Geometry of Steering Laws in Cooperative Control

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The Center for Communicating Networked Control Systems

James Clerk Maxwell (1831-1879)

Maxwell and Control

Maxwell's (1857) essay on Saturn's rings contained the first use of the **characteristic polynomial** to assess stability. This later led to his use of the same method in designing a speed governor – a physical feedback control device – in a problem of accurate electrical measurements.

Maxwell studied the problem of Saturn's rings under Newtonian attraction. Coherence of a ring of **artificial** earth satellites can be achieved by **synthetic** interactions (realized via feedback control laws).

Maxwell and Control

It is extraordinary that, as we are gathered here to discuss geometric integration and control, the spacecraft Cassini-Huygens is taking the closest look ever at Saturn’s rings (and shepherd moons), a triumph of **action at a distance** via precise control. New data on bending waves, density waves and spokes in the ring system may lead to further theoretical work on the problem of Saturn’s rings.

Maxwell and Gyroscopic Interaction

Maxwell's equations for the electromagnetic field are complemented by the equation of Lorentz for the force on a charged particle in an electromagnetic field.

Force law

$$F = q(E + v \times B)$$

In a region where the electric field vanishes, the particle motion is governed by a purely gyroscopic Lagrangian given in terms of the magnetic vector potential. **Gyroscopic forces leave the kinetic energy invariant**.

Our subject

This talk is about the exploitation of purely gyroscopic interactions to achieve coherent patterns in the motion of particles. We do this in the language most natural to the problem, the language of **moving frames.** The next few slides constitute a brief review of some aspects of moving frames.

Frenet-Serret Frame

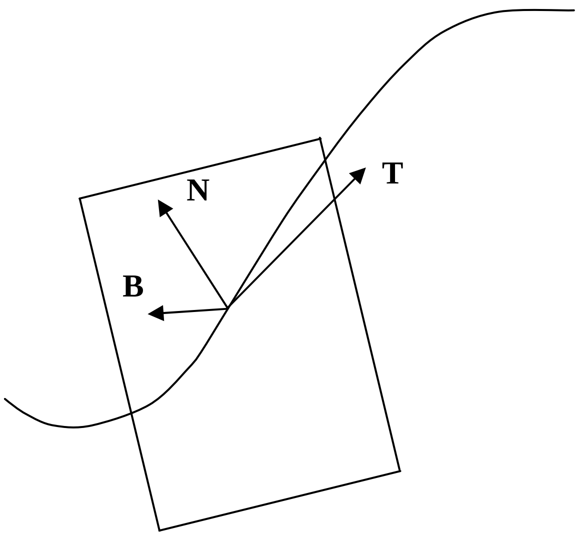
Regular Curve

$$s \mapsto \gamma(s)$$

$$C^3$$

and

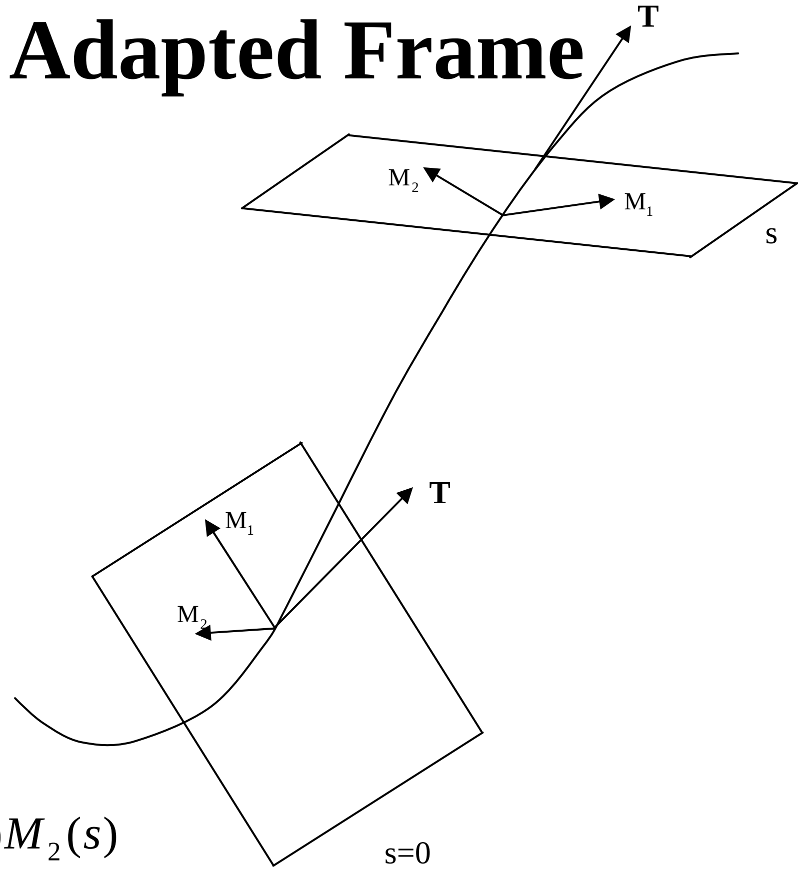
$$\gamma'' \neq 0$$



Frenet-Serret equations

$$\begin{aligned}
\gamma'(s) &= T(s) \\
T'(s) &= \kappa(s)N(s) \\
N'(s) &= -\kappa(s)T(s) + \tau(s)B(s) \\
B'(s) &= -\tau(s)N(s)
\end{aligned}$$

Natural Frenet or Relatively Parallel Adapted Frame



Regular Curve

$$s \mapsto \gamma(s)$$
$$C^2$$

Equations for relatively parallel adapted frame

$$\begin{aligned}
\gamma'(s) &= T(s) \\
T'(s) &= \qquad\qquad k_1(s)M_1(s) \quad + k_2(s)M_2(s) \\
M_1'(s) &= -k_1(s)T(s) \\
M_2'(s) &= -k_2(s)T(s)
\end{aligned}$$

R. L. Bishop, *Amer. Math. Month.* (1975), **82**(3):246-251

Control System on SE(3)

$$\begin{pmatrix} g & \gamma \\ 0 & 1 \end{pmatrix}' = \begin{pmatrix} g & \gamma \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \xi & e_1 \\ 0 & 0 \end{pmatrix}$$

where $g = [TNB]$ or $[T\, M_1 M_2]$

and $\xi = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & -k_1 & -k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix}$

and $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Inversion

$$\kappa(s) = \|\gamma''(s)\|$$

$$\tau(s) = \frac{\gamma'(s)\cdot\left(\gamma''(s)\times\gamma'''(s)\right)}{\left(\kappa(s)\right)^2}$$

(FS)

$$k_i(s) = \gamma''(s)\cdot M_i(0) - \int_0^s k_i(\sigma)\gamma''(s)\cdot\gamma'(\sigma)\;d\sigma$$

$$i = 1,2$$

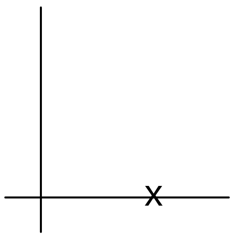
(RPAF)

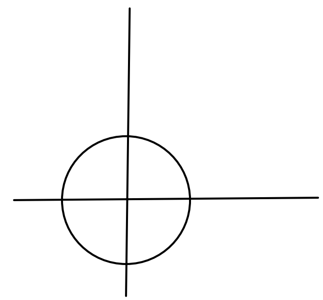
$$\kappa(s) = \sqrt{k_1^2(s)+k_2^2(s)}$$

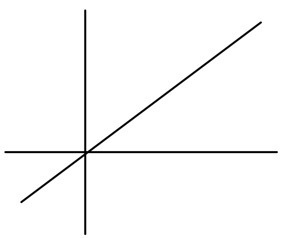
$$\theta'(s) = \tau$$

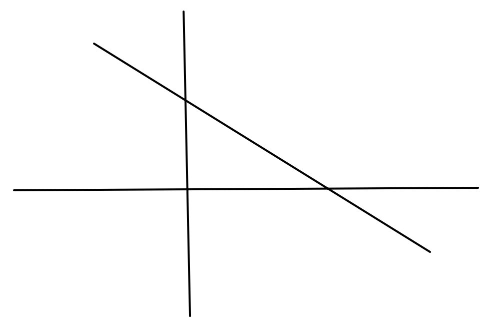
(where $\theta =$ arg $(\boldsymbol{k})$

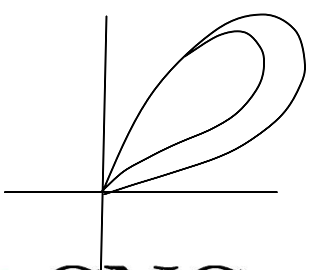
Normal (k-plane) Development of γ











Frame interaction

Isokinetic Motion

$$F = q(\mathrm{E} + \upsilon \times \mathrm{B})$$

$$\upsilon = \dot{x}$$

$$\mathrm{E} = 0$$

$$L = \tfrac{1}{2}\left\langle \upsilon, \upsilon \right\rangle + q\left\langle \upsilon, \mathrm{A}^{\#} \right\rangle$$

$$\mathrm{B} = \nabla \times \mathrm{A}$$

Outline

Motivation: Flying in spatial patterns (formations)

Model: Interaction laws for unit-speed particles

- Formations as shape equilibria
- Convergence to specific formations
  - Analysis for two particles (vehicles)
  - Simulations for *n* vehicles
- Interaction with surfaces - obstacle-avoidance and boundary-following

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- AFOSR: “Dynamics and Control of Agile Formations”
- ARO: “Communicating Networked Control Systems”

Red Arrows


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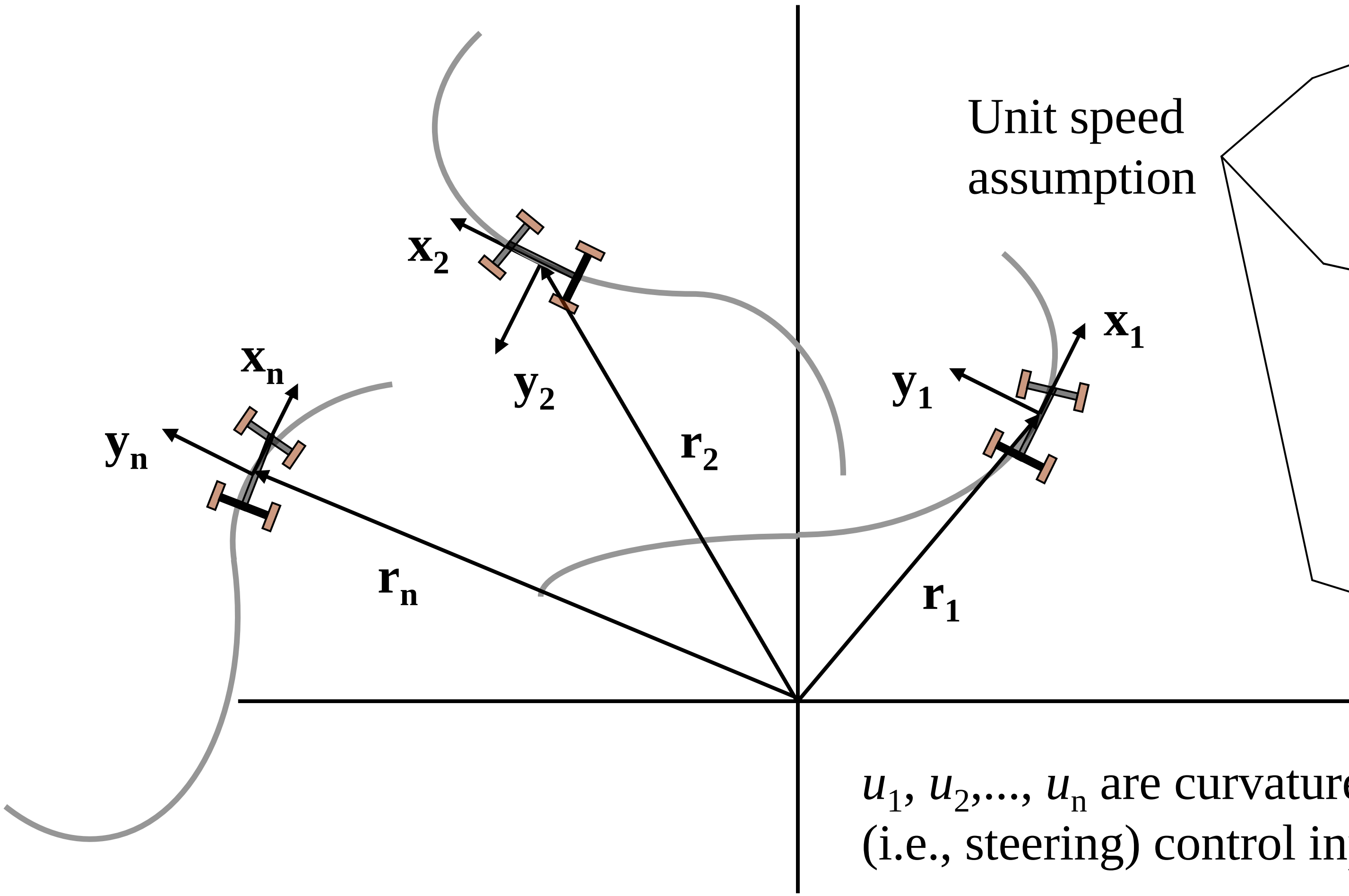
Planar Model (Natural Frenet Frames)



Unit speed assumption

$$\dot{\mathbf{r}}_1 = \mathbf{x}_1$$
$$\dot{\mathbf{x}}_1 = \mathbf{y}_1 u_1$$
$$\dot{\mathbf{y}}_1 = -\mathbf{x}_1 u_1$$

$$\dot{\mathbf{r}}_2 = \mathbf{x}_2$$
$$\dot{\mathbf{x}}_2 = \mathbf{y}_2 u_2$$
$$\dot{\mathbf{y}}_2 = -\mathbf{x}_2 u_2$$

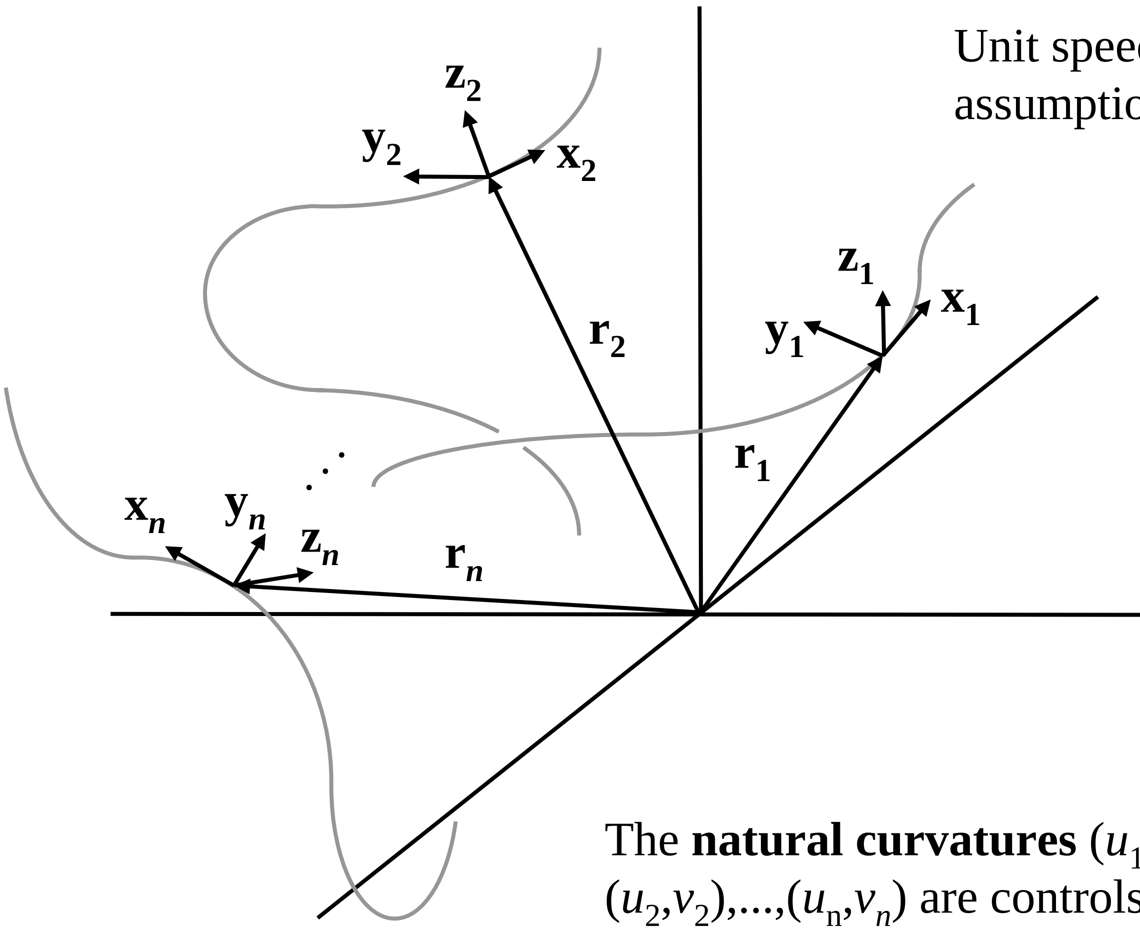
$$\vdots$$

$$\dot{\mathbf{r}}_n = \mathbf{x}_n$$
$$\dot{\mathbf{x}}_n = \mathbf{y}_n u_n$$
$$\dot{\mathbf{y}}_n = -\mathbf{x}_n u_n$$

$u_1, u_2, \ldots, u_n$ are curvature (i.e., steering) control inputs.

Specifying $u_1, u_2, \ldots, u_n$ as feedback functions of $(\mathbf{r}_1, \mathbf{x}_1, \mathbf{y}_1)$, $(\mathbf{r}_2, \mathbf{x}_2, \mathbf{y}_2), \ldots, (\mathbf{r}_n, \mathbf{x}_n, \mathbf{y}_n)$ defines a **control law**.

3-D Model (Natural Frenet Frames)



Unit speed assumption

$$
\begin{aligned}
\dot{\mathbf{r}}_1 &= \mathbf{x}_1 \\
\dot{\mathbf{x}}_1 &= \mathbf{y}_1 u_1 + \mathbf{z}_1 v_1 \\
\dot{\mathbf{y}}_1 &= -\mathbf{x}_1 u_1 \\
\dot{\mathbf{z}}_1 &= -\mathbf{x}_1 v_1 \\
\dot{\mathbf{r}}_2 &= \mathbf{x}_2 \\
\dot{\mathbf{x}}_2 &= \mathbf{y}_2 u_2 + \mathbf{z}_2 v_2 \\
\dot{\mathbf{y}}_2 &= -\mathbf{x}_2 u_2 \\
\dot{\mathbf{z}}_2 &= -\mathbf{x}_2 v_2 \\
&\vdots \\
\dot{\mathbf{r}}_n &= \mathbf{x}_n \\
\dot{\mathbf{x}}_n &= \mathbf{y}_n u_n + \mathbf{z}_n v_n \\
\dot{\mathbf{y}}_n &= -\mathbf{x}_n u_n \\
\dot{\mathbf{z}}_n &= -\mathbf{x}_n v_n
\end{aligned}
$$

The **natural curvatures** (u_1,v_1), (u_2,v_2),…,(u_n,v_n) are controls.

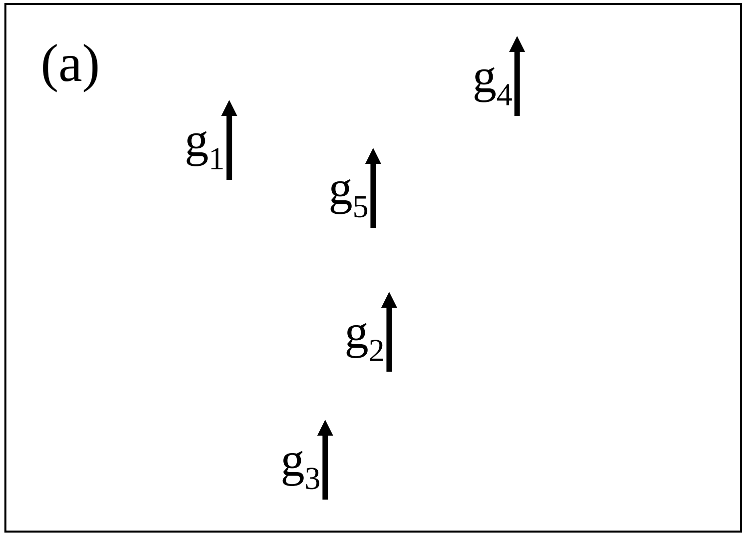
Relatively parallel adapted frames in the sense of R. L. Bishop, *Amer. Math. Month.* (1975), **82**(3):246-251

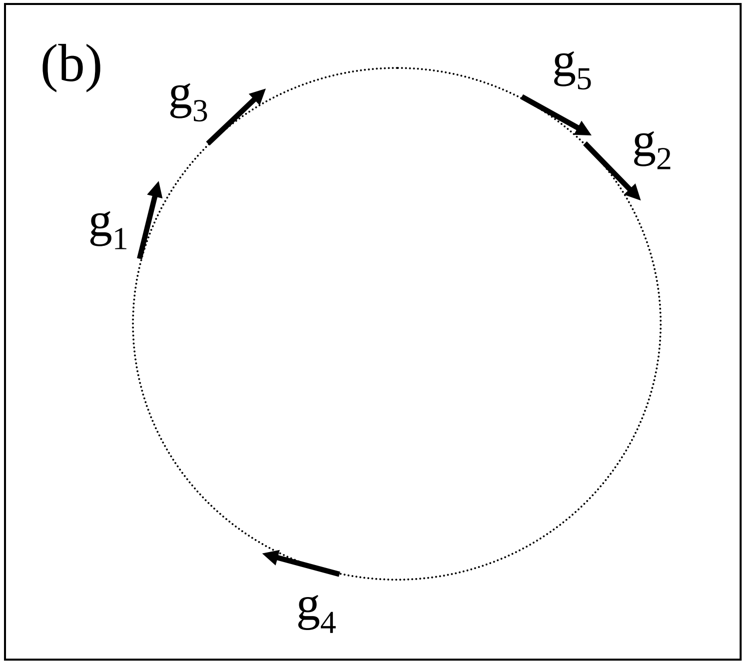
Characterization of Equilibrium Shapes

Proposition (Justh, Krishnaprasad): Suppose the controls $u_1, u_2, \ldots, u_n$ are invariant under rigid motions in **the plane**. For equilibrium shapes (i.e. relative equilibria of the unreduced dynamics), $u_1 = u_2 = \ldots = u_n$, and there are only two possibilities:

(a) $u_1 = u_2 = \ldots = u_n = 0$: all vehicles head in the same direction (with arbitrary relative positions), or

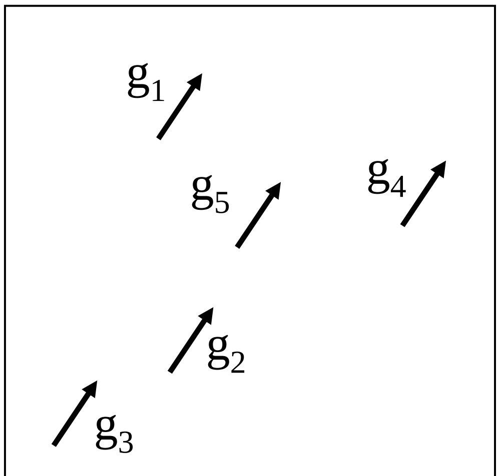
(b) $u_1 = u_2 = \ldots = u_n \neq 0$: all vehicles move on the same circular orbit (with arbitrary chordal distances between them).



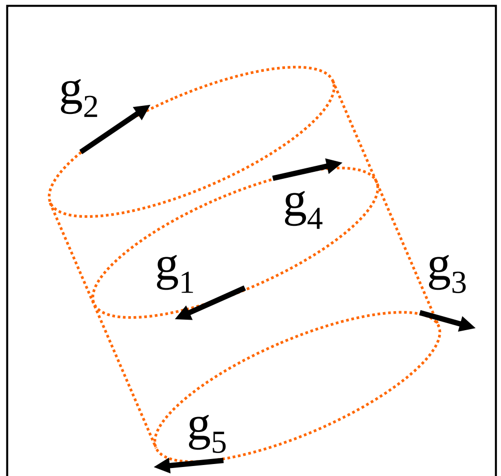


3-D Equilibrium Shapes

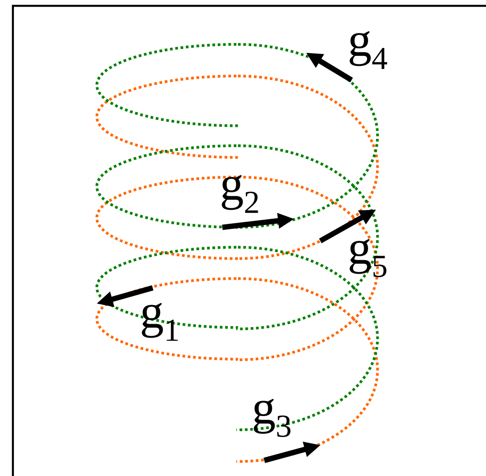
- The control laws are assumed to be invariant under rigid motions in three-dimensional space.
- Shape variables capture relative distances and angles between vehicles.
- Shape equilibria correspond to steady-state formations.



rectilinear formation

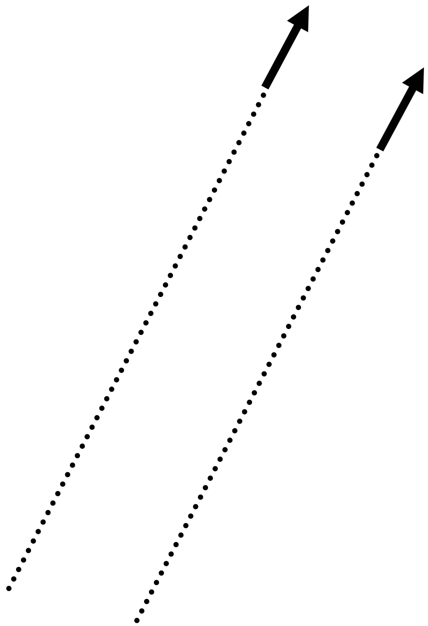


circling formation

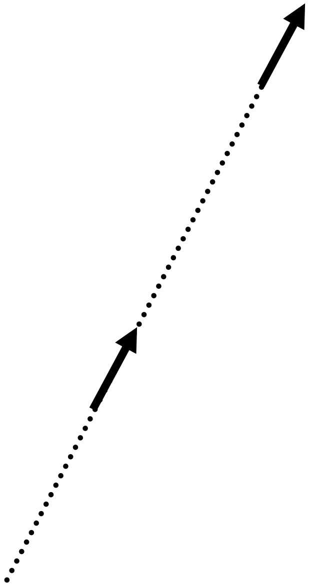


helical formation

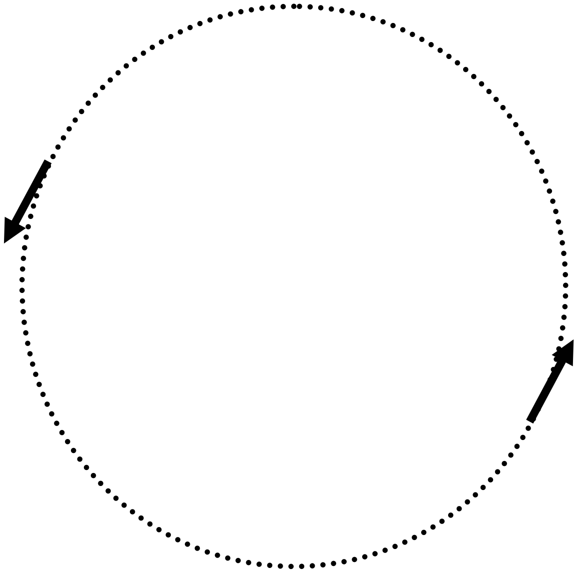
Pair Equilibria



Rectilinear formation
(motion perpendicular
to the baseline)



Collinear formation



Circling formation
(separation equals the
diameter of the orbit)

Planar Control Law for 2 particles

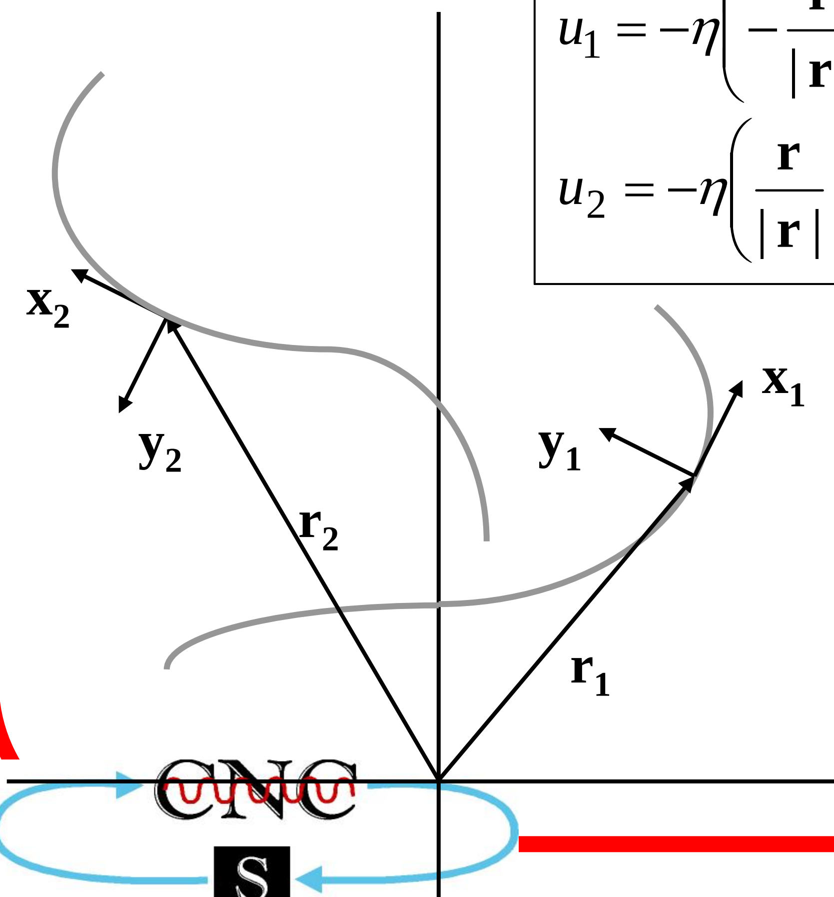
$$\dot{\mathbf{r}}_1 = \mathbf{x}_1 \qquad \dot{\mathbf{r}}_2 = \mathbf{x}_2 \qquad \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

$$\dot{\mathbf{x}}_1 = \mathbf{y}_1 u_1 \qquad \dot{\mathbf{x}}_2 = \mathbf{y}_2 u_2$$

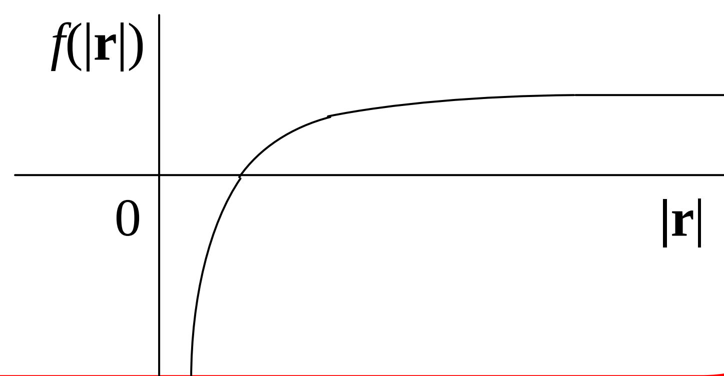
$$\dot{\mathbf{y}}_1 = -\mathbf{x}_1 u_1 \qquad \dot{\mathbf{y}}_2 = -\mathbf{x}_2 u_2$$

$$u_1 = -\eta\left(-\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_1\right)\left(-\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_1\right) - f(|\mathbf{r}|)\left(-\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_1\right) + \mu\mathbf{x}_2\cdot\mathbf{y}_1$$

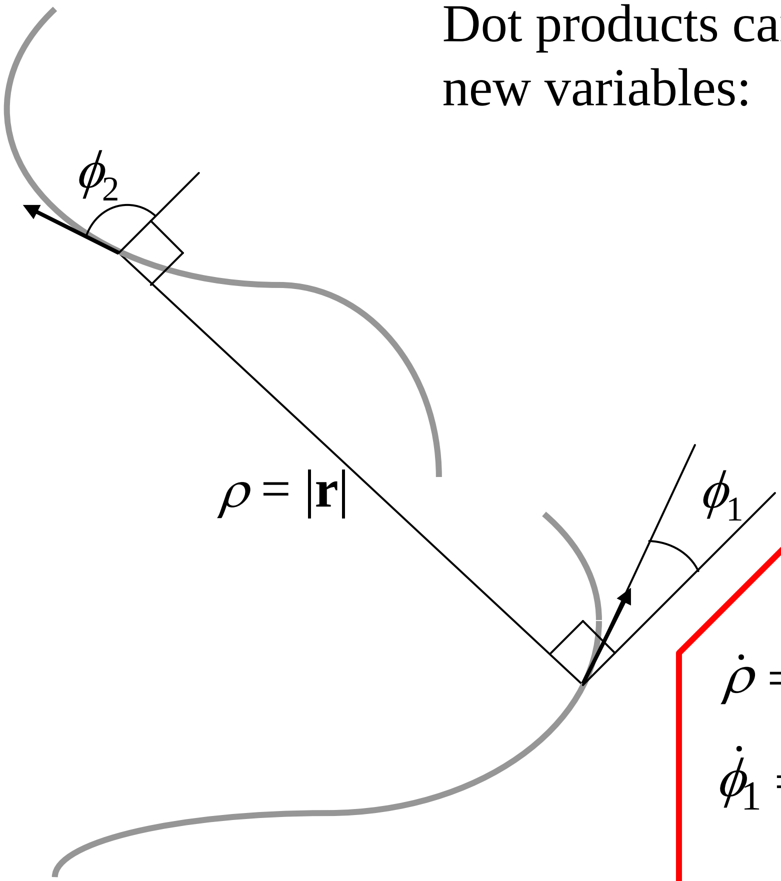
$$u_2 = -\eta\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_2\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right) - f(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right) + \mu\mathbf{x}_1\cdot\mathbf{y}_2$$



$$\mu > \frac{\eta}{2} > 0$$



Change of variables



Dot products can be expressed as sines and cosines in the new variables:

$$\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_1 = \sin\phi_1 \qquad \frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_1 = \cos\phi_1$$

$$\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_2 = \sin\phi_2 \qquad \frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2 = \cos\phi_2$$

$$\mathbf{x}_2\cdot\mathbf{y}_1 = \sin(\phi_2-\phi_1)$$

$$\mathbf{x}_1\cdot\mathbf{y}_2 = \sin(\phi_1-\phi_2)$$

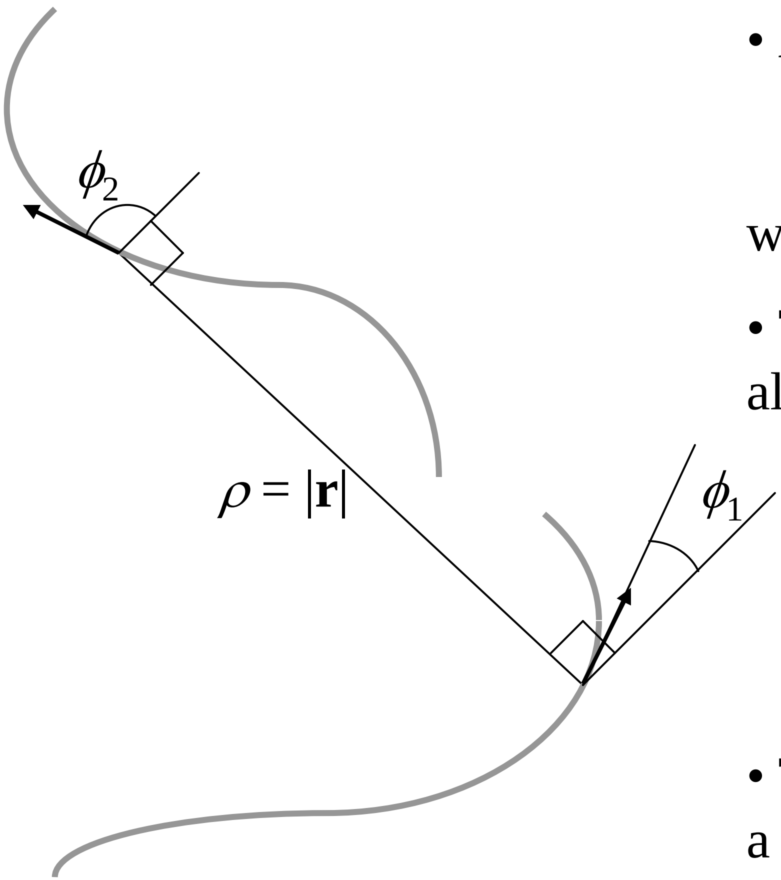
System after change of variables:

$$\dot{\rho} = \sin\phi_2 - \sin\phi_1$$

$$\dot{\phi}_1 = -\eta\sin\phi_1\cos\phi_1 + f(\rho)\cos\phi_1 + \mu\sin(\phi_2-\phi_1) + (1/\rho)(\cos\phi_2 - \cos\phi_1)$$

$$\dot{\phi}_2 = -\eta\sin\phi_2\cos\phi_2 - f(\rho)\cos\phi_2 + \mu\sin(\phi_1-\phi_2) + (1/\rho)(\cos\phi_2 - \cos\phi_1)$$

Lyapunov Function



- A Lyapunov function is

$$V_{pair} = -\ln(\cos(\phi_2 - \phi_1) + 1) + h(\rho)$$

where $f(\rho) = dh/d\rho$.

- The derivative of V_{pair} with respect to time along trajectories of the system is

$$\begin{aligned}\dot{V}_{pair} &= \frac{\partial V_{pair}}{\partial \phi_1}\dot{\phi}_1 + \frac{\partial V_{pair}}{\partial \varphi_2}\dot{\phi}_2 + \frac{\partial V_{pair}}{\partial \rho}\dot{\rho} \\ &\vdots \\ &\leq 0.\end{aligned}$$

- This Lyapunov function is the key to proving a convergence result for the two-particle system.

Convergence result

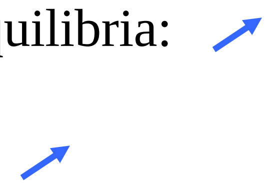
• Our Lyapunov function must be "radially unbounded," meaning $V_{pair} \to \infty$ as $\rho \to 0$ and as $\rho \to \infty$. (Some minor technical assumptions are also needed.)

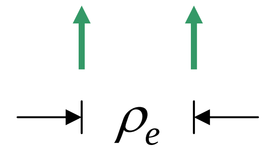
Proposition (Justh, Krishnaprasad): For any initial condition satisfying $|\phi_2 - \phi_1| \neq \pi$ and $\rho > 0$, the system converges to the set of equilibria, which has the form

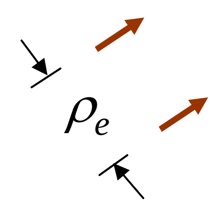
$$\left\{\left(\rho, \tfrac{\pi}{2}, \tfrac{\pi}{2}\right), \forall \rho > 0\right\} \cup \left\{\left(\rho, -\tfrac{\pi}{2}, -\tfrac{\pi}{2}\right), \forall \rho > 0\right\} \cup \left\{\left(\rho_e, 0, 0\right) \middle| f(\rho_e) = 0\right\}$$

Proof: Uses LaSalle's Invariance Principle.

• Examples of (relative) equilibria:







Intuition

Steering control equation for vehicle #2:

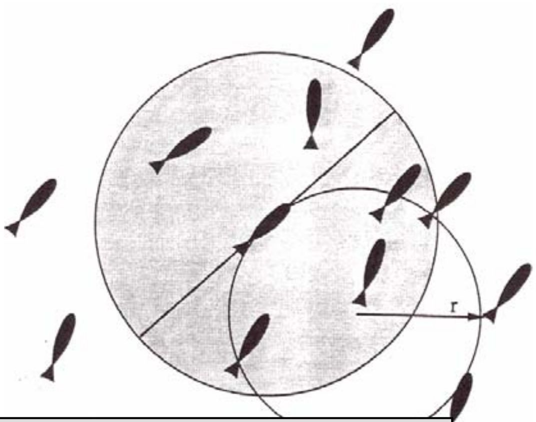
$$u_2 = -\eta\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_2\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right) - f(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right) + \mu\mathbf{x}_1\cdot\mathbf{y}_2$$

Align each vehicle perpendicular to the baseline between the vehicles.

Steer toward or away from the other vehicle to maintain appropriate separation.

Align with the other vehicle's heading.

- Biological analogy (swarming, schooling):
  - Decreasing responsiveness at large separation distances.
  - Switch from attraction to repulsion based on separation distance or density.
  - Mechanism for alignment of headings.



D. Grünbaum, "Schooling as a strategy for taxis in a noisy environment," in *Animal Groups in Three Dimensions*, J.K. Parrish and W.M. Hamner, eds., Cambridge University Press, 1997.

Key ideas for two-vehicle problem

- Unit-speed motion with steering control.

  - **Gyroscopic forces** preserve kinetic energy of each particle.

  - In mechanics, gyroscopic forces are associated with **vector potentials**.

- Shape variables: relative distances and angles.

- Lyapunov function $\Rightarrow$ convergence result for the shape dynamics.

- Equilibria of the shape dynamics = relative equilibria of the vehicle dynamics.

- Particle re-labeling symmetry.

- Lie group formulation:

  - The dynamics of each particle can be expressed as a left-invariant system evolving on $SE(2)$, the group of rigid motions in the plane.

  - $G=SE(2)$ is a symmetry group for the dynamics: the control law is invariant under rigid motions of the entire formation.

  - V_{pair} is also invariant under G. Therefore, we can consider the reduced system evolving on shape space = $(G \times G)/G = G$.

Gyroscopic Forces and Vector Potentials

Kinetic energy Scalar potential

Consider the Lagrangian:

$$L = \tfrac{1}{2}\dot{x}\cdot M\dot{x} + y(x)\cdot\dot{x} - x\cdot Kx,$$

$$x\in\mathbb{R}^n,\;\; M = M^T > 0,\;\; K = K^T.$$

Vector potential (linear-in-velocity term)

Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt}\big(M\dot{x} + y(x)\big) - \left(\frac{\partial y}{\partial x}\right)^T \dot{x} + Kx$$

$$= M\ddot{x} + \left(\frac{\partial y}{\partial x}\right)\dot{x} - \left(\frac{\partial y}{\partial x}\right)^T \dot{x} + Kx$$

$$= M\ddot{x} + Q(x)\dot{x} + Kx$$

$$= 0,$$

$$(*)\quad Q(x) = \left(\frac{\partial y}{\partial x}\right) - \left(\frac{\partial y}{\partial x}\right)^T \Rightarrow Q(x) = -Q^T(x).$$

Note: Lagrangian with linear-in-velocity term $\Rightarrow$ skew term in the dynamics, but the converse only holds if $Q(x)$ can be expressed as in $(*)$ for some $y(x)$.

Gyroscopically interacting particles

For a single particle:

$$\mathbf{r} = \text{position}, \quad \mathbf{v} = \dot{\mathbf{r}}, \quad \mathbf{a} = \ddot{\mathbf{r}},$$
$$m = \text{mass} = 1,$$
$$H = \text{kinetic energy} = \tfrac{1}{2}\|\mathbf{v}\|^2,$$
$$m\mathbf{a} = \mathbf{F} \Leftrightarrow \ddot{\mathbf{r}} = \begin{bmatrix} 0 & -u \\ u & 0 \end{bmatrix} \dot{\mathbf{r}}.$$

Note: **F** is a **gyroscopic force** (Recall Lorentz force law for a charged particle in a magnetic field)

$$\Longrightarrow$$

$$\dot{H} = 0,$$
$$\dot{\mathbf{r}} = \begin{pmatrix} \sqrt{2H}\cos\theta \\ \sqrt{2H}\sin\theta \end{pmatrix},$$
$$\dot{\theta} = u.$$

$$\Downarrow$$ Restrict to the level-set of H given by H=1/2.

Note that u may be a complicated function of time, and may involve feedback.

$$\dot{\mathbf{r}} = \mathbf{x}$$
$$\dot{\mathbf{x}} = \mathbf{y}u$$
$$\dot{\mathbf{y}} = -\mathbf{x}u$$

Frenet-Serret equations

For multiple particles, the kinetic energy of **each particle** is conserved, and the particles interact via **gyroscopic forces**.

Shape space for n particles

Frenet-Serret Equations

$$\dot{\mathbf{r}}_j = \mathbf{x}_j$$
$$\dot{\mathbf{x}}_j = \mathbf{y}_j u_j$$
$$\dot{\mathbf{y}}_j = -\mathbf{x}_j u_j$$
$$j = 1, 2, \ldots, n.$$

Group variables

$$g_j = \left[\begin{array}{cc:c} \mathbf{x}_j & \mathbf{y}_j & \mathbf{r}_j \\ & & \\ \hdashline 0 & 0 & 1 \end{array}\right]$$

$j = 1, 2, \ldots, n.$

$g_1, g_2, \ldots, g_n \in G = SE(2)$, the group of rigid motions in the plane.

Dynamics

$$\dot{g}_j = g_j \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} u_j \right)$$

$$= g_j \xi_j, \quad \xi_j \in se(2), \quad j = 1, 2, \ldots, n.$$

Configuration space

$$S = \overbrace{G \times G \times \cdots \times G}^{n \text{ copies}}$$

Shape variables capture **relative** positions and orientations.

Assume the controls $u_1, u_2, \ldots, u_n$ are functions of shape variables only.

Shape variables

$$\tilde{g}_j = g_1^{-1} g_j, \quad j = 2, \ldots, n.$$

Shape space

$$R = \overbrace{G \times G \times \cdots \times G}^{n-1 \text{ copies}}$$

Two-particle law: Lie group setting

- Dynamics on configuration space $S=G\times G$, where $G=SE(2)$:

$$g_1 = \left[\begin{array}{cc|c} \mathbf{x}_1 & \mathbf{y}_1 & \mathbf{r}_1 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} \cos\theta_1 & -\sin\theta_1 & \mathbf{r}_1 \\ \sin\theta_1 & \cos\theta_1 & \\ \hline 0 & 0 & 1 \end{array}\right], \quad \dot{g}_1 = g_1\xi_1 = g_1(A_0 + A_1 u_1).$$

$$\xi_1, \xi_2 \in se(2)$$

$$g_2 = \left[\begin{array}{cc|c} \mathbf{x}_2 & \mathbf{y}_2 & \mathbf{r}_2 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} \cos\theta_2 & -\sin\theta_2 & \mathbf{r}_2 \\ \sin\theta_2 & \cos\theta_2 & \\ \hline 0 & 0 & 1 \end{array}\right], \quad \dot{g}_2 = g_2\xi_2 = g_2(A_0 + A_1 u_2).$$

$$A_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Shape variable: $g = g_1^{-1} g_2$
- Dynamics on shape space $R=G$: $\dot{g} = g\xi$,

$$\xi = \xi_2 - g^{-1}\xi_1 g = \xi_2 - Ad_{g^{-1}}\xi_1 \in se(2).$$

- Controls as functions of the shape variable g:

$$u_1(g) = -\eta\left(\frac{g_{13}g_{23}}{r^2}\right) + f(r)\left(\frac{g_{23}}{r}\right) + \mu g_{21}, \quad g = [g_{ij}], \quad r = \sqrt{g_{13}^2 + g_{23}^2},$$

$$u_2(g) = -\eta\left(\frac{g^{13}g^{23}}{r^2}\right) + f(r)\left(\frac{g^{23}}{r}\right) + \mu g^{21}, \quad g^{-1} = [g^{ij}].$$

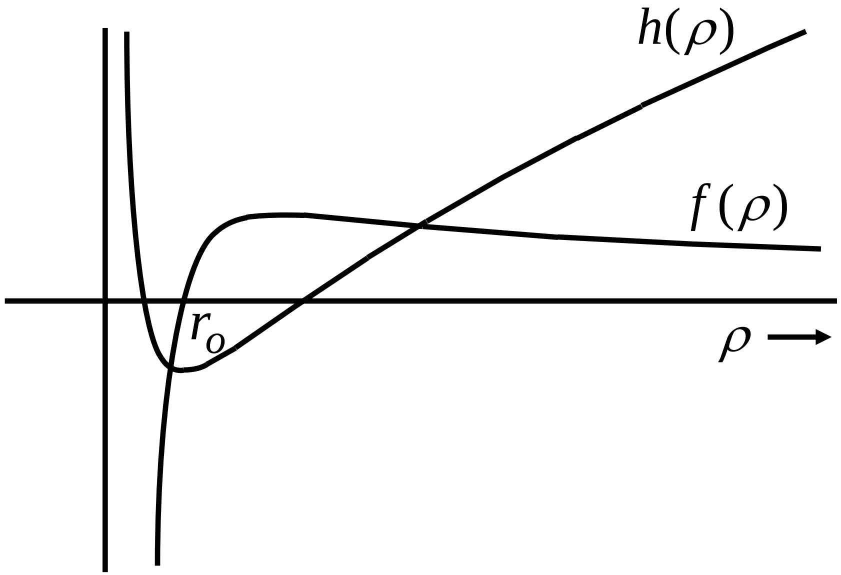
$$V_{pair} = V_{pair}(g).$$

Lyapunov Function

$$V = -\ln\left(1 + \mathbf{x}_2 \cdot \mathbf{x}_1\right) + h(|\,\mathbf{r}_2 - \mathbf{r}_1\,|)$$

Penalize heading-direction misalignment

Penalize inter-vehicle distances which are too large or small



- V depends only on shape variables.
- Idea: show that for suitable choice of control law, $dV/dt \leqslant 0$.
- For two vehicles, global convergence results are obtained (Justh and Krishnaprasad, TR 2002, CDC 2003, SCL 2004, CDC 2004)

Generalization to 3-D Law

Natural curvatures for vehicle #1:

$$u_1 = -\eta\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_1\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_1\right) + f(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_1\right) + \mu\mathbf{x}_2\cdot\mathbf{y}_1$$

$$v_1 = -\eta\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_1\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{z}_1\right) + f(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{z}_1\right) + \mu\mathbf{x}_2\cdot\mathbf{z}_1$$

Natural curvatures for vehicle #2:

$$u_2 = -\eta\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_2\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right) - f(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{y}_2\right) + \mu\mathbf{x}_1\cdot\mathbf{y}_2$$

$$v_2 = \underbrace{-\eta\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{x}_2\right)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{z}_2\right)}_{\text{Baseline alignment}} \underbrace{- f(|\mathbf{r}|)\left(\frac{\mathbf{r}}{|\mathbf{r}|}\cdot\mathbf{z}_2\right)}_{\text{Collision avoidance}} \underbrace{+ \mu\mathbf{x}_1\cdot\mathbf{z}_2}_{\text{Heading alignment}}$$

Lie Group Setting

Frenet-Serret Equations

$$\dot{\mathbf{r}}_j = \mathbf{x}_j$$
$$\dot{\mathbf{x}}_j = \mathbf{y}_j u_j$$
$$\dot{\mathbf{y}}_j = -\mathbf{x}_j u_j$$
$$j = 1, 2, \ldots, n.$$

Group variables

$$g_j = \left[\begin{array}{cc|c} \mathbf{x}_j & \mathbf{y}_j & \mathbf{r}_j \\ \hline 0 & 0 & 1 \end{array}\right]$$

$j = 1, 2, \ldots, n.$

$g_1, g_2, \ldots, g_n \in G = SE(2)$, the group of rigid motions in the plane.

Dynamics

$$\dot{g}_j = g_j \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} u_j \right)$$
$$= g_j \xi_j, \quad \xi_j \in se(2), \quad j = 1, 2, \ldots, n.$$

Configuration space

$$S = \overbrace{G \times G \times \cdots \times G}^{n \text{ copies}}$$

Assume the controls $u_1, u_2, \ldots, u_n$ are functions of shape variables only.

Shape variables capture **relative** vehicle positions and orientations.

Shape variables

$$\tilde{g}_j = g_1^{-1} g_j, \quad j = 2, \ldots, n.$$

Shape space

$$R = \overbrace{G \times G \times \cdots \times G}^{n-1 \text{ copies}}$$

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Lie Group Setting (3-D Trajectories)

- Represent each vehicle trajectory as a function on the Lie group SE(3) of rigid motions:

$$g_1 = \begin{bmatrix} \mathbf{x}_1 & \mathbf{y}_1 & \mathbf{z}_1 & \mathbf{r}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} \mathbf{x}_2 & \mathbf{y}_2 & \mathbf{z}_2 & \mathbf{r}_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3)$$

- Define the shape variable:

$$g = g_1^{-1} g_2 = \begin{bmatrix} \mathbf{x}_1 \cdot \mathbf{x}_2 & \mathbf{x}_1 \cdot \mathbf{y}_2 & \mathbf{x}_1 \cdot \mathbf{z}_2 & (\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{x}_1 \\ \mathbf{y}_1 \cdot \mathbf{x}_2 & \mathbf{y}_1 \cdot \mathbf{y}_2 & \mathbf{y}_1 \cdot \mathbf{z}_2 & (\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{y}_1 \\ \mathbf{z}_1 \cdot \mathbf{x}_2 & \mathbf{z}_1 \cdot \mathbf{y}_2 & \mathbf{z}_1 \cdot \mathbf{z}_2 & (\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{z}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Left-invariant systems on SE(3):

$$\dot{g}_1 = g_1 \xi_1 = g_1 (A_0 + A_1 u_1 + A_2 v_1), \quad \xi_1 \in se(3)$$

$$\dot{g}_2 = g_2 \xi_2 = g_2 (A_0 + A_1 u_2 + A_2 v_2), \quad \xi_2 \in se(3)$$

$$\dot{g} = g \xi = g(\xi_2 - \mathrm{Ad}_{g^{-1}} \xi_1) = g(\xi_2 - g^{-1} \xi_1 g), \quad \xi \in se(3)$$

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Formation Control for *n* vehicles

Generalization of the two-vehicle formation control law to *n* vehicles:

$$\dot{\mathbf{r}}_j = \mathbf{x}_j$$

$$\dot{\mathbf{x}}_j = \mathbf{y}_j u_j$$

$$\dot{\mathbf{y}}_j = -\mathbf{x}_j u_j$$

$$u_j = \frac{1}{n}\sum_{k \neq j}\left[F(\mathbf{r}_j - \mathbf{r}_k, \mathbf{x}_j, \mathbf{y}_j, \mathbf{x}_k, \mathbf{y}_k) - f(|\mathbf{r}_j - \mathbf{r}_k|)\left(\frac{\mathbf{r}_j - \mathbf{r}_k}{|\mathbf{r}_j - \mathbf{r}_k|} \cdot \mathbf{y}_j\right)\right]$$

$$j = 1, 2, \ldots, n$$

At present, it is **conjectured** (based on simulation results) that such control laws stabilize certain formations. However, analytical work is ongoing.

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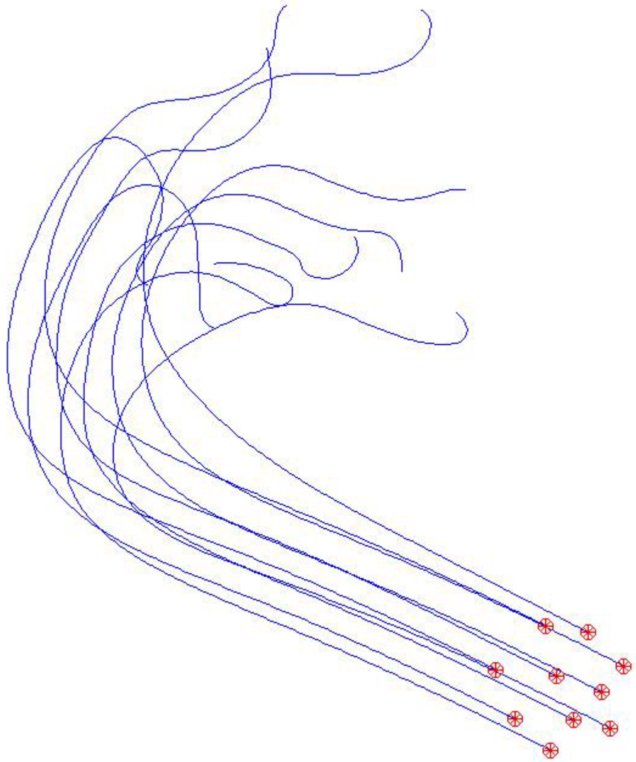
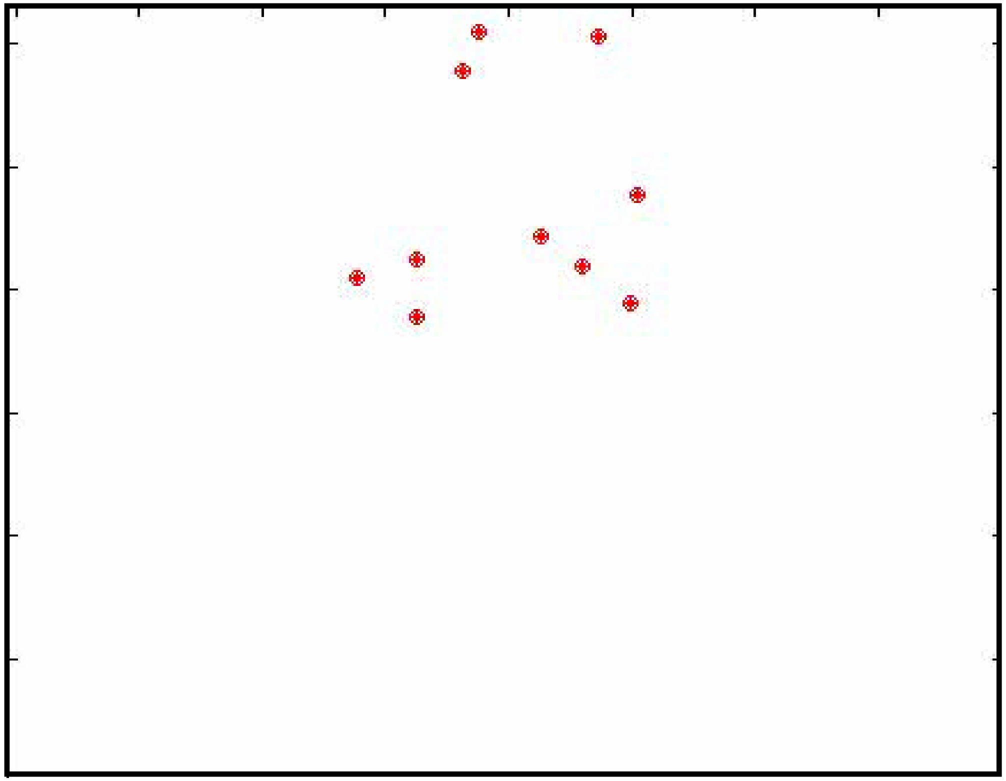
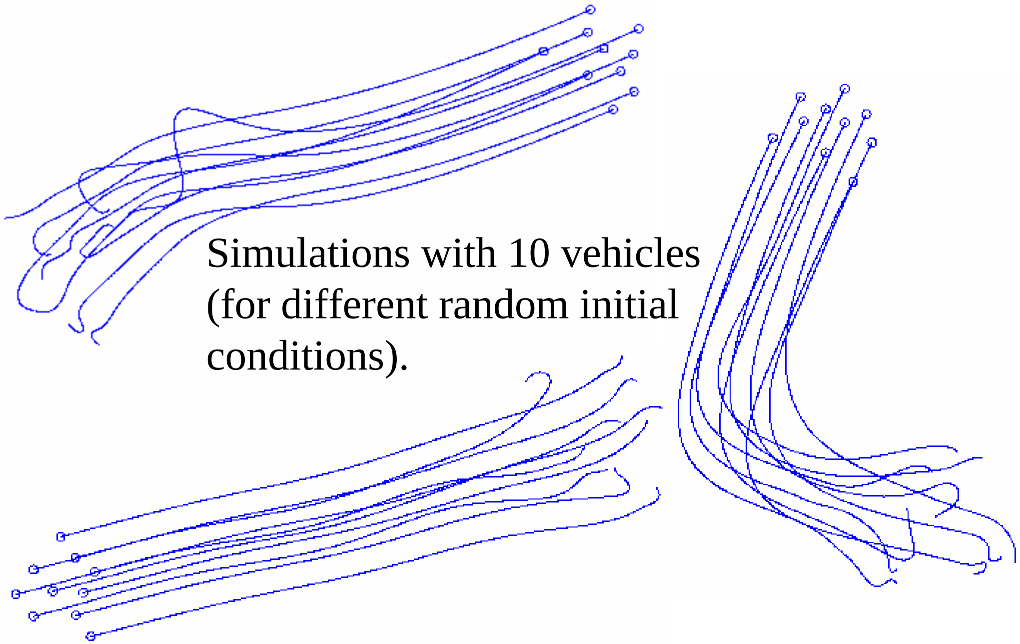
Rectilinear Control Law Simulation

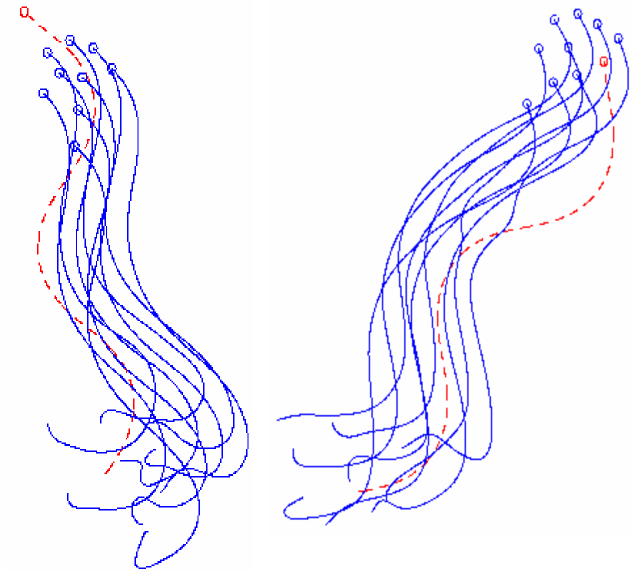




Rectilinear Control Law Simulations

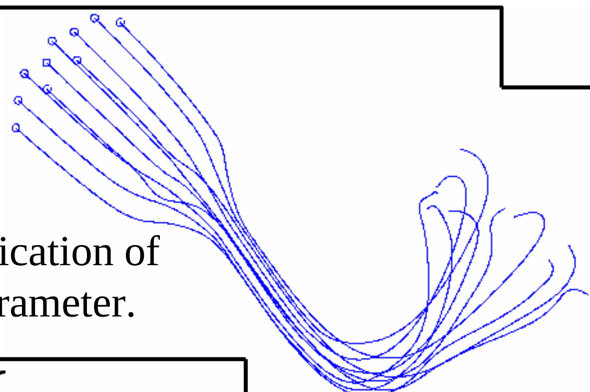


Simulations with 10 vehicles (for different random initial conditions).

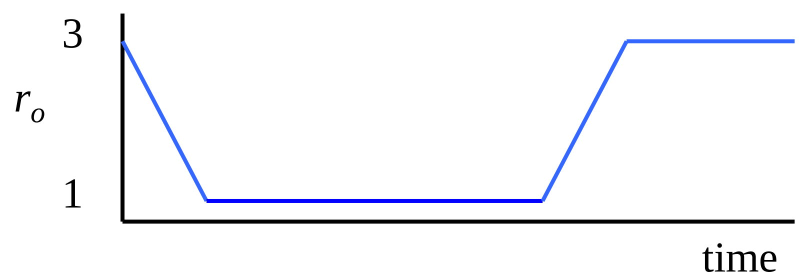


Leader-following behavior: the red vehicle follows a prescribed path (dashed line).

On-the-fly modification of the separation parameter.



Normalized Separation Parameter vs. Time



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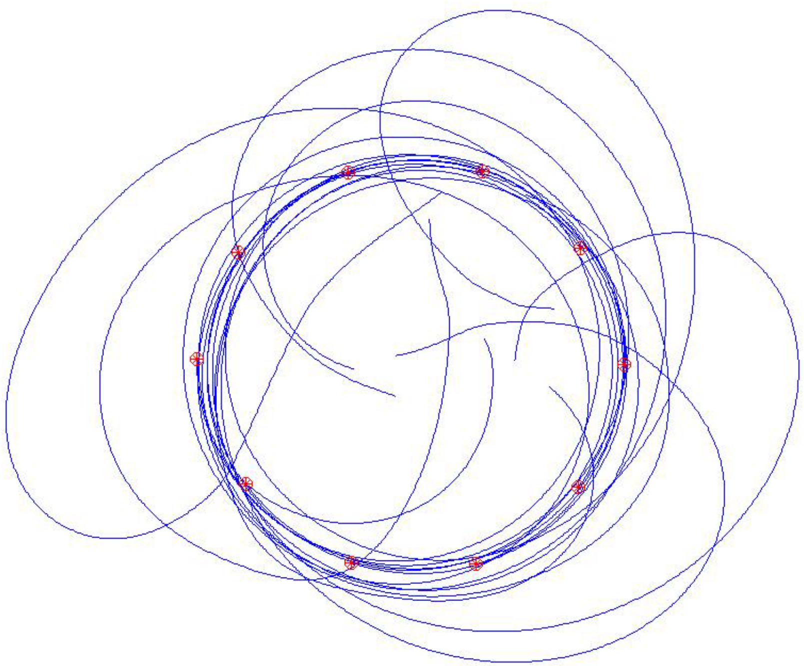
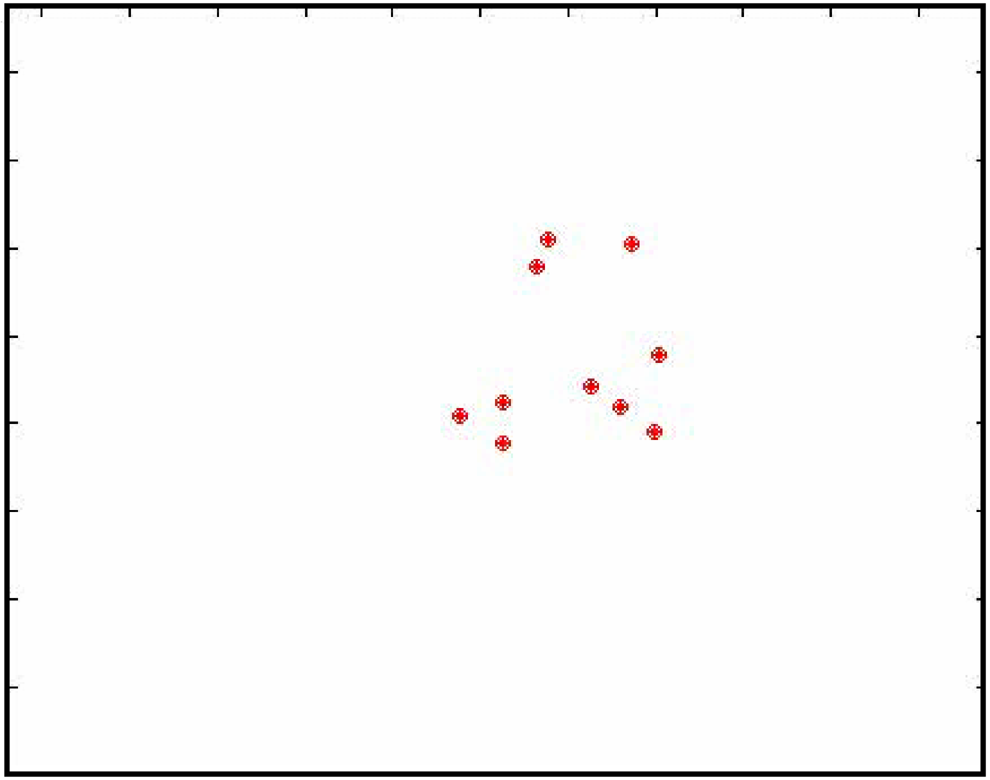
Circling Control Law Simulation





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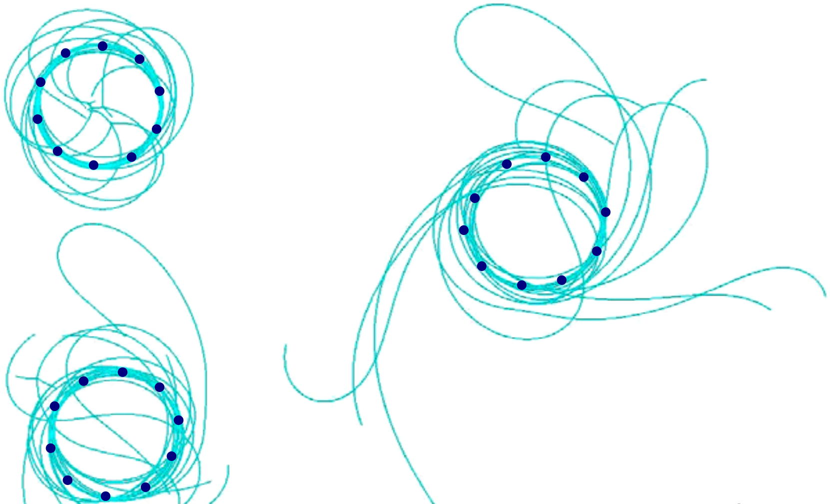
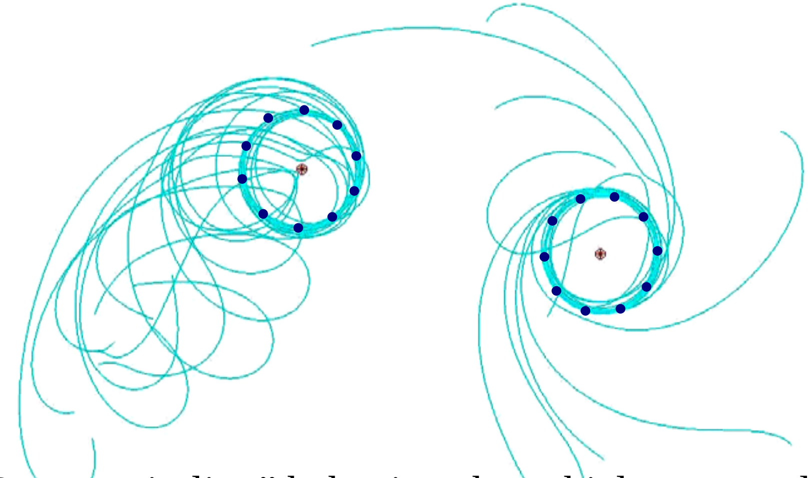
Circling Control Law

$$u_j = \frac{1}{n}\sum_{k \neq j}\left[\pm\eta\left(\frac{\mathbf{r}_j - \mathbf{r}_k}{|\mathbf{r}_j - \mathbf{r}_k|}\cdot \mathbf{x}_j\right) - f(|\mathbf{r}_j - \mathbf{r}_k|)\left(\frac{\mathbf{r}_j - \mathbf{r}_k}{|\mathbf{r}_j - \mathbf{r}_k|}\cdot \mathbf{y}_j\right)\right]$$



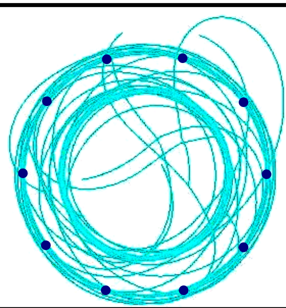
Simulations with 10 vehicles (for different random initial conditions).

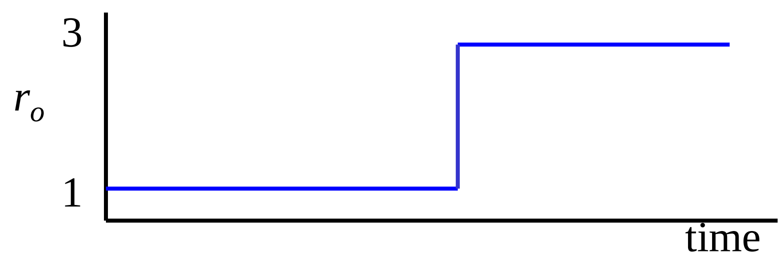


“Beacon-circling” behavior: the vehicles respond to a beacon, as well as to each other.

Normalized Separation Parameter vs. Time

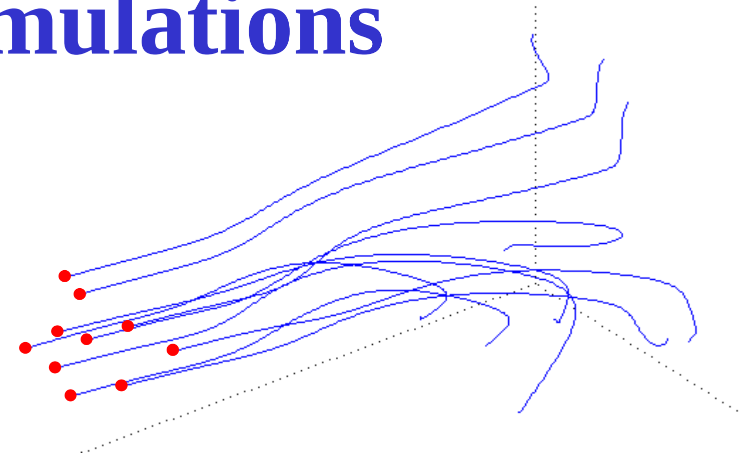
On-the-fly modification of the separation parameter.





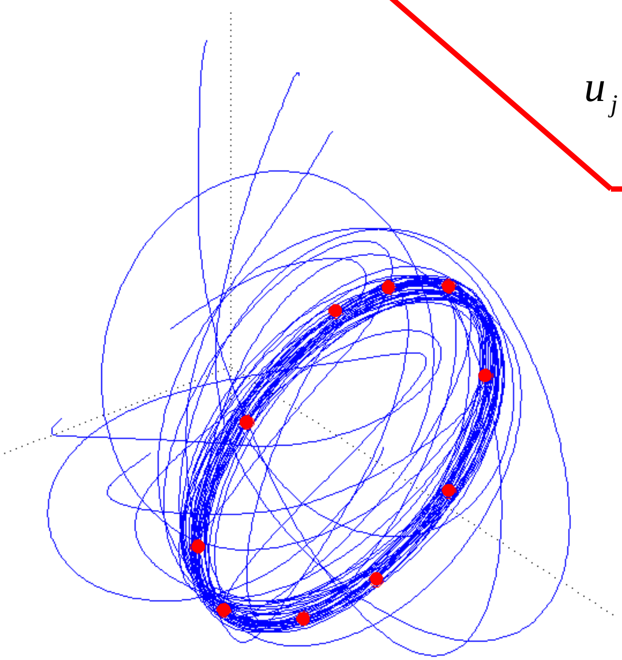
3-D Simulations

Rectilinear Law:



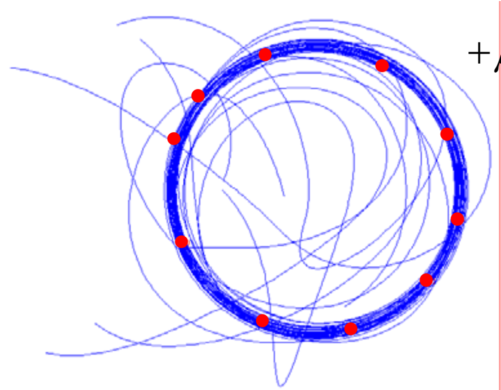
$$u_j = \frac{1}{n}\sum_{k\neq j}\left[-\eta\left(\frac{\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|}\cdot\mathbf{x}_j\right)\left(\frac{\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|}\cdot\mathbf{y}_j\right) - f(|\mathbf{r}_{jk}|)\left(\frac{\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|}\cdot\mathbf{y}_j\right) + \mu\mathbf{x}_k\cdot\mathbf{y}_j\right]$$

Circling Law:



$$u_j = \frac{1}{n}\sum_{k\neq j}\Bigg\{-\eta\left(\frac{\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|}\cdot\mathbf{x}_j\right)\left(\frac{\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|}\cdot\mathbf{y}_j\right) - f(|\mathbf{r}_{jk}|)\left(\frac{\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|}\cdot\mathbf{y}_j\right)$$

$$+\mu\Bigg[-\mathbf{x}_k\cdot\mathbf{y}_j + 2\left(\frac{\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|}\cdot\mathbf{x}_k\right)\left(\frac{\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|}\cdot\mathbf{y}_j\right)\Bigg]\Bigg]\Bigg\}$$

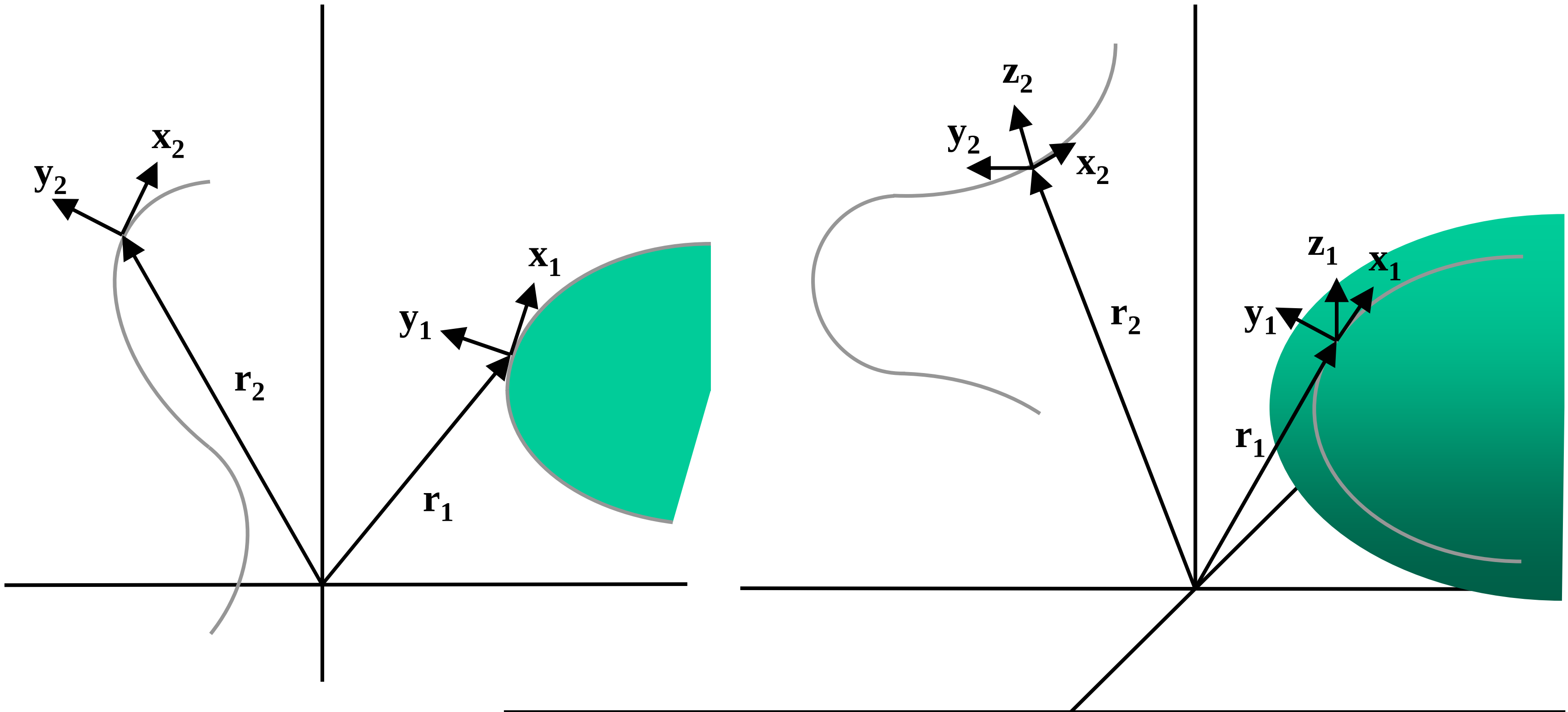


where $\mathbf{r}_{jk} = \mathbf{r}_j - \mathbf{r}_k$, $\mu > \frac{\eta}{2} > 0$,

$$f(|\mathbf{r}_{jk}|) = \alpha\left[1 - \left(\frac{r_o}{|\mathbf{r}_{jk}|}\right)^2\right], \quad \alpha > 0$$

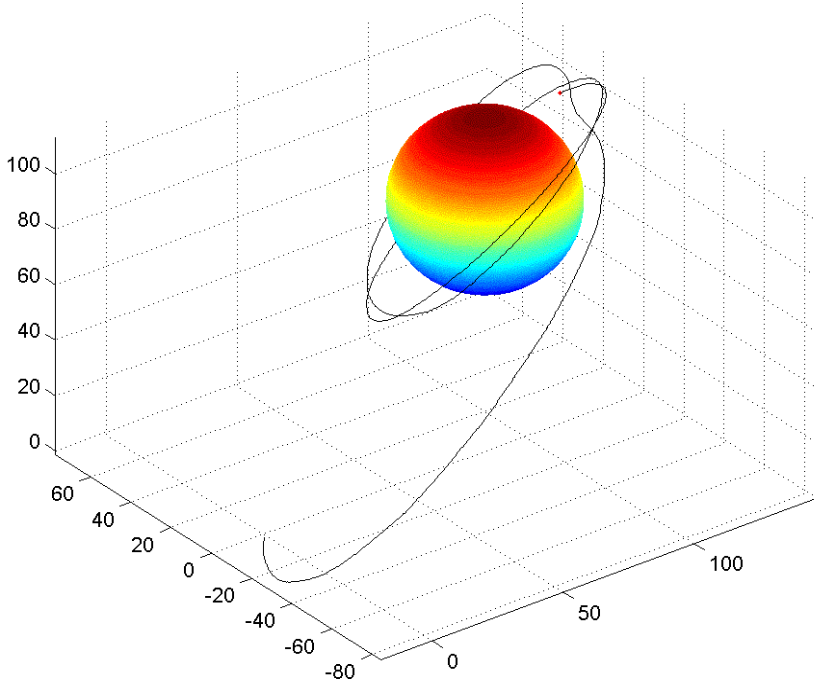
Obstacle Avoidance

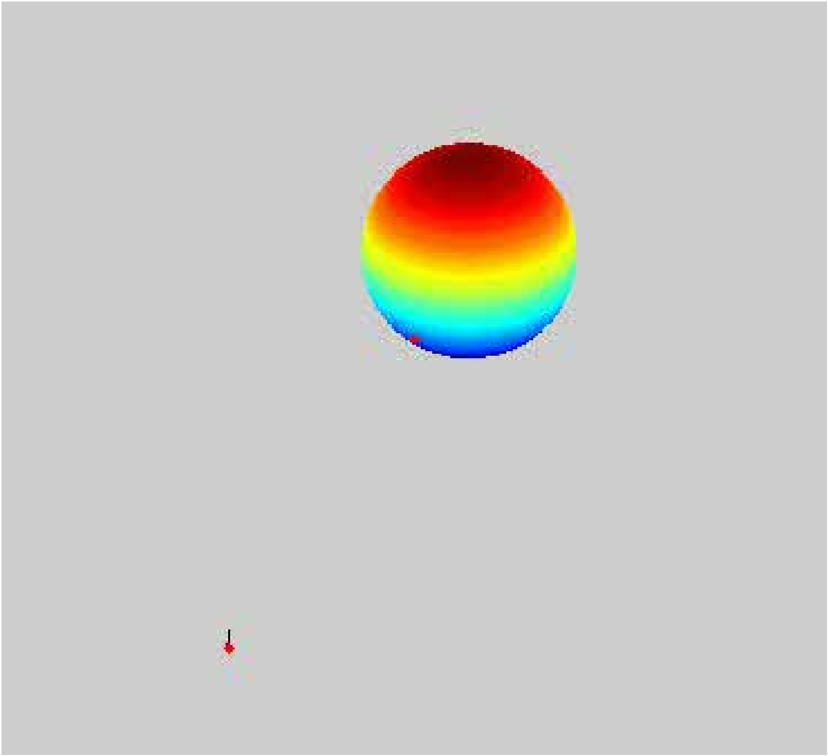
- Idea: control inputs for the moving vehicle are determined by the trajectory of the closest point on the obstacle surface.
- Goals: boundary following and non-collision.



F. Zhang, E.W. Justh, and P.S. Krishnaprasad, CDC 2004.

Boundary-Following Simulation





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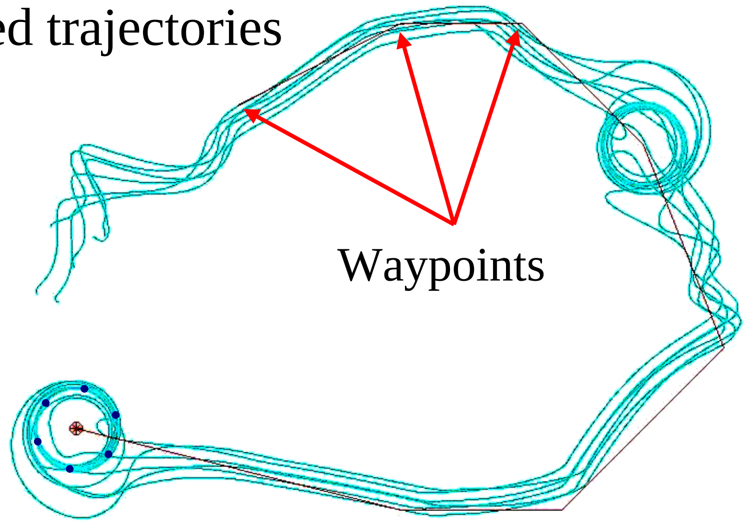
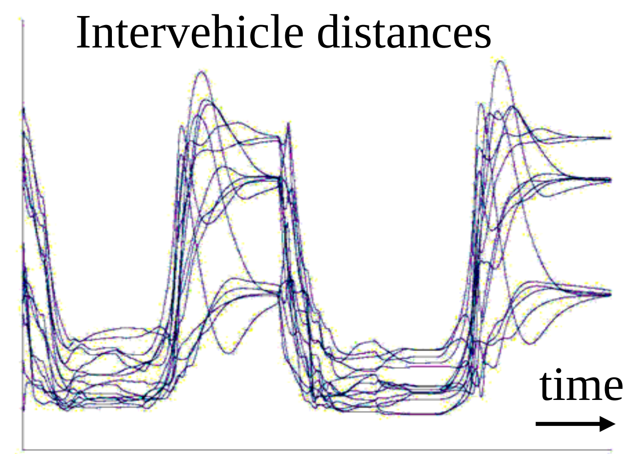
Performance Criteria

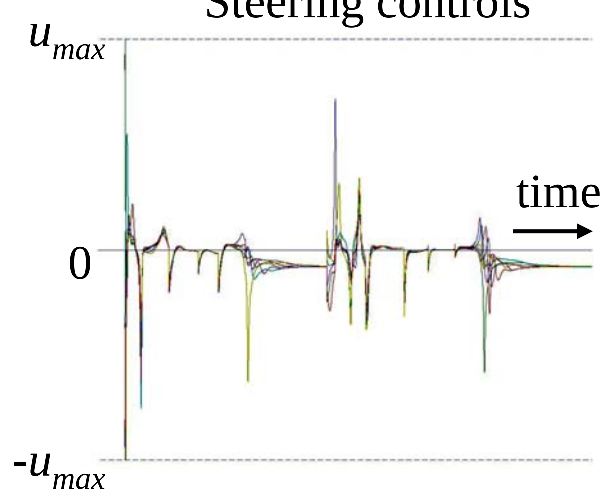
• Faithful following of waypoint-specified trajectories

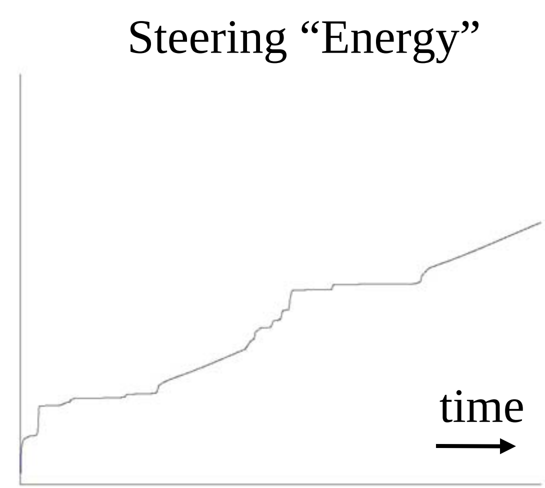


• Sufficient separation between vehicles (to avoid collisions)

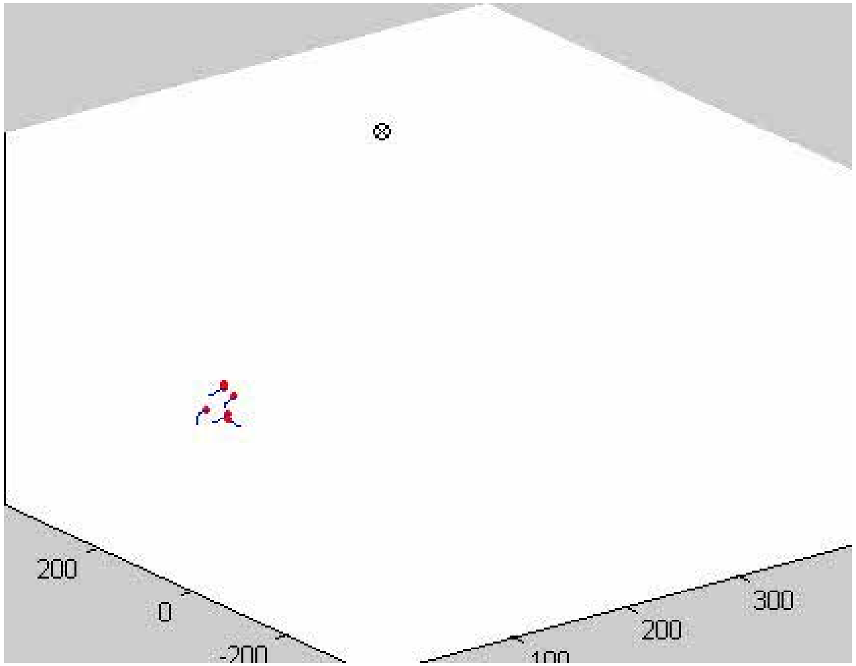


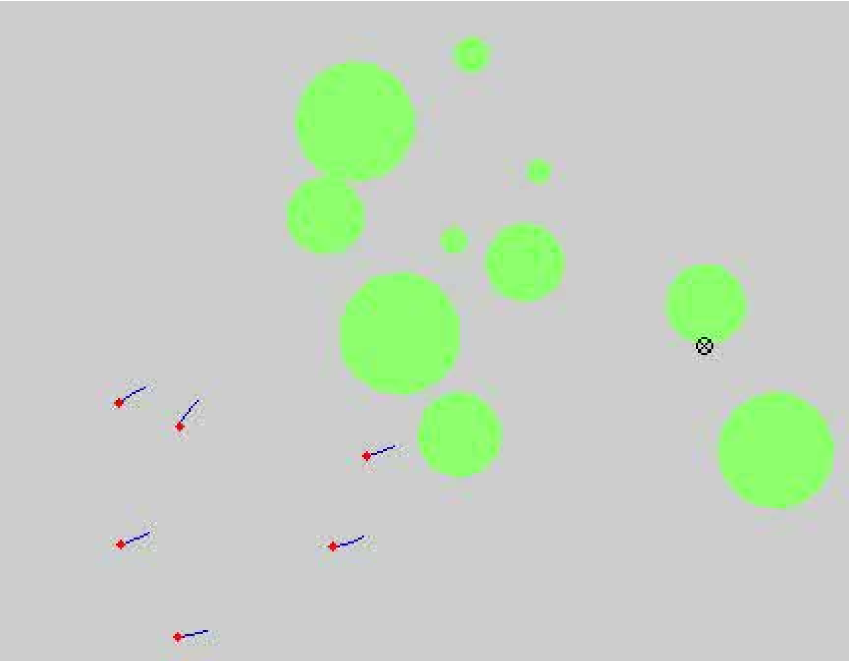
• Minimize steering: for UAVs, turning requires considerably more energy than straight, level flight. Maneuverability is also limited.





Toward Practical Missions





Convergence Result for $n > 2$

- We consider rectilinear relative equilibria, and the Lyapunov function

$$V = \sum_{j=1}^{n} \sum_{k<j} \left[-\ln\left(1 + \cos(\theta_j - \theta_k)\right) + h(|\mathbf{r}_j - \mathbf{r}_k|) \right]$$

- **Convergence Result** (Justh, Krishnaprasad): There exists a sublevel set Ω of V and a control law (depending only on shape variables) such that $\dot{V} \leq 0$ on Ω.

- With this Lyapunov function, we cannot prove global convergence for $n > 2$.

- Although we obtain an explicit formula for the controls u_j, $j=1,\ldots,n$, there is no guarantee that this particular choice of controls will result in convergence to a particular desired equilibrium shape in Ω.

Continuum Model

- Vector field (in polar coordinates):

$$\begin{pmatrix} d\mathbf{r}/dt \\ \\ d\theta/dt \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ u \end{pmatrix}.$$

- Continuity equation (Liouville equation):

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial (u\rho)}{\partial \theta} + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \cdot \nabla_{\mathbf{r}} \rho \right].$$

- Conservation of matter:

$$\int_G \rho(t,\mathbf{r},\theta)\, d\mathbf{r}\, d\theta = 1, \quad \forall t.$$

- Energy functional:

$$V_c(t) = \frac{1}{2}\int_G \int_G \left[-\ln\left(1+\cos(\theta-\tilde{\theta})\right) + h(|\mathbf{r}-\tilde{\mathbf{r}}|)\right] \rho(t,\mathbf{r},\theta)\rho(t,\tilde{\mathbf{r}},\tilde{\theta})\, d\mathbf{r}\, d\theta\, d\tilde{\mathbf{r}}\, d\tilde{\theta}.$$

- This continuum formulation only involves two scalar fields: the density $\rho(t,\mathbf{r},\theta)$ and the steering control $u(t,\mathbf{r},\theta)$.

- However, the underlying space is 3-dimensional (for planar formations).

- Incorporating time and/or spatial derivatives in the equation for u yields a coupled system of PDEs for ρ and u.

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1. E.W. Justh and P.S. Krishnaprasad, "A simple control law for UAV formation flying," Institute for Systems Research Technical Report TR 2002-38, 2002 (see http://www.isr.umd.edu).

2. E.W. Justh and P.S. Krishnaprasad, "Steering laws and continuum models for planar formations," *Proc. 42nd IEEE Conf. Decision and Control*, pp. 3609-3614, 2003.

3. E.W. Justh and P.S. Krishnaprasad, "Equilibria and steering laws for planar formations," *Systems and Control Letters*, Vol. 52, pp. 25-38, 2004.

4. E.W. Justh and P.S. Krishnaprasad, "Formation control in three dimensions," preprint, 2004.

5. F. Zhang, E.W. Justh, and P.S. Krishnaprasad, "Boundary following using gyroscopic control," Submitted to *IEEE Conf. Decision and Control*, 2004.

See also http://www.isr.umd.edu/~justh