Application of symmetry analysis to a PDE arising in the car windscreen design

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Outline

1. Introduction

1.1. Parameter identification problems modelled by PDEs

1.2. Symmetry reductions for a PDE

1.3. Symmetry reductions related to a parameter identification problem modelled by a PDE

2. A mathematical model arising in the car windscreen design

2.1. Introduction to the manufacture of car windscreens

2.2. Conditional symmetries associated with this model

2.2.1. Data and parameter invariant under the same group

2.2.2. Data and parameter invariant under different group

2.3. Equivalence transformations related to the studied model

2.4. Examples

1. Introduction

1.1. Parameter identification problems modelled by PDEs

Parameter identification problems

- arise in the inverse problems theory

- are concerned with the identification of physical parameters from observations of the evolution of a system

$$\mathcal{F}\left(x, w^{(m)}, E^{(n)}\right) = 0, \quad x \in \Omega \subset R^p \tag{1}$$

$E = E(x)$ is the *parameter* and $w = w(x)$ is the *data*

- assume that the parameter E and the data $w$are analytical functions

- the PDE (4) sometimes augmented with certain boundary conditions is called the inverse problem associated to a direct problem. The direct problem is the same equation but the unknown function is the data w, for which certain boundary conditions are required

• In general, these are *ill-posed problems*: they do not fulfill Hadamard's postulates for all admissible data: a solution exists, the solution is unique and the solution depends continuously on the given data.

• Arbitrary small changes in *data* may lead to arbitrary large changes in the solution

• The iterative approach of studying these problems is a functional-analytic setup with a special emphasis on *iterative regularization methods* [1]

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[1] H. W. Engl, M. Hanke and A. Neubauer, *Regularization of Inverse Problems*, Kluwer, Dordrecht, 1996

1.2 Symmetry reductions for a PDE

Consider a n-th order PDE

$$\mathcal{F}\left(x, u^{(n)}\right) = 0, \quad u = u(x), \quad x \in \Omega \subset \mathbf{R}^p \tag{2}$$

Symmetry reductions

$\longrightarrow$ analytical solutions of the PDE (2)

$\longrightarrow$ a reduced order model related to (2)

$$\tilde{\mathcal{F}}\left(z, v^{(n)}\right) = 0, \quad v = v(z), \quad z \in \tilde{\Omega} \subset \mathbf{R}^{p-1} \tag{3}$$

where $z = z(x)$ and $v = A(x, u(x))$.

1. *classical Lie symmetries* $\longrightarrow$ *symmetry group*

2. *nonclassical (conditional) symmetries*

3. If (2) depends on an arbitrary function:

$\longrightarrow$ *equivalence transformations*

1. *Classical Lie symmetries* - (local) Lie groups of transformations acting on the space of the independent variables x and the space of the dependent variable u with the property that they leave the PDE unchanged; the *classical Lie method* ([2] and [3])

2. *Nonclassical (conditional) symmetries* - (local) Lie groups of transformations acting on the space of the independent variables x and the space of the dependent variable u with the property that they might not leave the PDE unchanged, but using them we can find exact solutions of the studied model; e.g., the *nonclassical method* [4] and the *direct method* [5]

3. *Equivalence transformations* [6] - classical Lie symmetries acting on the space of the independent variables x, the space of the dependent variable u and the space of the arbitrary function; the *classical Lie method* applied to the studied PDE where the dependent variable u and the arbitrary function are both unknown functions [7]

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[2] S. Lie, *Gesammelte Abhandlungen*, 4, B. G. Teubner, Leipzig, 1929, pp. 320–384

[3] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts Math. 107, Springer Verlag, New York, 1986

[4] G. W. Bluman and J. D. Cole, *The general similarity solutions of the heat equation*, J. Math. Mech., 18 (1969), pp. 1025–1042

[5] P. A. Clarkson and M. Kruskal, *New similarity reductions of the Boussinesq equation*, J. Math. Phys., 30 (1989), pp. 2201–2213

[6] L. V. Ovsiannikov, *Group analysis of differential equations*, translated by W. F. Ames, Academic Press, New York, 1982

[7] N. H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations, Vol.1: Symmetries, Exact Solutions and Conservation Laws*, CRC Press, Boca Raton, 1994

1.3. Symmetry reductions related to a parameter identification problem modelled by a PDE

Consider

$$\mathcal{F}\left(x, w^{(m)}, E^{(n)}\right) = 0, \quad x \in \Omega \subset R^p \tag{4}$$

$E = E(x)$ is the *parameter* and $w = w(x)$ is the *data*

- how parameter identification problems can be analyzed with the tools of symmetry analysis theory (although the symmetry analysis theory is a common approach for studying PDEs)

1. E unknown, w given

   $\longrightarrow$ classical/nonclassical symmetries

2. E unknown, w arbitrary

   $\longrightarrow$ classical/nonclassical symmetries

3. E, w unknown

   $\longrightarrow$ *equivalence transformations*

- how to incorporate invariant data in non-invariant boundary conditions

! Ω bounded

! w satisfies certain boundary conditions

• $\partial\Omega = \{z = k\}$ where z is a similarity variable; w and Ω share the same invariance [8]

Prop. If G is a (local) Lie group of transformations on $\mathbf{R}^p$ and $H : M \to \mathbf{R}^l$ is a smooth function, then

H is G-invariant iff every level set $\{x \ : H(x) = k\}$, $k \in \mathbf{R}^l$ is a G-invariant subset of M

E.g. $w = g(z)$, $\partial\Omega = \{z = k\}$ is the k-level of z, i.e., $X(w) = 0$, X is the tangent vector to $\partial\Omega$

$\rightarrow$ rotational symmetries

• $\partial\Omega = \{z = k\}$ where z is not a similarity variable; w and Ω do not share the same invariance

E.g. $w(x, y) = z(x, y)^m - k^m$ and

$\partial\Omega = \{(x, y) : z(x, y) = k\}$, $m \geq 1$ natural, k fixed

- elliptical domains

- rounded square domains (Lamé ovals):

$x^{2n} + y^{2n} \leq 1$, $n \geq 2$ natural

$\rightarrow$ scaling symmetries

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[8] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*,

Appl. Math. Sci. Vol. 81, Springer, new York, 1989

2. A mathematical model arising in the car windscreen design
2.1. Introduction to the manufacture of car windscreens

The main processes used in the manufacture of car windscreens are the gravity sag bending process and the pressing process

a) The gravity sag bending process

- a piece of glass is placed over a rigid frame with the desired edge curvature and heated from below

- the glass becomes viscous due to the temperature rise and sags under its own weight

- the final shape depends on the viscosity distribution of the glass obtained from varying the temperature

b) The pressing process

- a piece of glass is heated and forced between two molds or dies of the desired shape

- the glass is allowed to cool and set into the final shape.

The two main considerations during these processes are the cost and the optical quality of the product.

The optical quality is affected by

- the quality of the glass: the initial glass used by manufacturers are produced to a high optical quality

- the required windscreen shape: the windscreen shapes are designed in order to minimize any adverse optical effects: for example, manufacturers try to avoid shapes with areas where Gaussian curvature becomes negative, which can give regions of poor visibility

- defects introduced during production: it is important that the windscreen does not contain distortions or imperfections, so that, it is essential that the production process to not affect the optical quality of the windscreen

When a windscreen in produced using both of the two processes, unavoidable defects due to the production process are introduced into the glass sheet

This is mainly due to stresses caused by the glass itself

In order to minimize these stresses, there is a final process after the initial production, the so-called annealing process, when the windscreen is gently heated for obtaining the desired shape

Sag bending process is found to introduce fewer defects into the glass than pressing process

Windscreens produced by sag bending require less annealing and so are cheaper and quicker to produce

! It is not obvious which shapes can be made using the gravity sag bending process

It has been shown that the sag bending process can be controlled (in a first approximation) in the terms of Young's modulus E, a spatially varying glass material parameter, and the displacement of the glass w can be described by the thin linear elastic plate theory ([9], [10] and [11])

The model is based on the *linear plate equation*

$$\begin{aligned}(E\,(w_{xx}+\nu w_{yy}))_{xx}+2(1-\nu)\,(Ew_{xy})_{xy}\\ +\,(E\,(w_{yy}+\nu w_{xx}))_{yy}=\frac{12(1-\nu^2)f}{h^3}\quad \text{on}\quad \Omega,\end{aligned}\tag{5}$$

where

$w = w(x,y)$ is the displacement of the glass sheet (the target shape) occupying a domain $\Omega \subset R^2$

$E = E(x,y)$ is Young's modulus: a positive function that can be influenced by adjusting the temperature in the process of heating the glass

f is the gravitational force

$\nu \in \left(0, \frac{1}{2}\right]$ is the glass Poisson ratio

h is thickness of the plate

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[9] D. Krause and H. Loch, *Mathematical Simulation in Glass Technology*, Springer Verlag Berlin Heidelberg, new York, 2002

[10] P. S. Manservisi, *Control and Optimization of the sag bending process in glass Windscreen design*, in L. Arkeryd, J. Bergh, P. Brenner, R. Petterson, Progress in Industrial Mathematics at ECMI98, Teubner, Stuttgart, 1999, pp. 97–105

[11] O. S. Narayanaswamy, *Stress and structural relaxation in tempering glass*, J. Am. Ceramic Soc., 61 (1978), pp. 146–152

$$
\left(E\left(w_{xx}+\nu w_{yy}\right)\right)_{xx}+2(1-\nu)\left(E w_{xy}\right)_{xy}
$$

$$
+\left(E\left(w_{yy}+\nu w_{xx}\right)\right)_{yy}=\frac{12(1-\nu^2)f}{h^3} \quad (5)
$$

- the direct problem (or the forward problem): for a given Young modulus E, find the displacement w of a glass sheet occupying a domain Ω before the heating process

- the PDE (5) is an elliptic fourth order linear PDE for the function $w = w(x, y)$

- two problems related to (5) have been studied: the clamped plate case and the simply supported plate case [12]

- we study the clamped case, in which the following boundary conditions are required: the plate is placed over a rigid frame,

$$
w(x, y)|_{\partial\Omega} = 0, \tag{6}
$$

and respectively,

$$
\frac{\partial w}{\partial n}|_{\partial\Omega} = 0, \tag{7}
$$

which means the (outward) normal derivative of w must be zero, i.e., the plate is not allowed to freely rotate around the tangent to $\partial\Omega$

- the associated inverse problem consists of finding Young's modulus E for a given data w in (5)

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[12] E. H. Mansfield, *The bending and stretching of plates*, Cambridge University Press, 1989

$$(E\,(w_{xx}+\nu w_{yy}))_{xx}+2(1-\nu)\,(Ew_{xy})_{xy}$$

$$+\,(E\,(w_{yy}+\nu w_{xx}))_{yy}=\frac{12(1-\nu^2)f}{h^3} \quad (5)$$

- the PDE (5) is a linear second order PDE for Young's modulus that can be written as

$$(w_{xx}+\nu w_{yy})E_{xx}+2(1-\nu)w_{xy}E_{xy}+(w_{yy}+\nu w_{xx})E_{yy}$$
$$+\,2(\Delta w)_x E_x+2(\Delta w)_y E_y+(\Delta^2 w)E=1, \quad (8)$$

after the scaling transformations $w \to \frac{1}{k}w$ or $E \to \frac{1}{k}E$, with $k=\frac{12(1-\nu^2)f}{h^3}$

- in (8), Δ denotes the Laplace operator

- the main problem in the car windscreen design is that the prescribed target shape w is frequent s.t. the discriminant

$$D=(1-\nu)^2 w_{xy}^2-(w_{xx}+\nu w_{yy})(w_{yy}+\nu w_{xx})$$

of the equation (8) changes sign in the domain Ω, so that (8) is a mixed type PDE

- this is one of the reason for which optical defects might occur during the process

- the equation (8) would naturally call for boundaries conditions for E on $\partial\Omega$ in the purely elliptic case (when $D<0$), and Cauchy data on a suitable (non-characteristic part) $\Gamma \subset \partial\Omega$ in the purely hyperbolic part (for $D>0$)

There is a recent interest in studying this inverse problem

- Mansfield [12] showed that a constant Young's modulus corresponds to a data which satisfies the nonhomogeneous biharmonic equation $\Delta^2 w = ct$.

- Salazar and Westbrook [13] studied the case when the data and the parameter are given by radial functions

- Temple [14] gives a survey on this subject and considered new examples of data for rectangular frames

- Kügler [15], [16] used a derivative free iterative regularization method for analyzing the problem on rectangular frames

- Engl and Kügler [17] studied a simplified model for the inverse problem on circular domains

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[12] E. H. Mansfield, *The bending and stretching of plates*, Cambridge University Press, 1989

[13] D. Salazar and R. Westbrook, *Inverse problems of mixed type in linear plate theory*, Eur. J. Appl. Math., to appear

[14] D. Temple, *An inverse system: An analysis arising from Windscreen manufacture*, M. Sc. Thesis, Department of Mathematics, Oxford University, Oxford, United Kingdom, 2002

[15] P. Kügler, *A derivative free LandWeber method for parameter identification in elliptic partial differential equations With application to the manufacture of car Windscreen s*, Ph.D. Thesis, Institute for Industrial Mathematics, Johannes Kepler University, Linz, Austria, 2003

[16] P. Kügler, *A parameter identification problem of mixed type related to the manufacture of car windscreen s*, SIAM J. Appl. Math., to appear

[17] H. W. Engl and P. Kügler, *The influence of the equation type on iterative parameter identification problems Which are elliptic or hyperbolic in the parameter*, Eur. J. Appl. Math., 14, 2 (2003), pp. 129–163

Our aim is to find the symmetry reductions related to the PDE (8) hidden by the nonlinearity that occurs between the data and the parameter

$\rightarrow$ group of transformations that leave the equation unchanged, which also relate the inverse and the direct problem

$\rightarrow$ knowledge of the invariants of these group actions allows us to write the target shape and the parameter in terms of them, and therefore, to reduce the order of the studied model

$\rightarrow$ we find again the obvious result that a Young's modulus constant corresponds to a data which is a solution of an non-homogeneous biharmonic equation

$\rightarrow$ the circular case considered by Salazar and Westbrook is a particular case of our study

$\rightarrow$ other target shapes which are not radial functions can be considered

- the equation (8) is invariant under scaling transformations

- target shapes modelled by homogeneous functions can be analyzed as well.

- in particular, we are interested in target shapes modelled by homogeneous polynomials defined on elliptical domains or square domains with rounded corners

2.2. Conditional symmetries associated with this model

The direct method (Clarkson and Kruskal [5]) applied to a second order PDE

$$\mathcal{F}\left(x, y, E^{(2)}\right) = 0, \tag{9}$$

consists of seeking solutions of the form

$$E(x, y) = \Phi(x, y, F(z)), \quad z = z(x, y), \quad (x, y) \in \Omega. \tag{10}$$

- $z = z(x, y)$ is called similarity variable
- $\{(x, y) : z(x, y) = k\}$ are called similarity curves

- after substituting (10) into (9), we require that the result to be an ODE for the arbitrary function $F = F(z)$

$\rightarrow$ certain conditions are imposed upon the functions Φ, z and their partial derivatives

The particular case [18]

$$E(x, y) = F(z(x, y)) \tag{11}$$

consists of looking for solutions depending only on the similarity variable z

- If z is an invariant of the group action then the solutions of the form (11) are as well

- Assume $z = z(x, y)$ is s.t. $\|\nabla z\| \neq 0$ on $\bar{\Omega}$

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[5] P. A. Clarkson and M. Kruskal, *New similarity reductions of the Boussinesq equation*, J. Math. Phys., 30 (1989), pp. 2201–2213

[18] R. Z. Zhdanov, *On conditional symmetries of multidimensional nonlinear equations of quantum field theory*, Symmetry in Nonlinear Mathematical Physics, 1 (1997), pp. 53–61

Assume that Young's modulus takes the form

$$E(x,y) = F(z(x,y)) \qquad (11)$$

and substitute it into (8)

$\longrightarrow$ we get the following relation

$$F''(z)\left[z_x^2(w_{xx}+\nu w_{yy}) + 2z_x z_y(1-\nu)w_{xy} + z_y^2(w_{yy}+\nu w_{xx})\right]$$
$$+ F'(z)\left[z_{xx}(w_{xx}+\nu w_{yy}) + 2(1-\nu)z_{xy}w_{xy} + +z_{yy}(w_{yy}+\nu w_{xx})\right.$$
$$\left.+2z_x(\Delta w)_x + 2z_y(\Delta w)_y\right] + F(z)(\Delta^2 w) = 1, \quad (12)$$

which must be an ODE for $F = F(z)$

$\longrightarrow$ the coefficients of the partial derivatives of F are functions of z only (i.e., these coefficients are also invariant under the same group action)

Denote them by

$$\begin{aligned}
\Gamma_1(z) &= z_x^2(w_{xx}+\nu w_{yy}) + 2z_x z_y(1-\nu)w_{xy} + z_y^2(w_{yy}+\nu w_{xx}),\\
\Gamma_2(z) &= z_{xx}(w_{xx}+\nu w_{yy}) + 2(1-\nu)z_{xy}w_{xy} + z_{yy}(w_{yy}+\nu w_{xx})\\
&\quad +2z_x(\Delta w)_x + 2z_y(\Delta w)_y,\\
\Gamma_3(z) &= \Delta^2 w.
\end{aligned} \qquad (13)$$

$\longrightarrow$ the PDE (8) is reduced to the second order linear ODE

$$\Gamma_1(z)F''(z) + \Gamma_2(z)F'(z) + \Gamma_3(z)F(z) = 1 \qquad (14)$$

2.2.1. Data and parameter invariant under the same group

If the target shape is invariant under the same group action as Young's modulus then

$$w(x,y) = G(z(x,y)) \tag{15}$$

- substituting (15) into the relations (13) we have

$$\begin{aligned}
\Gamma_1 &= G''(z_x^2+z_y^2)^2 + G'\left[(z_x^2+\nu z_y^2)z_{xx} + 2(1-\nu)z_x z_y z_{xy} + (z_y^2+\nu z_x^2)z_{yy}\right],\\
\Gamma_2 &= 2G'''(z_x^2+z_y^2)^2 + G''\left\{[7z_x^2+(\nu+2)z_y^2]z_{xx} + 2(5-\nu)z_x z_y z_{xy}\right.\\
&\quad \left.+[7z_y^2+(\nu+2)z_x^2]z_{yy}\right\} + G'\left\{(\Delta z)^2 + 2(1-\nu)(z_{xy}^2 - z_{xx}z_{yy})\right.\\
&\quad \left.+2\left[z_x(\Delta z)_x + z_y(\Delta z)_y\right]\right\},\\
\Gamma_3 &= G''''(z_x^2+z_y^2)^2 + 2G'''\left[(3z_x^2+z_y^2)z_{xx} + 4z_x z_y z_{xy} + (z_x^2+3z_y^2)z_{yy}\right.\\
&\quad +G''\left\{3(\Delta z)^2 + 4(z_{xy}^2 - z_{xx}z_{yy}) + 4\left[z_x(\Delta z)_x + z_y(\Delta z)_y\right]\right\} + G'\Delta^2 z.
\end{aligned} \tag{16}$$

- the coefficients of the partial derivatives of the function G must depend only on z, i.e.,

$$\begin{aligned}
\Gamma_1 &= \alpha^4 G'' + a_1 G',\\
\Gamma_2 &= 2\alpha^4 G''' + a_2 G'' + a_3 G',\\
\Gamma_3 &= \alpha^4 G'''' + 2a_4 G''' + a_5 G'' + a_6 G',
\end{aligned}$$

where

$$\begin{aligned}
\alpha^2(z) &= z_x^2 + z_y^2,\\
a_1(z) &= (z_x^2+\nu z_y^2)z_{xx} + 2(1-\nu)z_x z_y z_{xy} + (z_y^2+\nu z_x^2)z_{yy},\\
a_2(z) &= \left[7z_x^2+(\nu+2)z_y^2\right] z_{xx} + 2(5-\nu)z_x u_y z_{xy} + \left[7z_y^2+(\nu+2)z_x^2\right] z_{yy},\\
a_3(z) &= (\Delta z)^2 + 2(1-\nu)(z_{xy}^2 - z_{xx}z_{yy}) + 2\left[z_x(\Delta z)_x + z_y(\Delta z)_y\right],\\
a_4(z) &= (3z_x^2+z_y^2)z_{xx} + 4z_x z_y z_{xy} + (z_x^2+3z_y^2)z_{yy},\\
a_5(z) &= 3(\Delta z)^2 + 4(z_{xy}^2 - z_{xx}z_{yy}) + 4\left[z_x(\Delta z)_x + z_y(\Delta z)_y\right],\\
a_6(z) &= \Delta^2 z.
\end{aligned} \tag{17}$$

From the first relation in (17) which is a 2D eikonal equation, we get

$$\begin{aligned}
&z_x^2 z_{xx} + 2 z_x z_y z_{xy} + z_y^2 z_{yy} = \alpha^3(z)\alpha'(z),\\
&z_{xx} = \alpha(z)\alpha'(z) - \tfrac{z_y}{z_x} z_{xy},\\
&z_{yy} = \alpha(z)\alpha'(z) - \tfrac{z_x}{z_y} z_{xy}.
\end{aligned}$$

The last two equations imply

$$z_y^2 z_{xx} - 2 z_x z_y z_{xy} + z_x^2 z_{yy} = \alpha^3(z)\alpha'(z) - \alpha^4(z)\frac{z_{xy}}{z_x z_y}. \tag{18}$$

Assume that there is a function $\beta = \beta(z)$ such that

$$z_{xy} = \beta(z) z_x z_y \tag{19}$$

Indeed, since the left hand side in (18) depends only on z, one can easily check that

if z satisfies both the 2D eikonal equation in (17) and (19), then all the functions $a_i = a_i(z)$ defined by (17) are written in terms of α and β

$\longrightarrow$ the problem of finding z is reduced to that of integrating the 2D eikonal equation and the PDE system

$$\begin{cases} z_{xx} = \alpha\alpha' - \beta z_y^2 \\ z_{xy} = \beta z_x z_y \\ z_{yy} = \alpha\alpha' - \beta z_x^2. \end{cases} \tag{20}$$

The system (20) is compatible if it holds the following relation

$$\alpha\alpha'' + \alpha'^2 - 3\beta\alpha\alpha' + \alpha^2\left(\beta^2 - \beta'\right) = 0$$

Denote $\mu = \frac{1}{2}\alpha^2$. In this case, the above compatibility condition can be written as

$$\mu'' - 3\beta\mu' + 2\mu\left(\beta^2 - \beta'\right) = 0. \tag{21}$$

On the other hand, if the function β is given by

$$\beta(z) = -\frac{\lambda''(z)}{\lambda'(z)}, \tag{22}$$

where λ is a non-constant function, then the equation (19) turns into

$$(\lambda(z))_{xy} = 0$$

The general solution of this equation is given by

$$\lambda(z(x, y)) = a(x) + b(y), \tag{23}$$

with a and b being arbitrary functions. Substituting β from (22) into the compatibility condition (21) and after integrating once, we get

$$\mu'\lambda' + 2\mu\lambda'' = k, \tag{24}$$

where k is an arbitrary constant.

Case 1. If $k \neq 0$ then after integrating (24) and substituting back $\mu = \frac{1}{2}\alpha^2$, we get

$$\alpha^2(z) = \frac{2k\lambda(z) + C_1}{\lambda'^2(z)}. \tag{25}$$

The relation (23) implies $\lambda'(z)z_x = a'(x)$, and $\lambda'(z)z_y = b'(y)$

Substituting these relations, (23) and (25) into the 2D eikonal equation, it follows that $a = a(x)$ and $b = b(y)$ are solutions of the following respective ODEs

$$a'^2(x) - 2ka(x) = C_2, \quad \text{and} \quad b'^2(y) - 2kb(y) = C_3,$$

with $C_2 + C_3 = C_1$ (here C_i are real constants)

The above ODEs admit the non-constants solutions

$$a(x) = \frac{1}{2k}\left[k^2(x - C_4)^2 - C_2\right] \quad \text{and} \quad b(y) = \frac{1}{2k}\left[k^2(y - C_5)^2 - C_3\right],$$

and so, (23) takes the form

$$\lambda(z(x,y)) = \frac{k}{2}\left[(x - C_4)^2 + (y - C_5)^2\right] - \frac{C_1}{2k} \tag{26}$$

Notice that $\frac{1}{k_1}\lambda$ or $\lambda + k_2$ defines the same function β as the function λ does. Moreover, since the PDE (8) is invariant under translations in the (x,y)-space, we can consider

$$\lambda(z(x,y)) = x^2 + y^2 \tag{27}$$

If $\sqrt{\lambda}$ is a bijective function on a suitable interval, and if we denote by $\Phi = \left(\sqrt{\lambda}\right)^{-1}$ its inverse function, then z written in the polar coordinates (r,θ) (where $x = r\cos(\theta)$, $y = r\sin(\theta)$) is given by

$$z(x,y) = \Phi(r) \tag{28}$$

For simplicity, we consider $\Phi = Id$, and from that we get

$$E = F(r), \quad w = G(r), \quad z(x,y) = r \tag{29}$$

Hence, the ODE (14) turns into

$$\left(G'' + \frac{\nu}{r}G'\right)F'' + \left(2G''' + \frac{\nu+2}{r}G'' - \frac{1}{r^2}G'\right)F' + \left(G'''' + \frac{2}{r}G''' - \frac{1}{r^2}G'' + \frac{1}{r^3}G'\right)F = 1, \tag{30}$$

which can be reduced to the first order ODE

$$\left(G'' + \frac{\nu}{r}G'\right)F' + \left(G''' + \frac{1}{r}G'' - \frac{1}{r^2}G'\right)F = \frac{r^2 - r_0^2}{2r} + \frac{\gamma}{r}, \tag{31}$$

where $r_0 \in [0,1]$ with the property that

$$\gamma = \left[\left(rG'' + \nu G'\right)F' + \left(rG''' + G'' - \frac{1}{r}G'\right)F\right]|_{r=r_0}$$

is finite

The smoothness condition $G'(0) = 0$ implies that the equation (31) can be written as [12]

$$\left(G'' + \frac{\nu}{r}G'\right) F' + \left(G''' + \frac{1}{r}G'' - \frac{1}{r^2}G'\right) F = \frac{r}{2}. \qquad (32)$$

Case 2. If $k = 0$, similarly we get

$$z(x, y) = \Phi(k_1 x + k_2 y), \qquad (33)$$

where k_1 and k_2 are real constants such that $k_1^2 + k_2^2 > 0$

In this case, for $\Phi = Id$, the parameter and the data are written as

$$E = F(z) \quad w = G(z), \; z(x, y) = k_1 x + k_2 y, \qquad (34)$$

and the ODE (14) turns into

$$G''(z)F''(z) + 2G'''(z)F'(z) + G''''(z)F(z) = \frac{1}{(k_1^2 + k_2^2)^2}, \qquad (35)$$

with $\{z|G''(z) = 0\}$ the associated set of singularities.

Integrating the above ODE on the set $\{z|G''(z) \neq 0\}$ we obtain that Young's modulus is given by

$$E(x, y) = \frac{(k_1 x + k_2 y)^2 + C_1(k_1 x + k_2 y) + C_2}{2(k_1^2 + k_2^2)^2 G''(k_1 x + k_2 y)},$$

where C_i are arbitrary constants.

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[12] E. H. Mansfield, *The bending and stretching of plates*, Cambridge University Press, 1989

2.2.2. Data and parameter invariant under different groups

Consider two functionally independent functions on Ω, say $z = z(x,y)$ and $v = v(x,y)$, and let

$$w = H(v(x,y)) \tag{36}$$

be the target shape

- In this case, the data and the parameter do not share the same invariance

- Similarly to the above, substituting (36) into the relations (13) we get

$$\begin{aligned}
\Gamma_1 &= H''\left[(z_x v_x + z_y v_y)^2 + \nu(z_y v_x - z_x v_y)^2\right] \\
&\quad + H'\left[z_x^2 v_{xx} + 2 z_x z_y v_{xy} + z_y^2 v_{yy} + \nu\left(z_x^2 v_{yy} - 2 z_x z_y v_{xy} + z_y^2 v_{xx}\right)\right] \\
\Gamma_2 &= H'''(v_x^2 + v_y^2)(z_x v_x + z_y v_y) + H''\left[v_x^2 z_{xx} + 2 v_x v_y u_{xy} + v_y^2 u_{yy}\right. \\
&\quad + \nu\left(v_y^2 u_{xx} - 2 v_x v_y z_{xy} + v_x^2 z_{yy}\right) + 2 z_x v_x v_{xx} + 2(z_x v_y + z_y v_x) v_{xy} + 2 z_y v_y v_{yy} \\
&\quad \left. + (z_x v_x + z_y v_y)(\Delta v)\right] + H'\left[z_{xx} v_{xx} + 2 z_{xy} v_{xy} + z_{yy} v_{yy} + \nu\left(z_{xx} v_{yy} - 2 z_{xy} v_{xy}\right.\right. \\
&\quad \left.\left. + z_{yy} v_{xx}\right) + z_x (\Delta v)_x + z_y (\Delta v)_y\right] \\
\Gamma_3 &= H''''(v_x^2 + v_y^2)^2 + 2H'''\left[(3v_x^2 + v_y^2) v_{xx} + 4 v_x v_y v_{xy} + (v_x^2 + 3 v_y^2) v_{yy}\right] \\
&\quad + H''\left[3 v_{xx}^2 + 4 v_{xy}^2 + 3 v_{yy}^2 + 2 v_{xx} v_{yy} + 4 v_x (\Delta v)_x + 4 v_y (\Delta v)_y\right] + H' \Delta^2 v.
\end{aligned} \tag{37}$$

Recall that Γ_i's are functions of $z = z(x,y)$ only. Since each right hand side in the above relations contains the function $H = H(v)$ and its derivatives, We require that the coefficients of the derivatives of H to be functions of v. It follows that Γ_i must be constant and denote them by γ_i

Therefore, the last condition in (37) becomes

$$\Delta^2(w) = \gamma_3, \tag{38}$$

which is the biharmonic equation

According to the above assumption, we seek solutions of (38) that are functions of v only

Similarly, we get

$$v(x,y) = \Psi(r) \quad \text{or} \quad v(x,y) = \Psi(k_1 x + k_2 y) \tag{39}$$

Thus, for $\Psi = Id$, the target shape is written as

$$w(x,y) = H(r) \quad \text{or} \quad w(x,y) = H(k_1 x + k_2 y). \tag{40}$$

Since z and v are functionally independent, we get

$$z(x,y) = k_1 x + k_2 y, \quad v(x,y) = \sqrt{x^2 + y^2} \tag{41}$$

or

$$z(x,y) = \sqrt{x^2 + y^2}, \quad v(x,y) = k_1 x + k_2 y \tag{42}$$

One can prove that if the coefficients γ_i are constant, and if z and v are given by (41) or (42), respectively, then $\gamma_1 = \gamma_2 = 0$, and $\gamma_3 \neq 0$

On the other hand, the solutions of the biharmonic equation (38) of the form (40) are the following

$$w(x,y) = \frac{\gamma_3}{64} z^4 + C_1 z^2 + C_2 \ln(z) + C_3 z^2 \ln(z) + C_4, \ z = \sqrt{x^2 + y^2},$$

and respectively,

$$w(x,y) = \frac{\gamma_3}{24(k_1^2 + k_2^2)^2} v^4 + C_1 v^3 + C_2 v^2 + C_3 v + C_4, \ v = k_1 x + k_2 y,$$

and these correspond to the constant Young's modulus

$$E(x,y) = \frac{1}{\gamma_3} \tag{43}$$

Notice that only particular solutions of the biharmonic equation have been found in this case (i.e., solutions invariant under rotations and translations)

Since this PDE is also invariant under scaling transformations, which act not only on the space of the independent variables but on the data space as well, it is obvious to extend our study and to seek for other type of symmetry reductions.

2.3. Equivalence transformations related to the studied model

Consider a one-parameter Lie group of transformations acting on an open set $\mathcal{D} \subset \Omega \times \mathcal{W} \times \mathcal{E}$, where

$\mathcal{W}$ is the space of the data functions and

$\mathcal{E}$ is the space of the parameter functions

given by

$$\left\{ \begin{array}{lcl} x^* & = & x + \varepsilon \zeta(x, y, w, E) + \mathcal{O}(\varepsilon^2) \\ y^* & = & y + \varepsilon \eta(x, y, w, E) + \mathcal{O}(\varepsilon^2) \\ w^* & = & w + \varepsilon \phi(x, y, w, E) + \mathcal{O}(\varepsilon^2) \\ E^* & = & E + \varepsilon \psi(x, y, w, E) + \mathcal{O}(\varepsilon^2), \end{array} \right. \tag{44}$$

where ε is the group parameter. Let

$$V = \zeta(x, y, w, E)\partial_x + \eta(x, y, w, E)\partial_y + \phi(x, y, w, E)\partial_w + \psi(x, y, w, E)\partial_E \tag{45}$$

be its associated general infinitesimal generator

- (44) is called an equivalence transformation associated to the PDE (8) if this leaves the equation invariant

- the equivalence transformations can be found by applying the classical Lie method to (8), with E and w both considered as unknown functions

$\rightarrow$ following this method we obtain

$$\left\{ \begin{array}{lcl} \zeta(x, y, w, E) & = & k_1 + k_5 x - k_4 y \\ \eta(x, y, w, E) & = & k_2 + k_4 x + k_5 y \\ \phi(x, y, w, E) & = & k_3 + k_7 x + k_6 y + (4k_5 - k_8) w \\ \psi(x, y, w, E) & = & k_8 E \end{array} \right. \tag{46}$$

where k_i are real constants

- the vector field (45) is written as $V = \sum_{i=1}^{8} k_i V_i$, where

$$\begin{gathered} V_1 = \partial_x, \quad V_2 = \partial_y, \quad V_3 = \partial_w, \\ V_4 = -y\partial_x + x\partial_y, \quad V_5 = x\partial_x + y\partial_y + 4w\partial_w, \\ V_6 = y\partial_w, \quad V_7 = x\partial_w, \quad V_8 = -w\partial_w + E\partial_E. \end{gathered} \tag{47}$$

The equivalence transformations related to the PDE (8) are generated by the infinitesimal generators (47)

Thus, the equation is invariant under translations in the x-space, y-space, w-space, rotations in the space of the independent variables (x, y), scaling transformations in the (x, y, w)-space, Galilean transformations in the (y, w) and (x, w) spaces, and scaling transformations in the (w, E)-space, respectively.

- Notice that the conditional symmetries (§2.2) represent particular cases of the equivalence transformations

- Since each one-parameter group of transformations generated by V_i is a symmetry group, if $(w = G(x, y), E = F(x, y))$ is a pair of known solutions of (8), so are the following

$$\begin{aligned} w^{(1)} &= G(x - \varepsilon_1, y), & E^{(1)} &= F(x - \varepsilon_1, y), \\ w^{(2)} &= G(x, y - \varepsilon_2), & E^{(2)} &= F(x, y - \varepsilon_2), \\ w^{(3)} &= G(x, y) + \varepsilon_3, & E^{(3)} &= F(x, y), \\ w^{(4)} &= G(\tilde{x}, \tilde{y}), & E^{(4)} &= F(\tilde{x}, \tilde{y}), \\ w^{(5)} &= e^{4\varepsilon_5} G\left(e^{-\varepsilon_5} x, e^{-\varepsilon_5} y\right), & E^{(5)} &= F\left(e^{-\varepsilon_5} x, e^{-\varepsilon_5} y\right) \\ w^{(6)} &= G(x, y) + \varepsilon_6 y, & E^{(6)} &= F(x, y), \\ w^{(7)} &= G(x, y) + \varepsilon_7 x, & E^{(7)} &= F(x, y), \\ w^{(8)} &= e^{-\varepsilon_8} G(x, y), & E^{(8)} &= e^{\varepsilon_8} F(x, y), \end{aligned} \tag{48}$$

where $\tilde{x} = x\cos(\varepsilon_4) + y\sin(\varepsilon_4)$, $\tilde{y} = -x\sin(\varepsilon_4) + y\cos(\varepsilon_4)$, and ε_i are real constants

- the general solution of (8) constructed from a known one is given by

$$w(x,y) = e^{4\varepsilon_5-\varepsilon_8} G\left(e^{-\varepsilon_5}(\tilde{x}-\tilde{k}_1), e^{-\varepsilon_5}(\tilde{y}-\tilde{k}_2)\right) + e^{4\varepsilon_5-\varepsilon_8}\varepsilon_6 y + e^{4\varepsilon_5-\varepsilon_8}\varepsilon_7 x + e^{4\varepsilon_5-\varepsilon_8}\varepsilon_3$$

$$E(x,y) = e^{\varepsilon_8} F\left(e^{-\varepsilon_5}(\tilde{x}-\tilde{k}_1), e^{-\varepsilon_5}(\tilde{y}-\tilde{k}_2)\right),$$

where $\tilde{k}_1 = \varepsilon_1 \cos(\varepsilon_4) + \varepsilon_2 \sin(\varepsilon_4)$, and $\tilde{k}_2 = \varepsilon_1 \sin(\varepsilon_4) - \varepsilon_2 \cos(\varepsilon_4)$.

- The equivalence transformations form a Lie group $\mathcal{G}$ with a 8-dimensional associated Lie algebra $\mathcal{A}$. Using the adjoint representation of $\mathcal{G}$, one can find the optimal system of one-dimensional subalgebras of $\mathcal{A}$ (more details can be found in [3], pp. 203-209). This optimal system is spanned by the vector fields given in Table 1

- Denote by z, I, and J the invariants related to the one-parameter group of transformations generated by each vector field V_i. Here F and G are arbitrary functions, (r,θ) are the polar coordinates, and a, b, c are non-zero constants.

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[3] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts Math. 107, Springer Verlag, New York, 1986

Generators	Invariants	$w=w(x,y)$	$E=E(x,y)$	ODE
V_1	$z=y$ $I=w$ $J=E$	$w=G(z)$	$E=F(z)$	(50)
V_2	$z=x$ $I=w$ $J=E$	$w=G(z)$	$E=F(z)$	(50)
V_4	$z=r$ $I=w$ $J=E$	$w=G(z)$	$E=F(z)$	(30)
V_5	$z=\frac{y}{x}$ $I=x^{-4}w$ $J=E$	$w=x^4G(z)$	$E=F(z)$	(52)
cV_3+V_4	$z=r$ $I=w-c\theta$ $J=E$	$w=c\theta+G(z)$	$E=F(z)$	(30)
V_5+cV_8	$z=\frac{y}{x}$ $I=x^{c-4}w$ $J=x^{-c}E$	$w=x^{4-c}G(z)$	$E=x^cF(z)$	(53)
V_4+cV_8	$z=r$ $I=e^{c\theta}w$ $J=e^{-c\theta}E$	$w=e^{-c\theta}G(z)$	$E=e^{c\theta}F(z)$	(54)
V_4+cV_5	$z=re^{-c\theta}$ $I=r^{-4}w$ $J=E$	$w=r^4G(z)$	$E=F(z)$	(56)
V_4+cX_5 $+bV_8$	$z=re^{-c\theta}$ $I=r^{\frac{b}{c}-4}w$ $J=r^{-\frac{b}{c}}E$	$w=r^{4-\frac{b}{c}}G(z)$	$E=r^{\frac{b}{c}}F(z)$	(57)
V_1+cV_6	$z=y$ $I=w-cxy$ $J=E$	$w=cxy+G(z)$	$E=F(z)$	(50)
V_2+cV_7	$z=x$ $I=w-cxy$ $J=E$	$w=cxy+G(z)$	$E=F(z)$	(50)
V_1+cV_8	$z=y$ $I=e^{cx}w$ $J=e^{-cx}E$	$w=e^{-cx}G(z)$	$E=e^{cx}F(z)$	(58)
V_2+cV_8	$z=x$ $I=e^{cy}w$ $J=e^{-cy}E$	$w=e^{-cy}G(z)$	$E=e^{cy}F(z)$	(58)

Table 1

$\rightarrow$ To reduce the order of the PDE (8) one can integrate the first order PDE system

$$\left\{ \begin{array}{lcl} \zeta(x,y,w,E)w_x + \eta(x,y,w,E)w_y &=& \phi(x,y,w,E) \\ \zeta(x,y,w,E)E_x + \eta(x,y,w,E)E_y &=& \psi(x,y,w,E), \end{array} \right. \tag{49}$$

that defines the characteristics of the vector field (45)

$\rightarrow$ The reduced ODEs listed in Table 1:

• The invariance of the equation (8) under the one-parameter groups of transformations generated by V_1, V_2, $V_1 + cV_6$ and $V_2 + cV_7$, respectively, leads us to the same ODE

$$F''(z)G''(z) + 2F'(z)G'''(z) + F(z)G''''(z) = 1, \tag{50}$$

with the general solution

$$F(z) = \frac{z^2 + C_1 z + C_2}{2G''(z)} \tag{51}$$

on the set $\{z | G''(z) \neq 0\}$

• The invariance under the scaling transformation generated by the vector field V_5 yields the reduced ODE

$$\begin{aligned} &\left[G''\left(z^2+1\right)^2 - 6z(z^2+1)G' + 12(z^2+\nu)G\right]F'' \\ &\quad + 2\left[\left(z^2+1\right)^2 G''' - 5z(z^2+1)G'' + 3(4z^2+\nu+1)G' - 12zG\right]F' \\ &\quad + \left[\left(z^2+1\right)^2 G'''' - 4z(z^2+1)G''' + 4(3z^2+1)G'' - 24zG' + 24G\right]F = 1. \end{aligned} \tag{52}$$

• The following ODE

$$\left[\left(z^2+1\right)^2 G'' + 2(c-3)z(z^2+1)G' + (c-3)(c-4)(z^2+\nu)G\right] F''$$
$$+\left\{2\left(z^2+1\right)^2 G''' + 2(2c-5)z(z^2+1)G'' + 2(c-3)[z^2(c-4)+\nu(c-1)-1]G'\right.$$
$$\left.-2(c-3)(c-4)zG\right\} F' + \left\{\left(z^2+1\right)^2 G'''' + 2(c-2)z(z^2+1)G''' + \left[(c-3)(c-4)z^2\right.\right.$$
$$\left.\left.-2(c-2)+\nu c(c-1)\right] G'' - 2(c-4)(c-3)zG' + 2(c-4)(c-3)G\right\} F = 1 \tag{53}$$

is obtained in the case 6. The reduced equation

$$\left[G'' + \frac{\nu}{r}G' + \frac{\nu c^2}{r^2}G\right] F'' + \left[2G''' + \frac{\nu+2}{r}G'' + \frac{2\nu c^2-1}{r^2}G' - \frac{c^2(1+2\nu)}{r^3}G\right] F' \tag{54}$$
$$+\left[G'''' + \frac{2}{r}G''' + \frac{c^2\nu-1}{r^2}G'' + \frac{1-c^2(2\nu+1)}{r^3}G' + \frac{2c^2(\nu+1)}{r^4}G\right] F = 1$$

is related to the case 7. This can be written as the first order ODE

$$\left(G'' + \frac{\nu}{r}G' + \frac{\nu c^2}{r^2}G\right) F' + \left(G''' + \frac{1}{r}G'' + \frac{c^2\nu-1}{r^2}G' - \frac{c^2(1+\nu)}{r^3}G\right) F = \frac{r^2-r_0^2}{2r} + \frac{\gamma^*}{r}, \tag{55}$$

where $r_0 \in [0,1]$ with the property that

$$\gamma^* = \left[F'\left(rG'' + \nu G' + \frac{\nu}{r}G\right) + F\left(rG''' + G'' + \frac{c^2\nu-1}{r}G' - \frac{c^2(1+\nu)}{r^2}G\right)\right]|_{r=r_0}$$

is finite

• In the cases 8 and 9, after the change of the variable $z = \exp(t)$, the reduced ODEs are the following

$$\left\{(c^2+1)^2G'' + (c^2+1)(\nu+7)G' + 4\left[\nu(3c^2+1)+c^2+3\right]G\right\}F''$$
$$+\{2(c^2+1)^2G''' + (c^2+1)(\nu+19)G'' + 2[16+(c^2+1)(3\nu+13)]G'$$
$$+8(\nu+7)G\}F' + \{(c^2+1)^2G'''' + 12(c^2+1)G'''$$
$$+4(5c^2+13)G'' + 96G' + 64G\}F = 1, \tag{56}$$

and respectively,

$$\left\{(c^2+1)^2G'' + \left(\frac{1}{c}+c\right)[c(\nu+7)-2b]G' + \left(\frac{4}{c}-\frac{b}{c^2}\right)[c^3(1+3\nu)-c^2\nu b\right.$$
$$\left.+c(\nu+3)-b]G\right\}F'' + \left\{2(c^2+1)^2G''' + \left(\frac{1}{c}+c\right)[c(\nu+19)-4b]G''\right.$$
$$+2\left[\frac{b^2}{c^2}+\nu b^2+c^2(3\nu+13)-4bc(\nu+1)-12\frac{b}{c}+3\nu+29\right]G'$$
$$\left.+\left(\frac{4}{c}-\frac{b}{c^2}\right)[2c(\nu+7)+b(\nu-5)]G\right\}F' + \left\{(c^2+1)^2G''''\right.$$
$$+2\left(c+\frac{1}{c}\right)(6c-b)G''' + \left[\frac{b^2}{c^2}+\frac{b}{c}(\nu-17)+20c^2-bc(\nu+7)+\nu b^2+52\right]G''$$
$$\left.+\left(\frac{6}{c}-\frac{b}{c^2}\right)[16c+b(\nu-5)]G' + 2\left(\frac{4}{c}-\frac{b}{c^2}\right)[8c+b(\nu-3)]G\right\}F = 1. \tag{57}$$

• In the cases 12 and 13 we get the same equation

$$\left(G''+\nu c^2G\right)F'' + 2\left(G'''+\nu c^2G'\right)F' + \left(G''''+\nu c^2G''\right)F = 1, \tag{58}$$

with the general solution given by

$$F(z) = \frac{z^2+C_1z+C_2}{G''(z)+\nu c^2G(z)} \tag{59}$$

on the set $\{z|G''(z)+\nu c^2G(z) \neq 0\}$, where C_1 and C_2 are arbitrary real constants

2.4. Examples

$\rightarrow$ Since the data w is the function that models the target shape of a car windscreen, we seek a data with relevant physical and geometrical properties, such as smoothness and a positive curvature graph at least in the center of Ω, for which the boundary condition

$$w(x,y)|_{\partial\Omega} = 0 \quad (6)$$

is satisfied

Moreover, if there is no free rotation of the plate around the tangent to $\partial\Omega$ then the condition

$$\frac{\partial w}{\partial n}|_{\partial\Omega} = 0 \quad (7)$$

is required.

Remarks. Assume that $\partial\Omega = \{(x,y)|\ z(x,y) = k\}$ is the k-level set of the function z (here k being a non-zero constant) and $||\nabla z|| > 0$ on $\bar{\Omega}$. If the target shape is given by

$$w(x,y) = a(x,y)G(z(x,y))$$

where $a = a(x,y)$ is a suitable function according to Table 1, then the boundary conditions (6) and (7) are equivalent to

$$G(k) = 0 \quad \text{and} \quad G'(k) = 0$$

Therefore, the data might have the form

$$w(x,y) = a(x,y)\,(z(x,y) - k)^2\,H(z(x,y))$$

This corresponds to the case when the data and Ω are invariant under the same symmetry reduction

In our case, this can be applied to rotational symmetries

→ For scaling invariance, we have to incorporate the non-invariant boundary conditions in invariant solutions

- the problem on elliptical domains and on square domains with rounded corners

For instance, the class of target shapes of the form

$$w(x,y) = z^m(x,y) - k^m$$

where $m \geq 1$ a natural number, satisfies (6)

In this case, we have

$$\frac{\partial w}{\partial n}|_{\partial\Omega} = mk^{m-1}\|\nabla z\||_{\partial\Omega}$$

If this quantity is small then the condition (7) is almost satisfied (i.e., there is a small rotation of the plate around the tangent to $\partial\Omega$)

In the following examples, we assume that the glass Poisson ration $\nu = 0.5$

E.g. 1. Rotational invariant data and parameter

Consider the target shape of the form [14]

$$w(x,y) = G(r) = -\frac{1}{6}(r-1)^2(2r+1), \quad r = \sqrt{x^2+y^2},$$

defined on the unit disc which satisfies the boundary conditions (6) and (7). Since $G'(0) = 0$, the reduced ODE is (32) and this has a singularity at $r = \frac{3}{5}$. Since $E > 0$, we consider the constant of integration $C_1 = 1$, and so,

$$E(x,y) = F(r) = -\frac{1}{11}\left(r + \frac{1}{2}\right) + (5r-3)^{-\frac{6}{5}}$$

The PDE (8) is elliptic for $r \in \left(\frac{3}{5}, \frac{3}{4}\right)$, hyperbolic for $r \in \left[0, \frac{3}{5}\right) \bigcup \left(\frac{3}{5}, 1\right]$, and parabolic if $r = \frac{3}{5}$ or $r = \frac{3}{4}$, respectively.

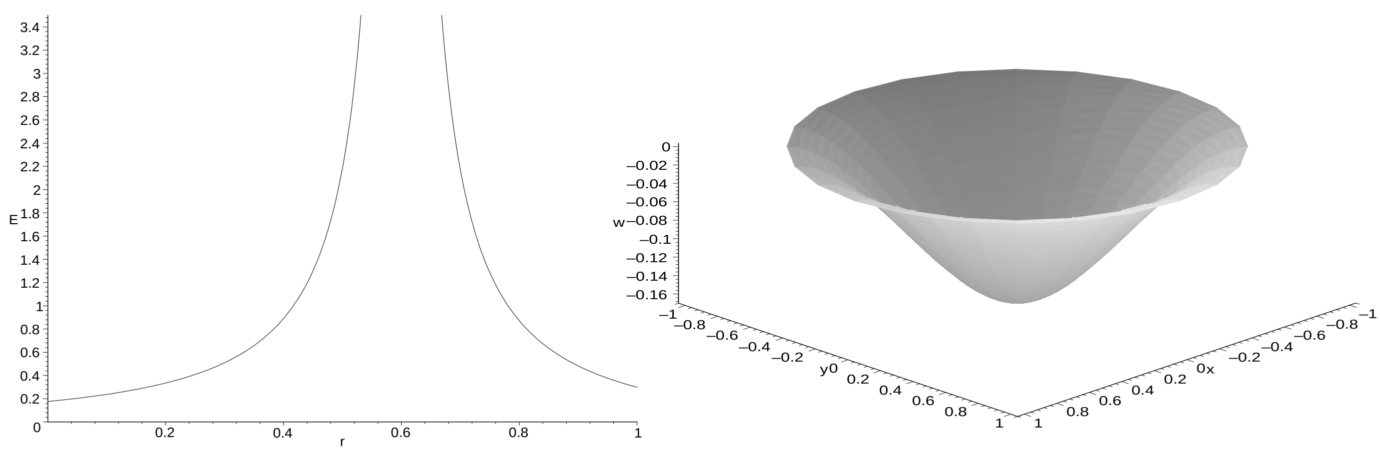

Fig. 1

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[14] D. Temple, *An inverse system: An analysis arising from Windscreen manufacture*, M. Sc. Thesis, Department of Mathematics, Oxford University, Oxford, United Kingdom, 2002

E.g. 2. Particular target shapes on rounded square domains

a) Suppose the Lamé oval $\partial\Omega = \{(x, y)|\ x^{2n} + y^{2n} = 1\}$ is the boundary of the domain (here $n \geq 2$ is a natural number). For

$$w(x, y) = \left(x^{2n} + y^{2n}\right)^m - 1, \tag{60}$$

$m \geq 1$ being a natural number, the equation (8) is elliptic on $\Omega - \{(0, 0)\}$ and parabolic in $(0, 0)$

These target shapes are invariant under $V_5 + cV_8 + (4 - c)V_3$, where $c = 4 - 2mn$. For $x > 0$ or $x < 0$, the functions (60) can be written as

$$w(x, y) = x^{2mn}G(z) - 1, \quad G(z) = \left(1 + z^{2n}\right)^m, \quad z = \frac{y}{x}.$$

According to the case 6 in Table 1, the associated Young's modulus has the form

$$E(x, y) = x^{4-2mn}F(z), \quad z = \frac{y}{x}.$$

Since $w(x, y) = w(y, x) = w(-x, y) = w(x, -y) = w(-x, -y)$, Young's modulus also shares these discrete symmetries. Thus, the reduced ODE (53) can be integrated for $z \in [0, 1]$. For $n = 2$ and $m = 1$, the data is a solution of the biharmonic equation and $E = 48^{-1}$.

For $n = 3$ and $m = 1$, the data and the numerical solution F satisfying $F(0) = 0.002$ and $F'(0) = 0$ are given in Fig. 2.a.

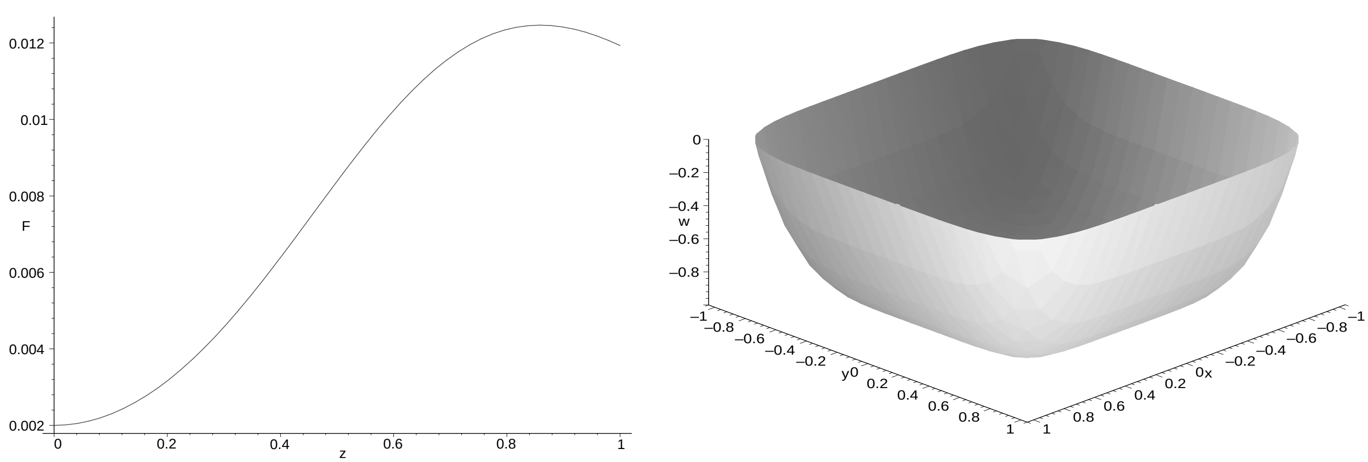

Fig. 2.a

b) Assume that $\partial\Omega = \{(x,y) | \; x^{2n} + y^2 = 1\}$, where $n \geq 1$ is a natural number. Consider the following class of target shapes

$$w(x,y) = x^{2n} + y^2 - 1,$$

is invariant under the vector field $V_2 + 2V_6$. Hence, the equation (8) is reduced to the ODE (50)

For $n = 3$ and $m = 1$, the associated Young's modulus is

$$E(x,y) = F(x) = \frac{x^2 + C_1 x + C_2}{2(30x^4 + 1)}.$$

Since $E > 0$, we can set $C_1 = 0$ and $C_2 = 1$ (see Fig. 2.b). The PDE (8) is elliptic on $\Omega - Oy$ and parabolic on the y-axis.

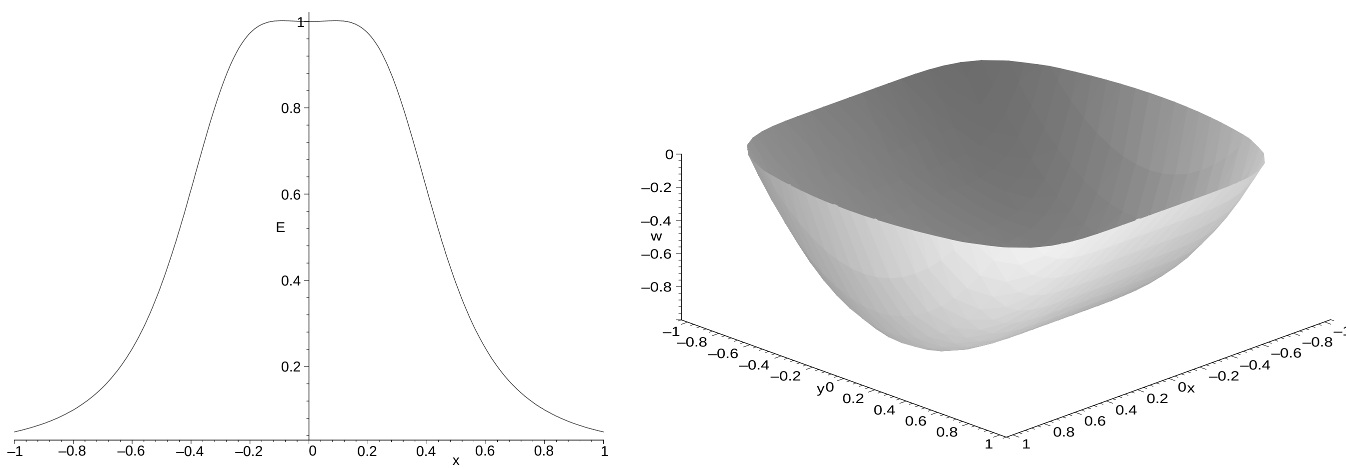

Fig. 2.b

E.g. 3. Particular target shapes on elliptic domains

Consider the data

$$w(x,y) = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^m - 1, \tag{61}$$

on the elliptic domain $\Omega = \{(x,y) | \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1\}$, where $m \geq 1$ is a natural number. These target shapes are invariant under $V_5 + cV_8 + (4-c)V_3$, where $c = 4 - 2m$.

The PDE (8) is elliptic on $\Omega - \{(0,0)\}$ and parabolic in the origin. For $x > 0$ or $x < 0$, (61) can be written as

$$w(x,y) = x^{2m} G(z) - 1, \quad G(z) = \left(\frac{1}{a^2} + \frac{z^2}{b^2}\right)^m, \quad z = \frac{y}{x}.$$

In this case, we look for solutions to (8) of the form

$$E(x,y) = x^{4-2m} F(z), \quad z = \frac{y}{x}$$

If $m = 2$, the data is a solution of the biharmonic equation, and the related Young's modulus is $E = 24\left(a^{-4} + b^{-4}\right) + 16a^{-2}b^{-2}$.

If $m \geq 3$, the reduced ODE is (52). For $m = 3$, the data and the numerical solution F of (52) satisfying $F(0) = 0.001$ and $F'(0) = 0$ are plotted in Fig. 3

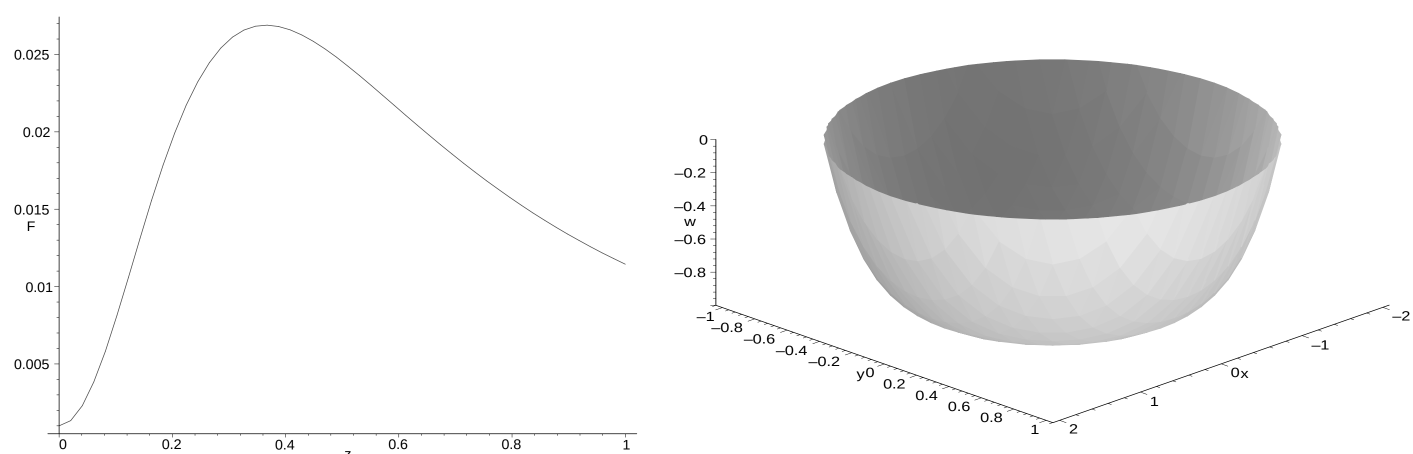

Fig. 3.

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