direct computation we see that

$$R \cdot \boldsymbol{h} = \tfrac{1}{2}\boldsymbol{\omega} \times \boldsymbol{h}.$$

□

The symmetric matrix D is the *deformation tensor*. Since it is symmetric, there is an orthonormal basis $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, $\hat{\boldsymbol{e}}_3$ in which D is diagonal, i.e. if $X = [\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3]$ then

$$X^{-1} D X = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}.$$

Now consider the motion of a fluid particle labelled by $\boldsymbol{x}+\boldsymbol{h}$ where $\boldsymbol{x}$ is fixed and $\boldsymbol{h}$ is small (for example suppose that only a short time has elapsed). Then the position of the particle is given by

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x}+\boldsymbol{h}) = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{h}) \\ \Leftrightarrow \qquad & \frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{h}) \\ \Leftrightarrow \qquad & \frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} \approx \boldsymbol{u}(\boldsymbol{x}) + D(\boldsymbol{x})\cdot\boldsymbol{h} + \tfrac{1}{2}\boldsymbol{\omega}(\boldsymbol{x})\times\boldsymbol{h}. \end{aligned}$$

Let us consider in turn each of the effects on the right shown. The term $\boldsymbol{u}(\boldsymbol{x})$ is simply uniform translational velocity (the particle being pushed by the ambient flow surrounding it). Now consider the second term $D(\boldsymbol{x})\cdot\boldsymbol{h}$. If we ignore the other terms then, approximately, we have

$$\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = D(\boldsymbol{x})\cdot\boldsymbol{h}.$$

Making a local change of coordinates so that $\boldsymbol{h} = X\hat{\boldsymbol{h}}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\hat{h}_1\\ \hat{h}_2\\ \hat{h}_3\end{pmatrix} = \begin{pmatrix} d_1 & 0 & 0\\ 0 & d_2 & 0\\ 0 & 0 & d_3\end{pmatrix}\begin{pmatrix}\hat{h}_1\\ \hat{h}_2\\ \hat{h}_3\end{pmatrix}.$$

We see that we have pure expansion or contraction (depending on whether d_i is positive or negative, respectively) in each of the characteristic directions $\hat{h}_i$, $i = 1, 2, 3$. Indeed the small linearized volume element $\hat{h}_1\hat{h}_2\hat{h}_3$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}(\hat{h}_1\hat{h}_2\hat{h}_3) = (d_1+d_2+d_3)(\hat{h}_1\hat{h}_2\hat{h}_3).$$

Note that $d_1+d_2+d_3 = \mathrm{Tr}(D) = \nabla\cdot\boldsymbol{u}$.

Let us now consider the effect of the third term $\frac{1}{2}\boldsymbol{\omega}\times\boldsymbol{h}$. Ignoring the other two terms we have

$$\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = \tfrac{1}{2}\boldsymbol{\omega}(\boldsymbol{x})\times\boldsymbol{h}.$$

Direct computation shows that

$$\boldsymbol{h}(t) = \Phi(t, \boldsymbol{\omega}(\boldsymbol{x}))\boldsymbol{h}(0),$$

where $\Phi(t, \boldsymbol{\omega}(\boldsymbol{x}))$ is the matrix that represents the rotation through an angle t about the axis $\boldsymbol{\omega}(\boldsymbol{x})$. Note also that $\nabla\cdot\big(\boldsymbol{\omega}(\boldsymbol{x})\times\boldsymbol{h}\big) = 0$.

Definition 5 (Circulation) Let $\mathcal{C}$ be a simple closed contour in the fluid at time $t = 0$. Suppose that $\mathcal{C}$ is carried along by the flow to the closed contour $\mathcal{C}_t$ at time t, i.e. $\mathcal{C}_t = \varphi_t(\mathcal{C})$. The *circulation* around $\mathcal{C}_t$ is defined to be the line integral

$$K = \oint_{\mathcal{C}_t} \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{r}.$$

Theorem 4 (Kelvin's circulation theorem) *For incompressible flow without external forces, the circulation K for any closed contour $\mathcal{C}_t$ is constant in time.*

10 Exercises

Exercise (trajectories and streamlines) For a given velocity flow field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ prescribed at position $\boldsymbol{x}$ and time t, the particle trajectories are given by the solutions to the system of ordinary differential equations

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}(t), t).$$

Streamlines are given by the solutions to the system of ordinary differential equations (where t is fixed)

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = \boldsymbol{u}\big(\boldsymbol{x}(s), t\big).$$

(a) Explain what particle trajectories and streamlines are, and their difference.

(b) For the two-dimensional flow in Cartesian coordinates,

$$(u, v) = \big(u_0, v_0 \cos(kx - \alpha t)\big),$$

where u_0, v_0, k and α are constants, find the general equation for a streamline. Show that the streamline passing through $(x, y) = (0, 0)$ at $t = 0$ is

$$y = \frac{v_0}{ku_0} \sin(kx).$$

Find the equation for the path of the particle which is at $(x, y) = (0, 0)$ at $t = 0$. Comment briefly on the contrast between the above streamline and particle path in the two separate limiting cases: first $\alpha \to 0$; second $k \to 0$.

Exercise (channel flow) Consider the two-dimensional channel flow (with U a given constant)

$$\boldsymbol{u} = \left(0, U\left(1 - \frac{x^2}{a^2}\right), 0\right),$$

between the two walls $x = \pm a$. Show that there is a *stream function* and find it. (*Hint:* a stream function ψ exists for a velocity field $\boldsymbol{u} = (u, v, w)$ when $\nabla \cdot \boldsymbol{u} = 0$ and we have an additional symmetry. Here the additional symmetry is uniformity with respect to z. You thus need to verify that if

$$u = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x},$$

then $\nabla \cdot \boldsymbol{u} = 0$ and then solve this system of equations to find ψ.)

Show that approximately 91% of the volume flux across $y = y_0$ for some constant y_0 flows through the central part of the channel $|x| \leqslant \frac{3}{4}a$.

Exercise (Couette flow) Let Ω be the region between two concentric cylinders of radii R_1 and R_2, where $R_1 < R_2$. Suppose the velocity field $\boldsymbol{u} = (u_r, u_\theta, u_z)$ of the fluid flow inside Ω, in cylindrical coordinates, is given by

$$u_r = 0, \qquad u_z = 0, \qquad \text{and} \qquad u_\theta = \frac{A}{r} + Br,$$

where

$$A = -\frac{R_1^2 R_2^2(\omega_2 - \omega_1)}{R_2^2 - R_1^2} \qquad \text{and} \qquad B = -\frac{R_1^2\omega_1 - R_2^2\omega_2}{R_2^2 - R_1^2}.$$

Show that the:

(a) velocity field $\boldsymbol{u} = (u_r, u_\theta, u_z)$ is a stationary solution of Euler's equations of motion for an ideal fluid with density $\rho \equiv 1$ (*hint:* you need to find a pressure field p that is consistent with the velocity field given);
(b) angular velocity of the flow on the two cylinders is ω_1 and ω_2.

Exercise (Venturi tube) Consider the Venturi tube shown in Fig. 6. Assume that the ideal fluid flow through the construction is homogeneous, incompressible and steady. The flow in the wider section of cross-sectional area A_1, has velocity u_1 and pressure p_1, while that in the narrower section of cross-sectional area A_2, has velocity u_2 and pressure p_2. Separately within the uniform wide and narrow sections, we assume the velocity and pressure are uniform themselves.

(a) Why does the relation $A_1 u_1 = A_2 u_2$ hold? Why is the flow faster in the narrower region of the tube compared to the wider region of the tube?
(b) Use Bernoulli's theorem to show that

$$\tfrac{1}{2}u_1^2 + \frac{p_1}{\rho_0} = \tfrac{1}{2}u_2^2 + \frac{p_2}{\rho_0},$$

where ρ_0 is the constant uniform density of the fluid.
(c) Using the results in parts (a) and (b), compare the pressure in the narrow and wide regions of the tube.
(d) Give a practical application where the principles of the Venturi tube is used or might be useful.

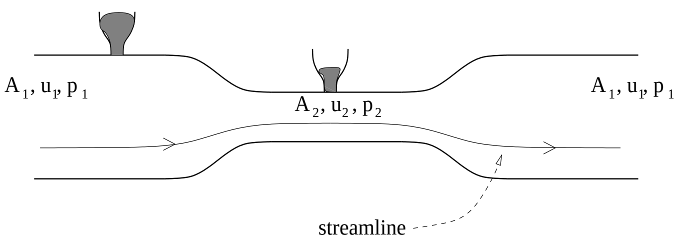

Fig. 6 *Venturi tube*: the flow in the wider section of cross-sectional area A_1 has velocity u_1 and pressure p_1, while that in the narrower section of cross-sectional area A_2 has velocity u_2 and pressure p_2. Separately within the uniform wide and narrow sections, we assume the velocity and pressure are uniform themselves.

Exercise (hurricane) We devise a simple model for a hurricane.

(a) Using the Euler equations for an ideal incompressible flow in cylindrical coordinates (see the bath or sink drain problem in the main text) show that at position (r, θ, z), for a flow which is independent of θ with $u_r = u_z = 0$, we have

$$\frac{u_\theta^2}{r} = \frac{1}{\rho_0}\frac{\partial p}{\partial r},$$
$$0 = \frac{1}{\rho_0}\frac{\partial p}{\partial z} + g,$$

where $p = p(r, z)$ is the pressure and g is the acceleration due to gravity (assume this to be the body force per unit mass). Verify that any such flow is indeed incompressible.

(b) In a *simple* model for a hurricane the air is taken to have uniform constant density ρ_0 and each fluid particle traverses a horizontal circle whose centre is on the fixed vertical z-axis. The (angular) speed u_θ at a distance r from the axis is

$$u_\theta = \begin{cases} \Omega r, & \text{for } 0 \leqslant r \leqslant a, \\ \Omega \frac{a^{3/2}}{r^{1/2}}, & \text{for } r > a, \end{cases}$$

where Ω and a are known constants.

(i) Now consider the flow given above in the inner region $0 \leqslant r \leqslant a$. Using the equations in part (a) above, show that the pressure in this region is given by

$$p = P_0 + \tfrac{1}{2}\rho_0 \Omega^2 r^2 - g\rho_0 z,$$

where P_0 is a constant. A free surface of the fluid is one for which the pressure is constant. Show that the shape of a free surface for $0 \leqslant r \leqslant a$ is a paraboloid of revolution, i.e. it has the form

$$z = Ar^2 + B,$$

for some constants A and B. Specify the exact form of A and B.

(ii) Now consider the flow given above in the outer region $r > a$. Again using the equations in part (a) above, and that the pressure must be continuous at $r = a$, show that the pressure in this region is given by

$$p = P_0 - \frac{\rho_0}{r}\Omega^2 a^3 - g\rho_0 z + \tfrac{3}{2}\rho_0 \Omega^2 a^2,$$

where P_0 is the same constant (reference pressure) as that in part (i) above.

Exercise (Clepsydra or water clock) A clepsydra has the form of a surface of revolution containing water and the level of the free surface of the water falls at a *constant* rate as the water flows out through a small hole in the base. The basic setup is shown in Fig. 7. Show that the container must have the form $z = C\,r^4$ in cylindrical polars, where C is a constant.

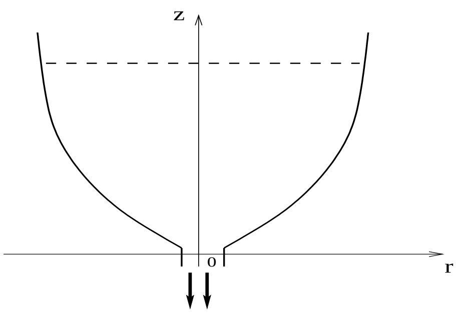

Fig. 7 Clepsydra (water clock).

Exercise (rigid body rotation) An ideal fluid of constant uniform density ρ_0 is contained within a fixed right-circular cylinder (with symmetry axis the z-axis). The fluid moves under the influence of a body force field $\boldsymbol{f} = (\alpha x + \beta y, \gamma x + \delta y, 0)$ per unit mass, where α, β, γ and δ are independent of the space coordinates. Use Euler's equations of motion to show that a rigid body rotation of the fluid about the z-axis, with angular velocity $\omega(t)$ given by $\dot{\omega} = \frac{1}{2}(\gamma - \beta)$ is a possible solution of the equation and boundary conditions. Show that the pressure is given by

$$p = p_0 + \frac{1}{2}\rho_0\big((\omega^2 + \alpha)x^2 + (\beta + \gamma)xy + (\omega^2 + \delta)y^2\big),$$

where p_0 is the pressure at the origin.

Exercise (evolution of vorticity) By taking the curl of Euler's equation for an ideal, incompressible, homogeneous fluid, subject to a conservative body force, show that the vorticity $\boldsymbol{\omega}$ satisfies

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}.$$

11 Notes

11.1 Streaklines

A *streakline* is the locus of all the fluid elements which at some time have past through a particular point, say (x_0, y_0, z_0). We can obtain the equation for a streakline through (x_0, y_0, z_0) by solving the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t),$$

assuming that at $t = t_0$ we have $\big(x(t_0), y(t_0), z(t_0)\big) = (x_0, y_0, z_0)$. Eliminating t_0 between the equations generates the streakline corresponding to (x_0, y_0, z_0). For example, ink dye injected at the point (x_0, y_0, z_0) in the flow will trace out a streakline.

11.2 Isentropic flows

A compressible flow is *isentropic* if there is a function π, called the *enthalpy*, such that

$$\nabla \pi = \frac{1}{\rho}\nabla p.$$

The Euler equations for an isentropic flow are thus

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= -\nabla \pi + \boldsymbol{f} \\ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) &= 0, \end{aligned}$$

in $\mathcal{D}$, and on $\partial\mathcal{D}$, $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ (or matching normal velocities if the boundary is moving).

For compressible ideal gas flow, the pressure is often proportional to ρ^γ, for some constant $\gamma \geqslant 1$, i.e.

$$p = C\,\rho^\gamma,$$

for some constant C. This is a special case of an isentropic flow, and is an example of an *equation of state*. In fact we can actually compute

$$\pi = \int^\rho \frac{p'(z)}{z}\,\mathrm{d}z = \frac{\gamma\,C\,\rho^\gamma}{\gamma - 1},$$

and the *internal energy* (see Chorin and Marsden, pages 14 and 15)

$$\epsilon = \pi - (p/\rho) = \frac{C\,\rho^\gamma}{\gamma - 1}.$$

A Evolution of the flow-map Jacobian

Our goal is to establish the following result for the Jacobian $J(\boldsymbol{x}, t)$ of the flow-map φ_t:

$$\frac{\partial}{\partial t} J(\boldsymbol{x}, t) = \big(\nabla \cdot \boldsymbol{u}(\varphi(\boldsymbol{x}, t), t)\big)\, J(\boldsymbol{x}, t).$$

Recall, for a fixed position $\boldsymbol{x} \in \mathcal{D}$ we denote by $\boldsymbol{\xi}(\boldsymbol{x}, t) = (\xi, \eta, \zeta)$ the position of the particle at time t, which at time $t = 0$ was at $\boldsymbol{x}$. We use φ_t to denote the map $\boldsymbol{x} \mapsto \boldsymbol{\xi}(\boldsymbol{x}, t)$, it is called the flow-map. Hence by definition $J(\boldsymbol{x}, t) = \det\big(\nabla\boldsymbol{\xi}(\boldsymbol{x}, t)\big)$. We know that a particle at position $\boldsymbol{\xi}(\boldsymbol{x}, t) = \big(\xi(\boldsymbol{x}, t), \eta(\boldsymbol{x}, t), \zeta(\boldsymbol{x}, t)\big)$, which started at $\boldsymbol{x}$ at time $t = 0$, evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi}(\boldsymbol{x}, t) = \boldsymbol{u}\big(\boldsymbol{\xi}(\boldsymbol{x}, t), t\big).$$

Since $\boldsymbol{x}$ is fixed this is equivalent to

$$\frac{\partial}{\partial t}\boldsymbol{\xi}(\boldsymbol{x}, t) = \boldsymbol{u}\big(\boldsymbol{\xi}(\boldsymbol{x}, t), t\big).$$

Taking the gradient with respect to $\boldsymbol{x}$ of this relation, and swapping over the gradient and $\partial/\partial t$ operations on the left, we see that

$$\frac{\partial}{\partial t}\nabla\boldsymbol{\xi}(\boldsymbol{x}, t) = \nabla\boldsymbol{u}\big(\boldsymbol{\xi}(\boldsymbol{x}, t), t\big).$$

Using the chain rule we have

$$\nabla_{\boldsymbol{x}}\boldsymbol{u}\big(\boldsymbol{\xi}(\boldsymbol{x}, t), t\big) = \Big(\nabla_{\boldsymbol{\xi}}\boldsymbol{u}\big(\boldsymbol{\xi}(\boldsymbol{x}, t), t\big)\Big) \cdot \big(\nabla_{\boldsymbol{x}}\boldsymbol{\xi}(\boldsymbol{x}, t)\big).$$

Combining the last two relations we see that

$$\frac{\partial}{\partial t}\nabla\boldsymbol{\xi} = (\nabla_{\boldsymbol{\xi}}\boldsymbol{u})\,\nabla\boldsymbol{\xi}.$$

Abel's Theorem then tells us that $J = \det\nabla\boldsymbol{\xi}$ evolves according to

$$\frac{\partial}{\partial t}\det\nabla\boldsymbol{\xi} = \big(\mathrm{Tr}(\nabla_{\boldsymbol{\xi}}\boldsymbol{u})\big)\,\det\nabla\boldsymbol{\xi}.$$

Since $\mathrm{Tr}(\nabla_{\boldsymbol{\xi}}\boldsymbol{u}) \equiv \nabla\cdot\boldsymbol{u}$ we have established the required result.

B Multivariable calculus identities

We provide here some useful multivariable calculus identities. Here ϕ and ψ are generic scalars, and $\boldsymbol{u}$ and $\boldsymbol{v}$ are generic vectors.

1. $\nabla \times \boldsymbol{u} = \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{pmatrix} = \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix}$.
2. $\nabla \cdot (\nabla \phi) = \nabla^2 \phi = \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2}$.
3. $\nabla \times (\nabla \phi) \equiv \boldsymbol{0}$.
4. $\nabla \cdot (\nabla \times \boldsymbol{u}) \equiv 0$.
5. $\nabla \times (\nabla \times \boldsymbol{u}) = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla^2 \boldsymbol{u}$.
6. $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$.
7. $\nabla(\boldsymbol{u} \cdot \boldsymbol{v}) = (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} + \boldsymbol{u} \times (\nabla \times \boldsymbol{v}) + \boldsymbol{v} \times (\nabla \times \boldsymbol{u})$.
8. $\nabla \cdot (\phi \boldsymbol{u}) = \phi(\nabla \cdot \boldsymbol{u}) + \boldsymbol{u} \cdot \nabla\phi$.
9. $\nabla \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\nabla \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\nabla \times \boldsymbol{v})$.
10. $\nabla \times (\phi \boldsymbol{u}) = \phi \nabla \times \boldsymbol{u} + \nabla\phi \times \boldsymbol{u}$.
11. $\nabla \times (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{u}\,(\nabla \cdot \boldsymbol{v}) - \boldsymbol{v}\,(\nabla \cdot \boldsymbol{u}) + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{v}$.

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