Nonholonomic Flows on Lie Groups

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Mainly work with D. Zenkov. Also Marsden, Murray, Krishnaprasad

- **Properties of Nonholonomic Systems:**
- **Energy Conservation**

Hamiltonian: Yes. Nonholonomic: Yes.

- **Momentum Conservation**

Hamiltonian: Yes, Noether's Theorem. Nonholonomic: No, Momentum Equation

- **Measure (volume) Preservation**

Hamiltonian: Yes. Nonholonomic: No, in general

- **Stability**

Hamiltonian: Never asymptotic. Nonholonomic: Can be asymptotic.

- **Key: Almost Poisson structure, Nonvariational**

•Nonholonomic Equations of Motion

See e.g. Bloch, Krishnaprsad, Marsden and Murray [1996] and Zenkov, Bloch and Marsden [1998], Bloch and Crouch [1995] and other references in these papers.

•The Lagrange-d'Alembert Principle

• Consider a system with a configuration space Q, local coordinates q^i and m nonintegrable constraints

$$\dot{s}^a + A_\alpha^a(r, s)\dot{r}^\alpha = 0$$

where $q = (r, s) \in \mathbb{R}^{n-p} \times \mathbb{R}^p$, which we write as $q^i = (r^\alpha, s^a)$, where $1 \leq \alpha \leq n - p$ and $1 \leq a \leq p$.

• Lagrangian $L(q^i, \dot{q}^i)$.

Equations of motion given by Lagrange-d'Alembert principle.

Definition 0.1 *The **Lagrange-d'Alembert equations of motion** for the system are those determined by*

$$\delta \int_a^b L(q^i, \dot{q}^i)\, dt = 0,$$

where we choose variations $\delta q(t)$ of the curve $q(t)$ that satisfy $\delta q(a) = \delta q(b) = 0$ and $\delta q(t)$ satisfies the constraints for each t where $a \leq t \leq b$.

• This principle is supplemented by the condition that the curve itself satisfies the constraints.

- Note that we take the variation *before* imposing the constraints; that is, we do not impose the constraints on the family of curves defining the variation.

- Equivalent to:

$$-\delta L = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right) \delta q^i = 0$$

for all variations $\delta q^i = (\delta r^\alpha, \delta s^a)$ satisfying the constraints at each point of the underlying curve $q(t)$, *i.e.* such that $\delta s^a + A^a_\alpha \delta r^\alpha = 0$.

Substituting:

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial r^\alpha} \right) = A^a_\alpha \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{s}^a} - \frac{\partial L}{\partial s^a} \right)$$

for all $\alpha = 1, \ldots, n - p$.

Combined with the constraint equations

$$\dot{s}^a = -A^a_\alpha \dot{r}^\alpha$$

for all $a = 1, \ldots, p$, give the complete equations of motion of the system.

Useful way of reformulating equations (0.2) is to define a *constrained Lagrangian* by substituting the constraints (0.3) into the Lagrangian:

$$L_c(r^\alpha, s^a, \dot{r}^\alpha) := L(r^\alpha, s^a, \dot{r}^\alpha, -A^a_\alpha(r, s)\dot{r}^\alpha).$$

The equations of motion can be written in terms of the constrained Lagrangian in the following way, as a direct coordinate calculation shows:

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A_\alpha^a \frac{\partial L_c}{\partial s^a} = -\frac{\partial L}{\partial \dot{s}^b}B_{\alpha\beta}^b \dot{r}^\beta,$$

where $B_{\alpha\beta}^b$ is defined by

$$B_{\alpha\beta}^b = \left(\frac{\partial A_\alpha^b}{\partial r^\beta} - \frac{\partial A_\beta^b}{\partial r^\alpha} + A_\alpha^a \frac{\partial A_\beta^b}{\partial s^a} - A_\beta^a \frac{\partial A_\alpha^b}{\partial s^a} \right).$$
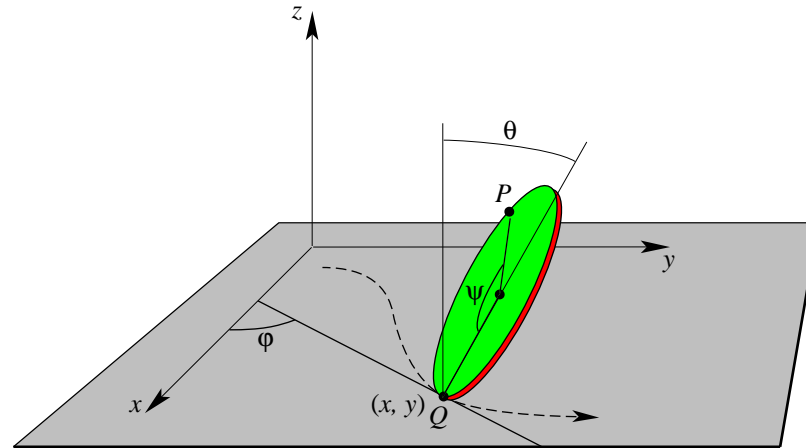
• The Falling Rolling Disk

Figure 0.1: The geometry for the rolling disk.

This is a system which exhibits stability but not asymptotic stability.

Denote mass, the radius, and the moments of inertia of the disk by m, R, A, B.

$$L = \frac{m}{2}\left[(\xi - R(\dot\phi \sin\theta + \dot\psi))^2 + \eta^2 \sin^2\theta + (\eta\cos\theta + R\dot\theta)^2\right]$$
$$+ \frac{1}{2}\left[A(\dot\theta^2 + \dot\phi^2 \cos^2\theta) + B(\dot\phi \sin\theta + \dot\psi)^2\right] - mgR\cos\theta,$$

where $\xi = \dot x \cos\phi + \dot y \sin\phi + R\dot\psi$ and $\eta = -\dot x \sin\phi + \dot y \cos\phi$, while the constraints are given by

$$\dot x = -\dot\psi R \cos\phi, \qquad \dot y = -\dot\psi R \sin\phi.$$
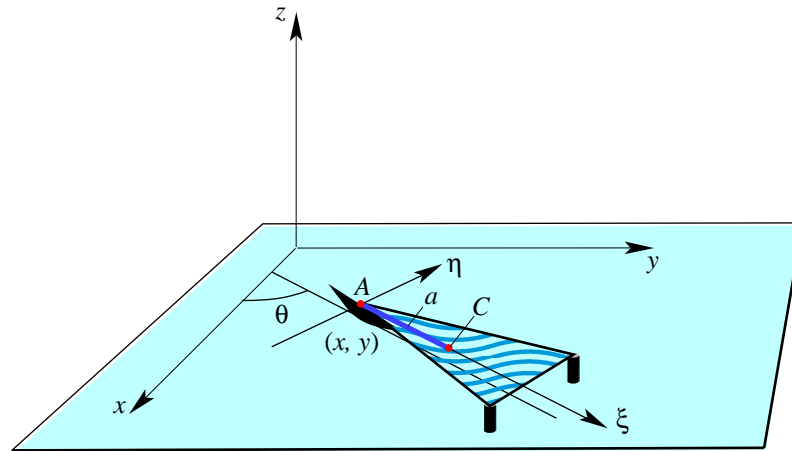
Other systems:

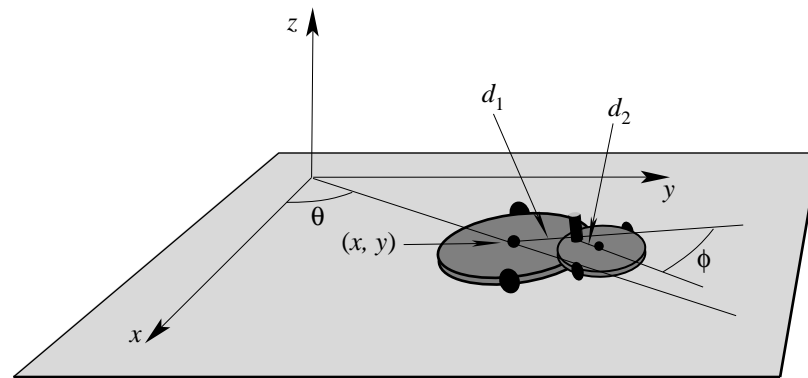Figure 0.2: The Chaplygin sleigh is a rigid body moving on two sliding posts and one knife edge.

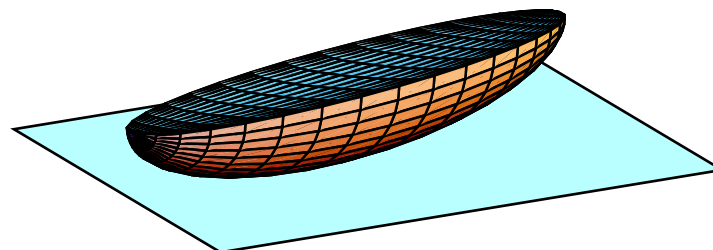Figure 0.3: The geometry for the roller racer.

Figure 0.4: The rattleback.

•The Chaplygin Sleigh

● Perhaps the simplest mechanical system which illustrates the possible dissipative nature of energy preserving nonholonomic systems.

Compare the sleigh equations to the Toda lattice equations.
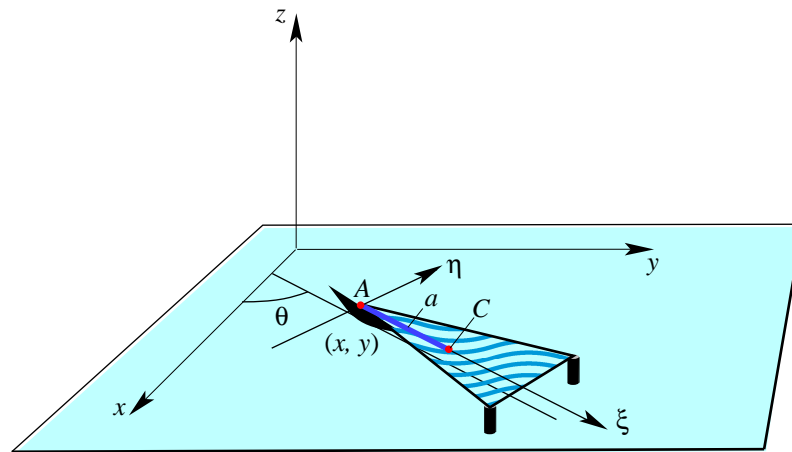

Figure 0.5: The Chaplygin sleigh is a rigid body moving on two sliding posts and one knife edge.

Equations:

$$\dot{v} = a\omega^2$$
$$\dot{\omega} = -\frac{ma^2}{I + ma^2}v\omega$$

Equations have a family of relative equilibria given by $(v, \omega)|v = \text{const}, \quad \omega = 0.$
Linearizing about any of these equilibria one finds one zero eigenvalue and one negative

eigenvalue.

In fact the solution curves are ellipses in $v - \omega$ plane with the positive v-axis attracting all solutions.
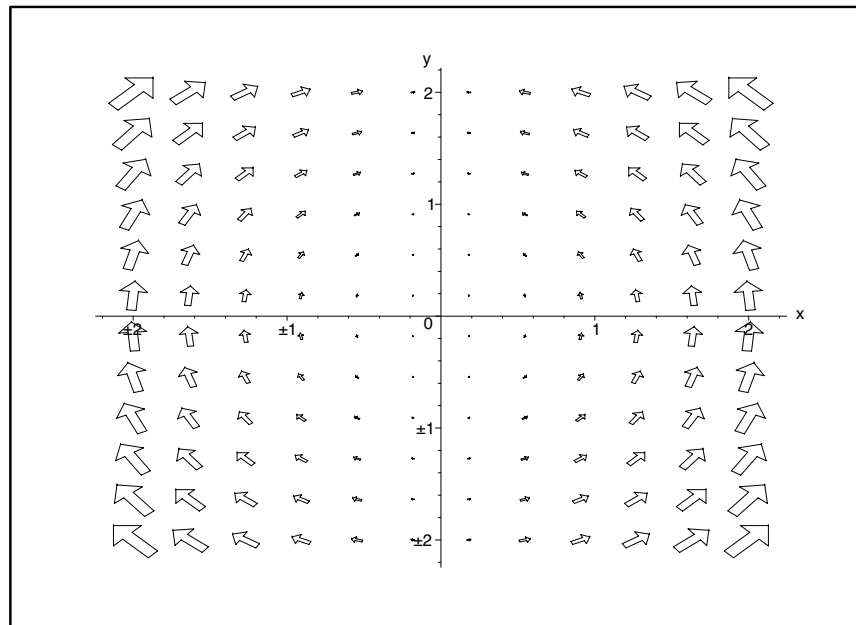
Figure 0.6: Chaplygin Sleigh/2d Toda phase portrait.

Normalizing, we have the equations

$$\dot{v} = \omega^2$$
$$\dot{\omega} = -v\omega.$$

Scaling time by a factor of two have: Chaplygin sleigh equations are equivalent to the two-dimensional Toda lattice equations except for the fact that there is no sign restriction on the

variable ω. Hence can be written in Lax pair form and solved by the method of factorization.

•The Toda Lattice

Interacting particles on the line.

Non-periodic finite Toda lattice as analyzed by Moser [1974]:

$$H(x, y) = \frac{1}{2} \sum_{k=1}^{n} y_k^2 + \sum_{k-1}^{n-1} e^{(x_k - x_{k+1})} .$$

Hamiltonian equations:

$$\dot{x}_k = \frac{\partial H}{\partial y_k} = y_k$$

$$\dot{y}_k = -\frac{\partial H}{\partial x_k} = e^{x_{k-1} - x_k} - e^{x_k - x_{k-1}} ,$$

where assume $e^{x_0 - x_1} = e^{x_n - x_{n+1}} = 0$.

Flaschka:

$$a_k = \frac{1}{2} e^{(x_k - x_{k+1})/2} \quad b_k = -\frac{1}{2} y_k .$$

Get:

$$\dot{a}_k = a_k(b_{k+1} - b_k), \quad k = 1, \cdots, n-1$$

$$\dot{b}_k = 2(a_k^2 - a_{k-1}^2), \quad k = 1, \cdots, n$$

with the boundary conditions $a_0 = a_n = 0$ and where the $a_i > 0$.

Matrix form:
$$\frac{d}{dt}L = [B, L] = BL - LB,$$
If N is the matrix $\text{diag}[1, 2, \cdots, n]$ the Toda flow can be written
$$\dot{L} = [L, [L, N]].$$

Shows flow also gradient (on a level set of its integrals).

• Double bracket form of Brockett [1988] (see Bloch [1990], Bloch Brockett and Ratiu [1990, 1992]).

•**The Two-dimensional Toda Lattice**

In two-dimensional case matrices in the Lax pair are
$$L = \begin{pmatrix} b_1 & a_1 \\ a_1 & -b_1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}.$$

Equations of motion:
$$\dot{b_1} = 2a_1^2$$
$$\dot{a_1} = -2a_1 b_1$$

For initial data $b_1 = 0$, $a_1 = c$, explicitly carrying out the factorization yields explicit solution
$$b_1(t) = -c\frac{\sinh 2ct}{\cosh 2ct}, \qquad a_1(t) = \frac{c}{\cosh 2ct}$$

Symmetries Symmetries play an important role in our analysis. Suppose we are given a nonholonomic system with Lagrangian $L : TQ \to \mathbb{R}$, and a (nonintegrable) constraint distribution $\mathcal{D}$. We can then look for a group G that acts freely and properly on the configuration space Q. It induces an action on the tangent space TQ and so it makes sense to ask that the Lagrangian L be *invariant*. Also, one can ask that the distribution be *invariant* in the sense that the action by a group element $g \in G$ maps the distribution $\mathcal{D}_q$ at the point $q \in Q$ to the distribution $\mathcal{D}_{gq}$ at the point gq. If these properties hold, we say that G is a *symmetry group*. The manifold Q/G is called the *shape space* of the system and the configuration space has the structure of a principal fiber bundle $\pi : Q \to Q/G$.

Geometry of Nonholonomic Systems with Symmetry

The group orbit through a point q, an (immersed) submanifold, is denoted

$$\mathrm{Orb}(q) := \{gq \mid g \in G\}.$$

Let $\mathfrak{g}$ denote the Lie algebra of the Lie group G. For an element $\xi \in \mathfrak{g}$, we denote by ξ_Q the vector field on Q arising from the corresponding infinitesimal generator of the group action, so these are also the tangent spaces to the group orbits. Define, for each $q \in Q$, the vector subspace $\mathfrak{g}^q$ to be the set of Lie algebra elements in $\mathfrak{g}$ whose infinitesimal generators evaluated at q lie in both $\mathcal{D}_q$ and $T_q(\mathrm{Orb}(q))$:

$$\mathfrak{g}^q := \{\xi \in \mathfrak{g} \mid \xi_Q(q) \in \mathcal{D}_q \cap T_q(\mathrm{Orb}(q))\} .$$

The corresponding bundle over Q whose fiber at the point q is given by $\mathfrak{g}^q$, is denoted by $\mathfrak{g}^{\mathcal{D}}$.

Reduced dynamics. Assuming that the Lagrangian and the constraint distribution are G-invariant, we can form the *reduced velocity phase space TQ/G* and the *reduced constraint space $\mathcal{D}/G$*. The Lagrangian L induces well defined functions, the *reduced Lagrangian*

$$l : TQ/G \to \mathbb{R}$$

and the *constrained reduced Lagrangian*

$$l_c : \mathcal{D}/G \to \mathbb{R},$$

satisfying $L = l \circ \pi_{TQ}$ and $L|_{\mathcal{D}} = l_c \circ \pi_{\mathcal{D}}$ where $\pi_{TQ} : TQ \to TQ/G$ and $\pi_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}/G$ are the projections. By general considerations, the Lagrange-d'Alembert equations induce well defined *reduced equations* on $\mathcal{D}/G$. That is, the vector field on the manifold $\mathcal{D}$ determined by the Lagrange-d'Alembert equations (including the constraints) is G-invariant, and so defines a reduced vector field on the quotient manifold $\mathcal{D}/G$. Call these equations the *Lagrange-d'Alembert-Poincaré equations.*

Let a local trivialization be chosen on the principle bundle $\pi : Q \to Q/G$, with a local representation having components denoted (r, g). Let r, an element of shape space Q/G, have coordinates denoted r^α, and let g be group variables for the fiber, G. In such a representation, the action of G is the left action of G on the second factor. The coordinates (r, g) induce the coordinates $(r, \dot{r}, \xi)$ on TQ/G, where $\xi = g^{-1}\dot{g}$. The Lagrangian L is invariant under the left action of G and so it depends on g and $\dot{g}$ only through the combination $\xi = g^{-1}\dot{g}$. Thus the reduced Lagrangian l is given by

$$l(r, \dot{r}, \xi) = L(r, g, \dot{r}, \dot{g}).$$

Therefore the full system of equations of motion consists of the following two groups:

1. The Lagrange-d'Alembert-Poincaré equation on $\mathcal{D}/G$ (see theorem 0.2).

2. The *reconstruction equation*

$$\dot{g} = g\xi.$$

The nonholonomic momentum in body representation. Choose a q-dependent basis $e_A(q)$ for the Lie algebra such that the first m elements span the subspace $\mathfrak{g}^q$ in the following way. First, one chooses, for each r, such a basis at the identity element $g = \text{Id}$, say

$$e_1(r), e_2(r), \ldots, e_m(r), e_{m+1}(r), \ldots, e_k(r).$$

Now define the *body fixed basis* by

$$e_A(r, g) = \text{Ad}_g\, e_A(r).$$

Then the first m elements will indeed span the subspace $\mathfrak{g}^q$ since the distribution is invariant. We denote the structure constants of the Lie algebra relative to this basis by C_{AB}^C.

To avoid confusion, we make the following index conventions:

1. The first batch of indices range from 1 to m corresponding to the symmetry directions along constraint space. These indices will be denoted $a, b, c, \ldots$.

2. The second batch of indices range from $m+1$ to k corresponding to the symmetry directions not aligned with the constraints. Indices for this range will be denoted by $a', b', c', \ldots$.

3. The indices $A, B, C, \ldots$ on the Lie algebra $\mathfrak{g}$ range from 1 to k.

4. The indices $\alpha, \beta, \ldots$ on the shape variables r range from 1 to σ. Thus, σ is the dimension of the shape space Q/G and so $\sigma = n - k$.

The summation convention for all of these indices will be understood.

Assume that the Lagrangian has the form of kinetic minus potential energy, and that the constraints and the orbit directions span the entire tangent space to the configuration space:

$$\mathcal{D}_q + T_q(\mathrm{Orb}(q)) = T_q Q.$$

Then it is possible to introduce a new Lie algebra variable Ω called the *body angular velocity* such that:

1. $\Omega = \mathcal{A}\dot{r} + \xi$, where the Lie algebra valued form $\mathcal{A} = \mathcal{A}_\alpha^A e_A(r) dr^\alpha$ is called the *nonholonomic connection* (see Bloch *et al.* [1996] for details).

2. The constraints are given by $\Omega \in \mathrm{span}\{e_1(r), \ldots, e_m(r)\}$ or $\Omega^{m+1} = \cdots = \Omega^k = 0$.

3. The reduced Lagrangian in the variables $(r, \dot{r}, \Omega)$ becomes

$$l(r^\alpha, \dot{r}^\alpha, \Omega^A) = \frac{1}{2} g_{\alpha\beta} \dot{r}^\alpha \dot{r}^\beta + \frac{1}{2} \mathbb{I}_{AB} \Omega^A \Omega^B + \lambda_{a'\alpha} \dot{r}^\alpha \Omega^{a'} - U(r). \qquad (0.8)$$

Here $g_{\alpha\beta}$ are coefficients of the kinetic energy metric induced on the manifold Q/G, $\mathbb{I}_{AC}$ are components of the *locked inertia tensor* defined by

$$\langle \mathbb{I}(r)\xi, \eta \rangle = \langle\!\langle \xi_Q, \eta_Q \rangle\!\rangle, \qquad \xi, \eta \in \mathfrak{g},$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is the kinetic energy metric. The coefficients $\lambda_{a'\alpha}$ are defined by

$$\lambda_{a'\alpha} = \frac{\partial^2 l}{\partial \xi^{a'} \partial r^\alpha} - \frac{\partial^2 l}{\partial \xi^{a'} \partial \xi^B} \mathcal{A}_\alpha^B.$$

The constrained reduced Lagrangian becomes especially simple in the variables $(r, \dot{r}, \Omega)$:

$$l_c = \frac{1}{2} g_{\alpha\beta} \dot{r}^\alpha \dot{r}^\beta + \frac{1}{2} \mathbb{I}_{ab} \Omega^a \Omega^b - U. \tag{0.9}$$

We remark that this choice of Ω block-diagonalizes the kinetic energy metric, *i.e.*, eliminates the terms proportional to $\Omega^a \dot{r}^\alpha$ in (0.9).

The *nonholonomic momentum in body representation* is defined by

$$p_a = \frac{\partial l}{\partial \Omega^a} = \frac{\partial l_c}{\partial \Omega^a}, \qquad a = 1, \ldots, m.$$

Notice that the nonholonomic momentum may be viewed as a collection of components of the ordinary momentum map along the constraint directions.

The Lagrange-d'Alembert-Poincaré equations. As in Bloch *et al.* [1996], the reduced equations of motion are given by the next theorem.

Theorem 0.2 *The following* **reduced nonholonomic Lagrange-d'Alembert-Poincaré equations** *hold for each* $1 \leq \alpha \leq \sigma$ *and* $1 \leq b \leq m$:

$$\frac{d}{dt}\frac{\partial l_c}{\partial \dot{r}^\alpha} - \frac{\partial l_c}{\partial r^\alpha} = -\mathcal{D}^c_{b\alpha}I^{bd}p_c p_d - \mathcal{K}_{\alpha\beta\gamma}\dot{r}^\beta \dot{r}^\gamma$$

$$- (\mathcal{B}^c_{\alpha\beta} - I_{c'a'}\mathbb{I}^{a'c}\mathcal{B}^{c'}_{\alpha\beta} + \mathcal{D}_{b\beta\alpha}I^{bc})p_c\dot{r}^\beta, \tag{0.10}$$

$$\frac{d}{dt}p_a = (C^c_{ba} - C^{c'}_{ba}I_{c'a'}\mathbb{I}^{a'c})I^{bd}p_c p_d + \mathcal{D}^c_{a\alpha}p_c\dot{r}^\alpha + \mathcal{D}_{a\alpha\beta}\dot{r}^\alpha \dot{r}^\beta. \tag{0.11}$$

Here and below $l_c(r^\alpha, \dot{r}^\alpha, \Omega^a)$ is the constrained Lagrangian, and I^{bd} and $I_{a'c'}$ are the inverse of the tensors $\mathbb{I}|_{\mathfrak{g}^q}$ and $\mathbb{I}^{-1}|_{(\mathfrak{g}^q)^*}$, respectively. We stress that in general $I^{bd} \neq \mathbb{I}^{bd}$ and $I_{a'c'} \neq \mathbb{I}_{a'c'}$.

The coefficients $\mathcal{B}^C_{\alpha\beta}$, $\mathcal{D}^c_{b\alpha}$, $\mathcal{D}_{b\alpha\beta}$, $\mathcal{K}_{\alpha\beta\gamma}$ are given by the formulae

$$\mathcal{B}^C_{\alpha\beta} = \frac{\partial \mathcal{A}^C_\alpha}{\partial r^\beta} - \frac{\partial \mathcal{A}^C_\beta}{\partial r^\alpha} - C^C_{BA}\mathcal{A}^A_\alpha \mathcal{A}^B_\beta + \gamma^C_{A\beta}\mathcal{A}^A_\alpha - \gamma^C_{A\alpha}\mathcal{A}^A_\beta,$$

$$\mathcal{D}^c_{b\alpha} = -(C^c_{Ab} - C^{c'}_{Ab}I_{c'a'}\mathbb{I}^{a'c})\mathcal{A}^A_\alpha + C^{c'}_{ab}\lambda_{c'\alpha}I^{ac} + \gamma^c_{b\alpha} - \gamma^{c'}_{b\alpha}I_{c'a'}\mathbb{I}^{a'c}, \qquad (0.12)$$

$$\mathcal{D}_{b\alpha\beta} = \lambda_{c'\beta}(\gamma^{c'}_{b\alpha} - C^{c'}_{Ab}\mathcal{A}^A_\alpha),$$

$$\mathcal{K}_{\alpha\beta\gamma} = \lambda_{c'\gamma}\mathcal{B}^{c'}_{\alpha\beta},$$

and the coefficients $\gamma^C_{b\alpha}$ are defined by

$$\frac{\partial e_b}{\partial r^\alpha} = \gamma^C_{b\alpha}e_C.$$

Equations (0.10) and (0.11) generalize the equations of motion in the orthogonal body frame (see Bloch *et al.* [1996]). Here we no longer assume that the body frame is orthogonal.

•Almost Poisson Systems

Recall:

Definition 0.3 *An almost Poisson manifold is a pair* $(M, \{,\})$ *where* M *is a smooth manifold and (i)* $\{,\}$ *defines an almost Lie algebra structure on the* C^∞ *functions on* M, *i.e. the bracket satisfies all conditions for a Lie algebra except that the Jacobi identity is not satisfied and (ii)* $\{,\}$ *is a derivation in each factor.*

If in addition Jacobi satisfied, Poisson manifold.

An almost Poisson structure on M will be Poisson if its *Jacobiator*, defined by

$$J(f,g,h) = \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\}g\}$$

vanishes.

• One can define an *almost Poisson* vector field on M by

$$\dot{z}_i = \pi_{ij}(z)\frac{\partial H}{\partial z_j}.$$

.

• "Hamiltonian" Formulation of Nonholonomic Systems

Nonholonomic systems are almost Poisson.

Start on the Lagrangian side with a configuration space Q and a Lagrangian L (possibly of the form kinetic energy minus potential energy, *i.e.*,

$$L(q, \dot{q}) = \frac{1}{2}\langle\langle \dot{q}, \dot{q} \rangle\rangle - V(q),$$

As above, our nonholonomic constraints are given by a distribution $\mathcal{D} \subset TQ$. We also let $\mathcal{D}^0 \subset T^*Q$ denote the annihilator of this distribution. Using a basis ω^a of the annihilator $\mathcal{D}^o$, we can write the constraints as

$$\omega^a(\dot{q}) = 0,$$

where $a = 1, \ldots, k$.

Recall that the cotangent bundle T^*Q is equipped with a canonical Poisson bracket and is expressed in the canonical coordinates (q, p) as

$$\{F, G\}(q, p) = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} = \left(\frac{\partial F^T}{\partial q}, \frac{\partial F^T}{\partial p}\right) J \begin{pmatrix} \frac{\partial G}{\partial q} \\ \frac{\partial G}{\partial p} \end{pmatrix}.$$

Here J is the canonical Poisson tensor

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

A constrained t phase space $\mathcal{M} = \mathbb{F}L(\mathcal{D}) \subset T^*Q$ is defined so that the constraints on the Hamiltonian side are given by $p \in \mathcal{M}$. In local coordinates,

$$\mathcal{M} = \left\{ (q, p) \in T^*Q \,\middle|\, \omega_i^a \frac{\partial H}{\partial p_i} = 0 \right\}.$$

Let $\{X_\alpha\}$ be a local basis for the constraint distribution $\mathcal{D}$ and let $\{\omega^a\}$ be a local basis for the annihilator $\mathcal{D}^0$. Let $\{\omega_a\}$ span the complementary subspace to $\mathcal{D}$ such that $\langle \omega^a, \omega_b \rangle = \delta_b^a$, where δ_b^a is the usual Kronecker delta. Here $a = 1, \ldots, k$ and $\alpha = 1, \ldots, n - k$. Define a coordinate transformation $(q, p) \mapsto (q, \tilde{p}_\alpha, \tilde{p}_a)$ by

$$\tilde{p}_\alpha = X_\alpha^i p_i, \qquad \tilde{p}_a = \omega_a^i p_i.$$

In the new (generally not canonical) coordinates $(q, \tilde{p}_\alpha, \tilde{p}_a)$, the Poisson tensor becomes

$$\tilde{J}(q, \tilde{p}) = \begin{pmatrix} \{q^i, q^j\} & \{q^i, \tilde{p}_j\} \\ \{\tilde{p}_i, q^j\} & \{\tilde{p}_i, \tilde{p}_j\} \end{pmatrix}.$$

Use $(q, \tilde{p}_\alpha)$ as induced local coordinates for $\mathcal{M}$. It is easy to show that

$$\frac{\partial \tilde{H}}{\partial q^j}(q, \tilde{p}_\alpha, \tilde{p}_a) = \frac{\partial H_\mathcal{M}}{\partial q^j}(q, \tilde{p}_\alpha),$$

$$\frac{\partial \tilde{H}}{\partial \tilde{p}_\beta}(q, \tilde{p}_\alpha, \tilde{p}_a) = \frac{\partial H_\mathcal{M}}{\partial \tilde{p}_\beta}(q, \tilde{p}_\alpha),$$

where $H_\mathcal{M}$ is the constrained Hamiltonian on $\mathcal{M}$ expressed in the induced coordinates. We can also truncate the Poisson tensor $\tilde{J}$ by leaving out its last k columns and last k rows and then describe the constrained dynamics on $\mathcal{M}$ expressed in the induced coordinates $(q^i, \tilde{p}_\alpha)$ as follows:

$$\begin{pmatrix} \dot{q}^i \\ \dot{\tilde{p}}_\alpha \end{pmatrix} = J_\mathcal{M}(q, \tilde{p}_\alpha) \begin{pmatrix} \frac{\partial H_\mathcal{M}}{\partial q^j}(q, \tilde{p}_\alpha) \\ \frac{\partial H_\mathcal{M}}{\partial \tilde{p}_\beta}(q, \tilde{p}_\alpha) \end{pmatrix}, \qquad \begin{pmatrix} q^i \\ \tilde{p}_\alpha \end{pmatrix} \in \mathcal{M}.$$

Here $J_\mathcal{M}$ is the $(2n - k) \times (2n - k)$ truncated matrix of $\tilde{J}$ restricted to $\mathcal{M}$ and is expressed in the induced coordinates.

The matrix $J_{\mathcal{M}}$ defines a bracket $\{\,\cdot\,,\cdot\,\}_{\mathcal{M}}$ on the constraint submanifold $\mathcal{M}$ as follows:

$$\{F_{\mathcal{M}}, G_{\mathcal{M}}\}_{\mathcal{M}}(q, \tilde{p}_\alpha) := \left(\frac{\partial F_{\mathcal{M}}^T}{\partial q^i} \frac{\partial F_{\mathcal{M}}^T}{\partial \tilde{p}_\alpha} \right) J_{\mathcal{M}}(q^i, \tilde{p}_\alpha) \begin{pmatrix} \frac{\partial G_{\mathcal{M}}}{\partial q^j} \\ \frac{\partial G_{\mathcal{M}}}{\partial \tilde{p}_\beta} \end{pmatrix},$$

for any two smooth functions $F_{\mathcal{M}}, G_{\mathcal{M}}$ on the constraint submanifold $\mathcal{M}$. Clearly, this bracket satisfies the first two defining properties of a Poisson bracket, namely, skew symmetry and the Leibniz rule, and one can show that it satisfies the Jacobi identity if and only if the constraints are holonomic. Furthermore, the constrained Hamiltonian $H_{\mathcal{M}}$ is an integral of motion for the constrained dynamics on $\mathcal{M}$ due to the skew symmetry of the bracket.

Following e.g. van der Schaft and Maschke [1994] and Koon and Marsden [1997] we can write the nonholonomic equations of motion as follows:

$$
\begin{pmatrix} \dot{s}^a \\ \dot{r}^\alpha \\ \dot{\tilde{p}}_\alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 & -A^a_\beta \\ 0 & 0 & \delta^\alpha_\beta \\ (A^b_\alpha)^T & -\delta^\beta_\alpha & -p_c B^c_{\alpha\beta} \end{pmatrix} \begin{pmatrix} \frac{\partial H_\mathcal{M}}{\partial s^b} \\ \frac{\partial H_\mathcal{M}}{\partial r^\beta} \\ \frac{\partial H_\mathcal{M}}{\partial \tilde{p}_\beta} \end{pmatrix}
$$

Jacobiator of the Poisson tensor vanishes precisely when the curvature of the nonholonomic constraint distribution is zero or the constraints are holonomic.

•Euler-Poincaré-Suslov Equations

Important special case of the reduced nonholonomic equations.

•**Example: Euler-Poincaré-Suslov Problem on** $SO(3)$ In this case the problem can be formulated as the standard Euler equations

$$I\dot{\omega} = I\omega \times \omega$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ are the system angular velocities in a frame where the inertia matrix is of the form $I = \text{diag}(I_1, I_2, I_3)$ and the system is subject to the constraint

$$a \cdot \omega = 0$$

where $a = (a_1, a_2, a_3)$.

The nonholonomic equations of motion are then given by

$$I\dot{\omega} = I\omega \times \omega + \lambda a$$

subject to the constraint. Solve for λ:

$$\lambda = -\frac{I^{-1}a \cdot (I\omega \times \omega)}{I^{-1}a \cdot a}.$$

If a is an eigenvector of the moment of inertia tensor flow is measure preserving.

Invariant Measures of the Euler-Poincaré-Suslov Equations An important special case of the reduced nonholonomic equations is the case when there is no shape space at all. In this case the system is characterized by the Lagrangian $L = \frac{1}{2}\mathbb{I}_{AB}\Omega^A\Omega^B$ and the left-invariant constraint

$$\langle a, \Omega \rangle = a_A\Omega^A = 0. \tag{0.20}$$

Here $a = a_A e^A \in \mathfrak{g}^*$ and $\Omega = \Omega^A e_A$, where e_A, $A = 1, \ldots, k$, is a basis for $\mathfrak{g}$ and e^A is its dual basis. Multiple constraints may be imposed as well. The two classical examples of such systems are the *Chaplygin Sleigh* and the *Suslov problem*. These problems were introduced by Chaplygin in 1895 and Suslov in 1902, respectively.

We can consider the problem of when such systems exhibit asymptotic behavior. Following Kozlov [1988] it is convenient to consider the unconstrained case first. In the absence of constraints the dynamics is governed by the basic Euler-Poincaré equations

$$\dot{p}_B = C_{AB}^C \mathbb{I}^{AD} p_C p_D = C_{AB}^C p_C \Omega^A \tag{0.21}$$

where $p_B = \mathbb{I}_{AB}\Omega^B$ are the components of the momentum $p \in \mathfrak{g}^*$. One considers the question of whether the (unconstrained) equations (0.21) have an absolutely continuous integral invariant $f d^k\Omega$ with summable density $\mathcal{M}$. If $\mathcal{M}$ is a positive function of class C^1 one calls the integral invariant an invariant measure. Kozlov [1988] shows

Theorem 0.4 *The Euler-Poincaré equations have an invariant measure if and only if the group G is unimodular.*

A group is said to be unimodular if it has a bilaterally invariant measure. A criterion

for unimodularity is $C_{AC}^C = 0$ (using the Einstein summation convention). Now we know (Liouville's theorem) that the flow of a vector differential equation $\dot{x} = f(x)$ is phase volume preserving if and only if div $f = 0$. In this case the divergence of the right hand side of equation (0.21) is $C_{AC}^C \mathbb{I}^{AD} p_D = 0$. The statement of the theorem now follows from the following theorem of Kozlov [1998]: *A flow due to a homogeneous vector field in $\mathbb{R}^n$ is measure-preserving if and only if this flow preserves the standard volume in $\mathbb{R}^n$.*

Now, turning to the case where we have the constraint (0.20) we obtain the *Euler-Poincaré-Suslov equations*

$$\dot{p}_B = C_{AB}^C \mathbb{I}^{AD} p_C p_D + \lambda a_B = C_{AB}^C p_C \Omega^A + \lambda a_B \tag{0.22}$$

together with the constraint (0.20). Here λ is the Lagrange multiplier. This defines a system on the subspace of the dual Lie algebra defined by the constraint. Since the constraint is assumed to be nonholonomic, this subspace is not a subalgebra. One can then formulate a condition for the existence of an invariant measure of the Euler-Poincaré-Suslov equations.

Theorem 0.5 *Equations (0.22) have an invariant measure if and only if*

$$K \operatorname{ad}_{\mathbb{I}^{-1}a}^* a + T = \mu a, \qquad \mu \in \mathbb{R}, \tag{0.23}$$

where $K = 1/\langle a, \mathbb{I}^{-1}a \rangle$ and $T \in \mathfrak{g}^*$ is defined by $\langle T, \xi \rangle = \operatorname{Tr}(\operatorname{ad}_\xi)$.

This theorem was proved by Kozlov [1988] for compact algebras and for arbitrary algebras by Jovanović [1998]. In coordinates, condition (0.23) becomes

$$K C_{AB}^C \mathbb{I}^{AD} a_C a_D + C_{BC}^C = \mu a_B.$$

For a compact algebra (0.23) becomes

$$[\mathbb{I}^{-1}a, a] = \mu a, \qquad \mu \in \mathbb{R}, \tag{0.24}$$

where we identified $\mathfrak{g}^*$ with $\mathfrak{g}$.

The proof of theorem 0.5 reduces to the computation of the divergence of the vector field in (0.22).

In the compact case only constraint vectors a which commute with $\mathbb{I}^{-1}a$ allow the measure to be preserved. This means that a and $\mathbb{I}^{-1}a$ must lie in the same maximal commuting subalgebra. In particular, if a is an eigenstate of the inertia tensor, the reduced phase volume is preserved. When the maximal commuting subalgebra is one-dimensional this is a necessary condition. This is the case for groups such as $SO(3)$.

We thus have the following result which reflects a symmetry requirement on the constraints:

Theorem 0.6 *A compact Euler-Poincaré-Suslov system is measure preserving (i.e. does not exhibit asymptotic dynamics) if the constraint vectors a are eigenvectors of the inertia tensor, or if the constrained system is $\mathbb{Z}_2$ symmetric about each of its principal axes. If the maximal commuting subalgebra is one-dimensional this condition is necessary.*

Invariant Measures of Systems with Internal Degrees of Freedom In this section we extend the result of Kozlov [1988] and Jovanović [1998] to nonholonomic systems with nontrivial shape space. One can think of these systems as the Euler-Poincaré-Suslov systems with internal degrees of freedom. Recall that the constraints are of the form $\Omega^{m+1} = \cdots = \Omega^k = 0$. To simplify the exposition, we consider below systems with a single constraint. The results are valid for systems with multiple constraints as well.

Consider a nonholonomic system with the reduced Lagrangian $l(r, \dot{r}, \Omega)$ and a constraint $\langle a(r), \Omega \rangle = 0$. The subspace of the Lie algebra defined by the constraint at the configuration q is denoted here by $\mathfrak{g}^q$. The orientation of this subspace in $\mathfrak{g}$ depends on the shape configuration of the system, r. The dimension of $\mathfrak{g}^q$ however stays the same. As discussed in section , we choose a special moving frame in which $\mathfrak{g}^q$ is spanned by the vectors $e_1(r), \ldots, e_{k-1}(r)$. The equation of the constraint in this basis becomes $\Omega^k = 0$. Recall that the horizontal part of the kinetic energy metric is $g_{\alpha\beta}(r)$.

Theorem 0.7 *The system associated with the reduced Lagrangian $l(r, \dot{r}, \Omega)$ and the constraint $\langle a(r), \Omega \rangle = 0$ has an integral invariant with a C^1 density $\mathcal{M}(r)$ if and only if*

(i) $\left(C^a_{ba} - C^k_{ba} \dfrac{\mathbb{I}^{ka}}{\mathbb{I}^{kk}} \right) - g^{\alpha\delta} \mathcal{D}_{b\alpha\delta} = 0,$

(ii) the form $\left[\mathcal{D}^b_{b\beta} - g^{\alpha\delta} \lambda_{k\delta} \mathcal{B}^k_{\alpha\beta} \right] dr^\beta$ *is exact.*

Systems with One-Dimensional Shape Space. Assume that condition (i) of theorem 0.7 is satisfied. In this case the equation for the density of the invariant measure becomes

$$d(\ln \mathcal{M}) = d(\ln g) + \mathcal{D}_b^b dr. \tag{0.25}$$

The solution of this equation is globally defined if the shape space is either noncompact (and thus diffeomorphic to $\mathbb{R}$), or compact and the average of the function $\mathcal{D}_b^b$ equals zero.

Systems with Conserved Momentum. If the nonholonomic momentum is a constant of motion, then condition (i) of theorem 0.7 is trivially satisfied. Moreover, condition (ii) now asks that the form

$$g^{\alpha\delta}\lambda_{k\delta}\mathcal{B}_{\alpha\beta}^k r^\beta \tag{0.26}$$

is exact. The system thus preserves the measure with the density

$$\mathcal{M} = \det g \exp\left(-\int g^{\alpha\delta}\lambda_{k\delta}\mathcal{B}_{\alpha\beta}^k r^\beta\right).$$

Examples

The Routh Problem. This mechanical system consists of a uniform sphere rolling without slipping on the inner surface of a vertically oriented surface of revolution. He described the family of stationary periodic motions and obtained a necessary condition for stability of these motions. Routh noticed as well that integration of the equations of motion may be reduced to integration of a system of two linear differential equations with variable coefficients and considered a few cases when the equations of motion can be solved by quadratures. Modern references that treat this system are Hermans [1995] and Zenkov [1995].

This problem is $SO(2) \times SO(2)$-invariant, where the first copy of $SO(2)$ represents rotations about the axis of the surface of revolution while the second copy of $SO(2)$ represents rotations of the sphere about its radius through the contact point of the surface and the sphere.

Let r be the latitude of this contact point, a be the radius of the sphere, $c(r) + a$ be the reciprocal of the curvature of the meridian of the surface, and $b(r)$ be the distance from the axis of the surface to the sphere's center. The shape metric is $c^2(r)\dot{r}^2/2$ while the momentum equations are

$$\dot{p}_1 = \frac{c(r)\sin r}{b(r)}p_1\dot{r} - \frac{2}{7}p_2\dot{r}, \qquad \dot{p}_2 = \left(1 - \frac{c(r)\cos r}{b(r)}\right)p_1\dot{r}.$$

The shape space is one-dimensional, the symmetry group $SO(2) \times SO(2)$ is commutative, and there are no terms proportional to $\dot{r}^2$ in the momentum equations. The trace term in (0.25) equals $c(r)\sin r/b(r)$, and thus the density of the invariant measure for the Routh problem is

$$\mathcal{M} = c^2(r)e^{\int \frac{c(r)\sin r}{b(r)}\,dr}. \tag{0.27}$$

The group action in this problem is singular: the intersection points of the surface of revolution and its axis have nontrivial isotropy subgroups. The shape coordinate r equals $\pm\pi/2$ at these points. As a result,

$$\lim_{r\to-\pi/2}\mathcal{M}(r) = \lim_{r\to\pi/2}\mathcal{M}(r) = \infty.$$

The Falling Disk. Consider a homogeneous disk rolling without sliding on a horizontal plane. This mechanical system is $SO(2) \times SE(2)$-invariant; the group $SO(2)$ represents the symmetry of the disk while the group $SE(2)$ represents the Euclidean symmetry of the overall system.

Classical references for the rolling disk are Vierkandt [1892], Korteweg [1899], and Appel [1900]. In particular, Vierkandt showed that on the reduced space $\mathcal{D}/SE(2)$—the constrained velocity phase space modulo the action of the Euclidean group $SE(2)$—most orbits of the system are periodic.

The shape of the system is specified by a single coordinate—the tilt of the disk denoted here by r. The momentum equations are

$$\dot{p}_1 = mR^2 \left(-\frac{\sin r}{A \cos r} p_1 + \left(\frac{\cos r}{mR^2 + B} + \frac{\sin^2 r}{A \cos r} \right) p_2 \right) \dot{r},$$

$$\dot{p}_2 = mR^2 \left(-\frac{1}{A \cos r} p_1 + \frac{\sin r}{A \cos r} p_2 \right) \dot{r}.$$

Hence, the trace terms $\mathcal{D}_b^b$ in (0.25) vanish, and the density of the invariant measure equals the component of the shape metric $g(r)$. The latter equals the moment of inertia of the disk with respect to the line through the rim of the disk and parallel to its diameter. Since the density of the measure is determined up to a constant factor, we conclude that the dynamics preserves the reduced phase space volume.

The 3D Chaplygin Sleigh with an Oscillating Mass. The three-dimensional Chaplygin sleigh is a free rigid body subject to the nonholonomic constraint $v^3 = 0$, where v^3 is the third component of the (linear) velocity relative to the body frame. The Lagrangian of this system is

$$\frac{1}{2}M\left[(v^1)^2 + (v^2)^2 + (v^3)^2\right] + \frac{1}{2}\left[I_1(\Omega^1)^2 + I_2(\Omega^2)^2 + I_3(\Omega^3)^2\right].$$

In this formula M is the mass of the body, I_j are the eigenvalues of its inertia tensor, and $(\Omega^1, \Omega^2, \Omega^3)$ and (v^1, v^2, v^3) are the angular and linear velocities relative to the body frame. The dynamics of this system is discussed in Neimark and Fufaev [1972].

We couple this system with an oscillator moving along the third coordinate axis of the body frame. The mass of this oscillator is m and the displacement from the origin is r. To keep the notation uniform with the general theory, we write the components of the linear velocity relative to the body frame as $(\Omega^4, \Omega^5, \Omega^6)$. The vector $(\Omega^1, \Omega^2, \Omega^3, \Omega^4, \Omega^5, \Omega^6)$ should be viewed as an element of the Lie algebra $se(3)$. The Lagrangian of this new system is

$$L = \frac{1}{2}\left[I_1(\Omega^1)^2 + I_2(\Omega^2)^2 + I_3(\Omega^3)^2\right] + \frac{M}{2}\left[(\Omega^4)^2 + (\Omega^5)^2 + (\Omega^6)^2\right]$$
$$+ \frac{m}{2}\left[(\Omega^4 + \Omega^2 r)^2 + (\Omega^5 - \Omega^1 r)^2 + (\Omega^6 + \dot{r})^2\right] - U(r). \qquad (0.28)$$

The configuration space is $\mathbb{R} \times SE(3)$, and the system is invariant under the left action of $SE(3)$ on the second factor. We have not specified the potential energy as its choice does not affect the existence of the invariant measure. The shape space is just the first factor of $\mathbb{R} \times SE(3)$ and is one dimensional, and thus the above theory is applicable. To show the existence of the

invariant measure, we note the following:

1. The constrained Lagrangian does not contain terms that simultaneously depend on $\dot{r}$ and p_a. The constraint is $\Omega^6 = 0$. Therefore, all the coefficients of the nonholonomic connection as well as its curvature form vanish. This implies that the terms $\mathcal{D}_{a\alpha\beta}$ and $\mathcal{K}_{\alpha\beta\gamma}$ vanish too. The differential form from condition (ii) of theorem 0.7 is therefore trivial.

2. The moving frame is r-independent. Therefore all of the coefficients $\gamma^B_{A\alpha}$ are trivial. Condition (i) of theorem 0.7 is satisfied since the group $SE(3)$ is unimodular and e_6 is the eigenvector of the inertia tensor.

3. The shape metric is r-independent.

The system's dynamics preserves the volume in the reduced phase space.

This can be verified by a straightforward computation of the divergence of the vector field that defines the equations of motion:

$$\ddot{r} = -\frac{\partial U_a}{\partial r},$$
$$\dot{p}_1 = -\Omega^2 p_3 + \Omega^3 p_2 - m\Omega^5 \dot{r},$$
$$\dot{p}_2 = -\Omega^3 p_1 + \Omega^1 p_3 + m\Omega^4 \dot{r},$$
$$\dot{p}_3 = -\Omega^1 p_2 + \Omega^2 p_1 - \Omega^4 p_5 + \Omega^5 p_4,$$
$$\dot{p}_4 = \Omega^3 p_5 - m\Omega^2 \dot{r},$$
$$\dot{p}_5 = -\Omega^3 p_4 + m\Omega^1 \dot{r}.$$

Chaplygin Sphere. This system consists of a sphere rolling without slipping on a horizontal plane. The center of mass of this sphere is at the geometric center, but the principal moments of inertia are distinct. Chaplygin [1903] proved integrability of this problem. Modern references for the Chaplygin sphere include Kozlov [1985] and Schneider [2002].

One may view this system as a nonholonomic version of the Euler top. The configuration space is diffeomorphic to $SO(3) \times \mathbb{R}^2$. We choose the Euler angles (θ, ϕ, ψ) and the Cartesian coordinates (x, y) as the configuration parameters of the Chaplygin sphere. The Lagrangian and constraints written in these coordinates become

$$L = \frac{I_1}{2}(\dot{\theta}\cos\psi + \dot{\phi}\sin\psi\sin\theta)^2 + \frac{I_2}{2}(-\dot{\theta}\sin\psi + \dot{\phi}\cos\psi\sin\theta)^2$$
$$+ \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \frac{M}{2}(\dot{x}^2 + \dot{y}^2)$$

and

$$\dot{x} - \dot{\theta}\sin\phi + \dot{\psi}\cos\phi\sin\theta = 0, \qquad \dot{y} + \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\sin\theta = 0,$$

respectively.

This system is $SE(2)$ invariant. The action by the group element (α, a, b) on the configuration space is given by

$$(\theta, \psi, \phi, x, y) \mapsto (\theta, \psi, \phi + \alpha, x\cos\alpha - y\sin\alpha + a, x\sin\alpha + y\cos\alpha + b).$$

The shape space for the Chaplygin sphere is diffeomorphic to the two-dimensional sphere. The nonholonomic momentum map has just one component and is moreover preserved. Straightforward computations show that the form (0.26) is exact. The conditions for measure existence

are therefore satisfied. The density of the invariant measure is computed in *overdetermined coordinates* in Chaplygin [1903] (see also Kozlov [1985]).

The invariant manifolds of the Chaplygin sphere are two-dimensional tori. The phase flow on these tori is measure preserving and thus there are angle variables (x, y) on each torus in which the flow equations become

$$\dot{x} = \frac{\lambda}{\mathcal{M}(x, y)}, \qquad \dot{y} = \frac{\mu}{\mathcal{M}(x, y)}.$$

See Kolmogorov [1953] and Kozlov [1985] for details. In general, these equations cannot be rewritten as

$$\dot{x} = \lambda, \qquad \dot{y} = \mu.$$

The flow however becomes quasi-periodic after a time substitution $dt = \mathcal{M}(x, y)d\tau$ (see Kozlov [1985] for details). This example thus shows that the flow on the nonholonomic invariant tori can be more complicated than a Hamiltonian flow.

It follows that adding a symmetry preserving potential to the Lagrangian of the Chaplygin sphere leaves the new system measure preserving with the same measure density. This was pointed out by Kozlov for a specific potential (see Kozlov [1985] for details).

A Spherical Robot: Controlled Chaplygin's Sphere and Chaplygin's Top

Robot viewed as a controlled Chaplygin's sphere or Chaplygin's top.
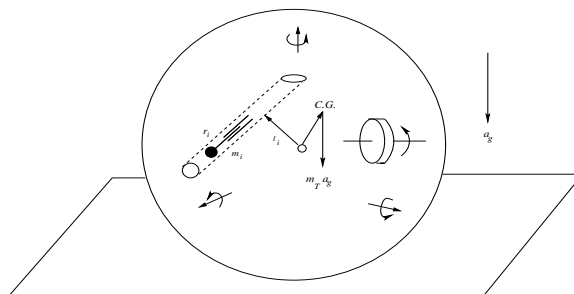

Figure 0.7: Schematic configuration of a spherical robot in a uniform gravitational field with a slider and a rotor

Assume the spherical base body can roll without sliding on a horizontal plane in a uniform gravitational field; the radius of the ball is a. See Figure 0.7 for its schematic configuration. Choose a base body coordinate frame with the origin at the center of the ball. Let $x \in \mathbb{R}^3$ denote the position of the center of the ball in the inertial frame, and let $R \in \mathrm{SO}(3)$ represent the base body attitude that maps from the base body coordinate frame onto the inertial frame. The spherical robot is controlled by internal actuators. Relative motion of the internal actuators with respect to the base body is described by generalized shape coordinates $r \in Q_s$, where Q_s is referred to as the shape space. Hence, the configuration space manifold is $\mathrm{SO}(3) \times \mathbb{R}^3 \times Q_s$. Note that we are only interested in (x_1, x_2), the horizontal position of the center of the ball. The general theory of developing equations of motion in the form of Euler-Poincarè is given in the next subsection.

General System

Let G be a general linear matrix Lie group and W be a vector space; let S denote the semidirect product $G \textcircled{S} W$. The shape space Q_s is an n-dimensional Abelian Lie group with local coordinates $r = (r_1, \cdots, r_n)$. The configuration manifold is $Q = S \times Q_s$. We assume that $G \times Q_s$ forms a trivial principal fiber bundle and that for every $q = (g, r) \in Q$ where $g \in G$ and $r \in Q_s$, the left action of G on $G \times Q_s$ is a smooth map $\Phi : G \times (G \times Q_s) \to (G \times Q_s)$ given by $\Phi_h q = (hg, r)$ for an arbitrary $h \in G$, where Φ is assumed to be free and proper. We assume that controls act on the shape space Q_s so that the shape is fully actuated.

Let the reduced Lagrangian be

$$l(\xi, Y, \Gamma, r, \dot{r}) = T(\xi, Y, r, \dot{r}) - V(\Gamma, r),$$

where $\xi = g^{-1}\dot{g} \in \mathfrak{g}$, $\Gamma = g^{-1}a_0 \in W$ for a fixed $a_0 \in W$. Let $y(t)$ be a (smooth) curve in W, $Y = g^{-1}\dot{y} \in W$. Assume the reduced kinetic energy $T(\xi, Y, r, \dot{r})$ can be written as

$$T(\xi, Y, r, \dot{r}) = \frac{1}{2} \Big(\xi, Y, \dot{r} \Big) \overline{M}(r) \begin{pmatrix} \xi \\ Y \\ \dot{r} \end{pmatrix},$$

where $\overline{M}(r)$ denotes a reduced inertia tensor on Q_s only. Assume the reduced constraint can be written as $Y = \xi\zeta(\Gamma)$, where $\zeta : W \to W$ is a smooth vector-valued function. Thus the constrained reduced Lagrangian becomes

$$l_c(\xi, \Gamma, r, \dot{r}) = T_c(\xi, \Gamma, r, \dot{r}) - V(\Gamma, r),$$

where

$$T_c(\xi, \Gamma, r, \dot{r}) = \frac{1}{2}\left(\xi, \dot{r}\right) M(\Gamma, r) \begin{pmatrix} \xi \\ \dot{r} \end{pmatrix}.$$

Here $M(\Gamma, r)$ defines another reduced inertia tensor dependent on Q_s and the dynamic parameter $\Gamma \in W$, which is referred to as "advected parameter" in the literature. Since $\Gamma = g^{-1}a_0$ evolves on G, one may view $M(\Gamma, r)$ as a function on $G \times Q_s$.

Equations of Motion of the Spherical Robot

In this section, we apply the general results given in the previous section to the spherical robot. For this example, the Lie group $G = SO(3)$ and the vector space $W = \mathbb{R}^3$.

Let m_0 be the mass of the base body, J_0 be the inertia tensor of the base body defined with respect to the base body coordinate frame, $m_i,\ i = 1, \cdots, N$, be the mass of the i-th auxiliary body, and $J_i(r)$ be the inertia tensor of the i-th auxiliary body defined with respect to the base body coordinate frame. Moreover, let ρ_0 denote the relative position vector of the center of mass of the ball and let $\rho_i(r)$ denote the relative position vector of the center of mass of the i-th auxiliary body. We denote the angular velocity of the i-th body relative to the base body coordinate frame by $C_i(r)\dot{r}$. That is, suppose the orientation of the i-th body in the base body frame is given by $R_i(r) \in SO(3)$, where $R_i(r)$ maps from a coordinate frame for the i-th body into the base body coordinate frame. Let $\widehat{\omega}_i(r, \dot{r}) = R_i^{-1}(r)\frac{d}{dt}R_i(r)$ denote the angular velocity of the i-th body relative to its own coordinate frame. Then $C_i(r)\dot{r} = R_i(r)\omega_i(r, \dot{r})$.

We obtain the reduced equations of motion for $SO(3) \times Q_s$

$$\frac{d}{dt}\left(\frac{\partial l_c}{\partial \omega}\right) - \frac{\partial l_c}{\partial \omega} \times \omega = -m_T a\left\{\left[\left(\frac{\partial \rho_c(r)}{\partial r}\dot{r}\right) \cdot \omega\right]\Gamma - \left[(\rho_c(r) \times \Gamma) \cdot \omega + \left(\frac{\partial \rho_c(r)}{\partial r}\dot{r}\right) \cdot \Gamma\right]\omega\right\}$$
$$+ m_T a_g \Gamma \times \rho_c(r), \tag{0.29}$$

$$\frac{d}{dt}\left(\frac{\partial l_c}{\partial \dot{r}}\right) - \frac{\partial l_c}{\partial r} = u_s, \tag{0.30}$$

where the shape dynamics are completely controlled.