Noether's Theorem for *SMOOTH*, DISCRETE and Finite Element Models

Elizabeth Mansfield

KENT
UNIVERSITY OF KENT
AT CANTERBURY

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Noether's Theorem

links SYMMETRIES and conservation laws for Euler Lagrange Systems.

What is a conservation law?

Answer: a divergence expression which is zero on solutions of the system.

The heat equation

$$u_t + (-u_x)_x = 0$$

is its own conservation law. Integrating,

$$\frac{\partial}{\partial t}\int_\Omega u + (-u_x)]_{\partial\Omega} = 0$$

where we assume u is sufficiently nice that we can interchange ∂_t and $\int$, and we have applied Stokes' Theorem. In words:

Rate of change of total heat in Ω	=	Net of comings and goings across the boundary

no sources or sinks

The usual examples

Symmetry	Conserved Quantity
leaves $L\mathrm{d}x$ invariant	the quantity behind the $\frac{\mathrm{D}}{\mathrm{D}t}$ in the Divergence
$\begin{cases} t^* = t + c \\ \text{translation in time} \end{cases}$	Energy
$\begin{cases} x_i^* = t + c \\ \text{translation in space} \end{cases}$	Linear Momenta
$\begin{cases} \mathbf{x}^* = \mathcal{R}\mathbf{x} \\ \text{rotation in space} \end{cases}$	Angular Momenta
$\begin{cases} a^* = \phi(a,b) \\ b^* = \psi(a,b) \\ \phi_a\psi_b - \phi_b\psi_a \equiv 1 \\ \text{Particle relabelling} \end{cases}$	Potential vorticity

Variational Complexes 1-2-3 !

are locally exact

SMOOTH cf. P.J. Olver, Applications . . .

$$
\begin{array}{ccccccccccc}
\xrightarrow{\text{Curl}} & \Lambda^2 & \xrightarrow{\text{Div}} & \Lambda^3 & \xrightarrow{\hat{d}} & \hat{\Lambda}_1 & \xrightarrow{\hat{d}} & \hat{\Lambda}_2 & \xrightarrow{\hat{d}} \\
 & & & & & \downarrow \pi & & \downarrow \pi & \\
 & & & & & \Lambda^1{}_* & \xrightarrow{\delta} & \Lambda^2{}_* & \xrightarrow{\delta}
\end{array}
$$

DISCRETE Hydon and ELM, J. FoCM

$$
\begin{array}{ccccccccccc}
\xrightarrow{\Delta} & \mathbf{Ex}^2 & \xrightarrow{\Delta} & \mathbf{Ex}^3 & \xrightarrow{\hat{d}} & \hat{\Lambda}_1 & \xrightarrow{\hat{d}} & \hat{\Lambda}_2 & \xrightarrow{\hat{d}} \\
 & & & & & \downarrow \pi & & \downarrow \pi & \\
 & & & & & \Lambda^1_* & \xrightarrow{\delta} & \Lambda^2_* & \xrightarrow{\delta}
\end{array}
$$

Finite Element ELM and GRW Quispel, CRM Proc.

$$
\begin{array}{ccccccccccc}
\xrightarrow{d} & \tilde{\mathcal{F}}^2 & \xrightarrow{d} & \tilde{\mathcal{F}}^3 & \xrightarrow{\hat{d}\circ\int} & \widehat{\mathcal{F}_1} & \xrightarrow{\hat{d}} & \widehat{\mathcal{F}_2} & \xrightarrow{\hat{d}} \\
 & & & & & \downarrow \pi & & \downarrow \pi & \\
 & & & & & \mathcal{F}^1_* & \xrightarrow{\delta} & \mathcal{F}^2_* & \xrightarrow{\delta}
\end{array}
$$

Exactness can be used to find conservation laws for non Euler-Lagrange systems via clever ansatze!

cf Hereman, Sanders, Sayers and Wang, *CRM Proceedings*; Hydon *J. Phys. A*

Exactness is proved by the use of so-called homotopy operators H_i,

$$\underset{\leftarrow}{\rightarrow} \Lambda^2 \underset{\underset{H_1}{\leftarrow}}{\overset{\text{Div}}{\rightarrow}} \Lambda^3 \underset{\underset{H_0}{\leftarrow}}{\overset{E}{\rightarrow}} \Lambda_1^* \underset{\leftarrow}{\rightarrow}$$

which satisfy

$$(\text{Div} H_1 + H_0 E)\omega = \omega, \qquad \text{all } \omega \in \Lambda^3$$

Thus if $E(\omega) = 0$, then $\omega = \text{Div}(H_1(\omega))$.

Idea: solve $E(\text{clever ansatz}) = 0$ for parameters and arbitrary functions. Then you have a conservation law using H_1.

More on $\hat{\mathsf{d}}$ and π

$\mathcal{SMOOTH}$

$$
\begin{aligned}
\hat{\mathsf{d}}(L\mathsf{d}x) &= \hat{\mathsf{d}}\left(\tfrac{1}{2}\left(u_x^2 + u_{xx}^2\right)\mathsf{d}x\right) \\
&= (u_x\mathsf{d}u_x + u_{xx}\mathsf{d}u_{xx})\mathsf{d}x \\
&= (-u_{xx}\mathsf{d}u + u_{xxxx}\mathsf{d}u)\mathsf{d}x \\
&\quad + \tfrac{\mathsf{D}}{\mathsf{D}x}\left(u_x\mathsf{d}u - 2u_{xx}\mathsf{d}u_x + \tfrac{\mathsf{D}}{\mathsf{D}x}\left(u_{xx}\mathsf{d}u\right)\right) \\
&= E(L)\mathsf{d}u\mathsf{d}x + \tfrac{\mathsf{D}}{\mathsf{D}x}\eta_L
\end{aligned}
$$

General Formula, explicit, exact, symbolic, for η_L known.

$E = \pi \circ \hat{\mathsf{d}}$, where π projects out the divergence term.

More than one dependent variable

$$
\begin{aligned}
\hat{\mathsf{d}}L(x,u,v,\ldots)\mathsf{d}x &= E^u(L)\mathsf{d}u\mathsf{d}xE^v(L)\mathsf{d}v\mathsf{d}x \\
&\quad + \tfrac{\mathsf{D}}{\mathsf{D}x}\eta_L
\end{aligned}
$$

More on $\hat{\mathsf{d}}$ and π

DISCRETE

$$
\begin{aligned}
\hat{\mathsf{d}}(L\mathsf{d}x) &= \hat{\mathsf{d}}\left(\tfrac{1}{2}u_n^2 + u_n u_{n+1}\right)\Delta_n \\
&= (u_n \mathsf{d}u_n + u_{n+2}\mathsf{d}u_n + u_n \mathsf{d}u_{n+2})\Delta_n \\
&= (u_n + u_{n+2} + u_{n-2})\mathsf{d}u_n \Delta_n \\
&\quad +(S - \mathsf{id})(\cdots) \\
&= E(L_n)\mathsf{d}u_n \Delta_n + \Delta(\eta_{L_n})
\end{aligned}
$$

General Formula, explicit, exact, symbolic, for η_{L_n} known.

$E = \pi \circ \hat{\mathsf{d}}$, where π projects out the total difference term.

More than one dependent variable

$$
\hat{\mathsf{d}}(L_n \Delta_n) = E^u(L_n)\mathsf{d}u_n \Delta_n + E^v(L_n)\mathsf{d}v_n \Delta_n + \Delta(\eta_{L_n})
$$

Variational Symmetries

Symmetries arise from Lie group actions.

EXAMPLE: $G = (\mathbb{R}, +)$

$$\epsilon \cdot x = x^* = \frac{x}{1-\epsilon x}, \qquad \epsilon \cdot u = u^*(x^*) = \frac{u(x)}{1-\epsilon x}$$

Group Action Property

$$\delta \cdot (\epsilon \cdot x) = \delta \cdot \left(\tfrac{x}{1-\epsilon x}\right) = \frac{\frac{x}{1-\epsilon x}}{1-\delta\frac{x}{1-\epsilon x}}$$
$$= \frac{x}{1-(\epsilon+\delta)x} = (\epsilon+\delta)\cdot x$$

and similarly for $u(x)$.

Prolonged Group Action

$$\epsilon \cdot u_x = u^*_{x^*} = \frac{\partial u^*(x^*)}{\partial x} \Big/ \frac{\partial x^*}{\partial x} = \frac{u_x}{(1-\epsilon x)^2}$$

and

$$\delta \cdot (\epsilon \cdot u_x) = \frac{\delta \cdot u_x}{(1-\epsilon(\delta \cdot x))^2} = \frac{u_x}{(1-(\delta+\epsilon)x)^2}$$

Action on Integrals

$$\epsilon \cdot \int_\Omega L(x, u, u_x, \ldots)\, \mathrm{d}x$$

def'n of $\epsilon\cdot$
$$= \int_{\epsilon\cdot\Omega} L(\epsilon \cdot x, \epsilon \cdot u, \epsilon \cdot u_x, \cdots)\, \mathrm{d}\epsilon \cdot x$$

change of variable
$$= \int_\Omega L(\epsilon \cdot x, \epsilon \cdot u, \epsilon \cdot u_x, \cdots) \frac{\mathrm{d}\epsilon \cdot x}{\mathrm{d}x}\, \mathrm{d}x$$

Use L^2 theory to get that a variational symmetry of a Lagrangian is a group action such that

$$L(x, u, u_x, \ldots) = L(\epsilon \cdot x, \epsilon \cdot u, \epsilon \cdot u_x, \cdots) \frac{\mathrm{d}\epsilon \cdot x}{\mathrm{d}x}$$

Infinitesimal Action on Integrals

Since the symmetry invariance condition

$$L(x,u,u_x,\ldots) = L(\epsilon\cdot x, \epsilon\cdot u, \epsilon\cdot u_x, \cdots)\frac{\mathrm{d}\epsilon\cdot x}{\mathrm{d}x}$$

is true all ϵ, then if everything is sufficiently smooth, applying $\frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0}$ to both sides, and noting that when $\epsilon = 0$ we have the identity action,

$$\begin{aligned} 0 &= \tfrac{\partial L}{\partial x}\xi + \tfrac{\partial L}{\partial u}\phi + \tfrac{\partial L}{\partial u_x}\phi^x + \cdots + L\xi_x \\ &= \tfrac{\mathrm{D}(L\xi)}{\mathrm{D}x} + \tfrac{\partial L}{\partial u}Q + \tfrac{\partial L}{\partial u_x}\tfrac{\mathrm{D}Q}{\mathrm{D}x} + + \tfrac{\partial L}{\partial u_{xx}}\tfrac{\mathrm{D}^2 Q}{\mathrm{D}x^2} + \cdots \end{aligned}$$

where

$$Q = \phi - u_x\xi, \quad \phi = \frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0}\epsilon\cdot u, \quad \xi = \frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0}\epsilon\cdot x$$

and $\frac{\mathrm{D}}{\mathrm{D}x}$ is the total derivative operator.

$$0 = \mathrm{Div}(L\xi) + \sum(\mathrm{D}^J Q^\alpha)\frac{\partial L}{\partial u^\alpha_J}$$

Almost to the punchline

Let

$$\mathbf{v}_Q = \sum_\alpha Q^\alpha \frac{\partial}{\partial u^\alpha}$$

Then the *prolongation* is defined by

$$\mathrm{pr}\mathbf{v}_Q = \sum_{\alpha,J} \mathsf{D}^J Q^\alpha \frac{\partial}{\partial u^\alpha_J}$$

Note

$$u^\alpha_J = \frac{\partial u^\alpha}{\partial x_1^{J_1} \cdots \partial x_p^{J_p}} = \mathsf{D}^J u^\alpha$$

Then

$$\sum \left(\mathsf{D}^J Q^\alpha\right) \frac{\partial L}{\partial u^\alpha_J} = \mathrm{pr}\mathbf{v}_Q \lrcorner \hat{\mathrm{d}} L$$

Recall that $\hat{\mathrm{d}}$ is one of the two operators comprising the Euler Lagrange operator, while the left hand side is a divergence if Q is the characteristic of a symmetry.

THE PUNCHLINE

$$\cdots \stackrel{\text{Curl}}{\longrightarrow} \Lambda^2 \stackrel{\text{Div}}{\longrightarrow} \Lambda^3 \stackrel{E}{\longrightarrow} \Lambda^1_* \stackrel{\pi\circ\hat{\mathrm{d}}}{\longrightarrow} \cdots$$

$$\Lambda^3 \ni L \qquad \Lambda^1_* \ni E^\alpha(L)\mathrm{d}u^\alpha \mathrm{d}x = \hat{\mathrm{d}}(L) + \mathrm{Div}(\eta_L)$$

$$\overleftarrow{\mathrm{pr}\mathbf{v}_Q \lrcorner}$$

$$Q \cdot E(L) = \mathbf{v}_Q \lrcorner \hat{\mathrm{d}}(L) + \mathrm{Div}(\mathrm{pr}\mathbf{v}_Q \lrcorner \eta_L)$$

If Q is the characteristic of a symmetry, we have that

$$\mathbf{v}_Q \lrcorner \hat{\mathrm{d}}(L) = \mathrm{Div}(L\xi)$$

and hence that

$$Q \cdot E(L) = \mathrm{Div}(\text{something})$$

Non-trivial example

Semi-geostrophic equations

Group
$$\begin{cases} a^* = \phi(a,b) & \phi_a\psi_b - \phi_b\psi_a = 1 \\ b^* = \psi(a,b) \end{cases}$$

Invariants
$$\begin{cases} h = (x_a y_b - x_b y_a)^{-1} \\ \partial_x = h(y_b\partial_a - y_a\partial_b) \\ \partial_y = h(-x_b\partial_a + x_a\partial_b) \end{cases}$$

Equations
$$\begin{cases} \mathrm{D}_t x = -\frac{g}{f^2}\mathrm{D}_t h_x - \frac{g}{f}h_y \\ \mathrm{D}_t y = -\frac{g}{f^2}\mathrm{D}_t h_y + \frac{g}{f}h_x \end{cases}$$

The Lagrangian has 4 arbitary functions which obey two conditions. The conserved quantity is *potential vorticity*

$$\frac{1}{h}\left(f + \frac{g}{f}(h_{xx} + h_{yy})\frac{g^2}{f^3}(h_{xx}h_{yy} - h_{xy}^2)\right)$$

DISCRETE Almost Punchline

This case is easier than the smooth case.

- Since n cannot vary in a smooth way, the "mesh variables" x_n are treated as dependent variables.

- The group action commutes with shift:

$$\epsilon \cdot S^j(u_n) = \epsilon \dot{u}_{n+j} = S^j \epsilon \cdot u_n$$

so no prolongation formulae are required.

For example,

$$\epsilon \cdot u_n = \frac{u_n}{1 - \epsilon x_n} \Longrightarrow \epsilon \cdot u_{n+j} = \frac{u_{n+j}}{1 - \epsilon x_{n+j}}$$

The symmetry condition is:

$$L_n(x_n, \cdots x_{n+j}, u_n, \cdots u_{n+k}) \\ = L_n(x_n^*, \cdots x_{n+j}^*, u_n^*, \cdots u_{n+k}^*)$$

where $(\,)^* \equiv \epsilon \cdot (\,)$.

Applying

$$\frac{\mathsf{d}}{\mathsf{d}\epsilon}\Big|_{\epsilon=0}$$

to both sides of the symmetry condition yields

$$0 = \sum_k \frac{\partial L_n}{\partial x_{n+k}} \frac{\mathsf{d}}{\mathsf{d}\epsilon}\Big|_{\epsilon=0} x^*_{n+k} + \frac{\partial L_n}{\partial u_{n+k}} \frac{\mathsf{d}}{\mathsf{d}\epsilon}\Big|_{\epsilon=0} u^*_{n+k}$$

Setting

$$Q^x_n = \frac{\mathsf{d}}{\mathsf{d}\epsilon}\Big|_{\epsilon=0} x^*_n, \qquad Q^u_n = \frac{\mathsf{d}}{\mathsf{d}\epsilon}\Big|_{\epsilon=0} u^*_n$$

then since

$$Q^x_{n+k} = S^k(Q^x_n), \qquad Q^u_{n+k} = S^k(Q^u_n)$$

the equation above can be written as

$$0 = X_Q \lrcorner \hat{\mathsf{d}} L_n, \qquad X_Q = \sum_{\alpha, J} S^J(Q^\alpha_n) \frac{\partial}{\partial u^\alpha_{n+J}}$$

DISCRETE Punchline

$$\overset{\Delta}{\to} \mathbf{Ex}^2 \overset{\Delta}{\to} \quad \mathbf{Ex}^3 \quad \overset{\pi\circ\hat{\mathrm{d}}}{\to} \quad \Lambda^1_* \to \Lambda^2_*$$

$$\begin{array}{ccc} \Uplus & & \Uplus \\ L_n & & E(L_n)\mathrm{d}u_n \\ & & = \hat{\mathrm{d}}(L_n) + \Delta(\eta_{L_n}) \\ & \overleftarrow{X}_{Q\lrcorner} & \end{array}$$

$$Q \cdot E(L_n) = X_{Q\lrcorner}\hat{\mathrm{d}}(L_n) + \Delta(X_{Q\lrcorner}\eta_{L_n})$$

Again, we get that if

$$X_{Q\lrcorner}\hat{\mathrm{d}}(L_n) = 0$$

then

$$Q \cdot E(L_n) = \Delta(\text{something}),$$

that is, a total difference expression which is zero on solutions of the discrete Euler Lagrange system.

Nice example T.D. Lee, Difference Equations and Conservation Laws, J. Stat. Phys., **46** (1987)

A difference model for $\int(\frac{1}{2}\dot{x}^2 - V(x))\,\mathrm{d}t$

Define

$$\bar{V}(n) = \frac{1}{x_n - x_{n-1}} \int_{x_{n-1}}^{x_n} V(x)\,\mathrm{d}x$$

and take

$$L_n = \left[\frac{1}{2}\left(\frac{x_n - x_{n-1}}{t_n - t_{n-1}}\right)^2 - \bar{V}(n)\right](t_n - t_{n-1})$$

The group action is $t_n^* = t_n + \epsilon$, with x_n invariant. The conserved quantity is thus "energy". Now, $Q_n^t = 1$ for all n, and $Q_n^x = 0$. The equations become

$$\begin{aligned} 0 &= E^t(L_n) &= \frac{\partial}{\partial t_n}L_n + S\left(\frac{\partial}{\partial t_{n-1}}L_n\right) \\ 0 &= X_Q \lrcorner \mathrm{d}(L_n) &= \frac{\partial}{\partial t_n}L_n + \frac{\partial}{\partial t_{n-1}}L_n \end{aligned}$$

as L_n is a function of $(t_n - t_{n-1})$.

It is easy to see in this case that

$$0 = (S - \mathrm{id})\left(\frac{\partial}{\partial t_n} L_n\right)$$

is implied by the two equations, to yield

$$\frac{1}{2}\left(\frac{x_n - x_{n-1}}{t_n - t_{n-1}}\right)^2 + \bar{V}(n) = c$$

Note that the energy in the smooth case is

$$1/2\dot{x}^2 + V.$$

Can regard the EL eqn for the mesh variables as an equation for a variable mesh.

INTERLUDE

If we know the group action for a particular conservation law, we can "design in" that conservation law into a discretisation by taking a Lagrangian composed of invariants. These necesarily satisfy $v_Q(I) = 0$ or $X_Q(I_n) = 0$. The Fels and Olver formulation of moving frames is particularly helpful here: a sample theorem is

Discrete rotation invariants in $\mathbb{Z}^2$

Let (x_n, y_n), (x_m, y_m) be two points in the plane. Then

$$I_{n,m} = x_n y_n + x_m y_m, \qquad J_{n,m} = x_n y_m - x_m y_n$$

are rotation invariants. Moreover, any disrete rotation invariant is a function of these.

Made up example

Suppose

$$L_n = \frac{1}{2} J_{n,n+1}^2 = \frac{1}{2}(x_n y_{n+1} - x_{n+1} y_n)^2$$

then

$$\begin{cases} E_n^x &= J_{n,n+1} y_{n+1} - J_{n-1,n} y_{n-1} \\ E_n^y &= -J_{n,n+1} x_{n+1} + J_{n-1,n} x_{n-1} \end{cases}$$

Now,

$Q_n = (Q_n^x, Q_x^y) = (-y_n, x_n) = \frac{\mathrm{d}}{\mathrm{d}\theta}\Big|_{\theta=0} (x_n^*, y_n^*)$

and thus

$$\begin{aligned} Q_n \cdot E_n &= J_{n,n+1}(-y_n y_{n+1} - x_n x_{n+1}) \\ &\quad + J_{n-1,n}(y_n y_{n-1} + x_n x_{n-1}) \\ &= -J_{n,n+1} I_{n,n+1} + J_{n-1,n} I_{n-1,n} \\ &= -(S - \mathrm{id})(J_{n-1,n} I_{n-1,n}) \end{aligned}$$

gives the conserved quantity.

Note that $I_{n,m} = I_{m,n}$ and $J_{n,m} = -J_{m,n}$

Less easy example

Hereman et al., Densities, Symmetries and Recursion operators for nonlinear DDEs, CRM Proceedings

The Toda lattice in polynomial form is

$$\begin{cases} \dot{u_n} &= v_{n-1} - v_n \\ \dot{v_n} &= v_n(u_n - u_{n+1}) \end{cases}$$

The scaling symmetry is the basis for the ansatz used to obtain the differential-difference conservation laws, which are of the form

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho_n + (S - \mathrm{id})J_n = 0$$

for example

$$\rho_n = \frac{1}{3}u_n^3 + u_n(v_{n-1} - v_n),\, J_n = u_{n-1}u_n v_{n-1} + v_{n-1}^2$$

These results use the ansatz plus homotopy operator method outlined earlier.

Summary of the Pattern

$$\left\{\begin{matrix}\mathrm{Div}\\ \Delta\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\Lambda^3\\ \mathbf{Ex}^3\end{matrix}\right\} \xrightarrow{\pi\circ\hat{\mathrm{d}}} \Lambda^1_* \longrightarrow \Lambda^2_* \longrightarrow$$

$$\begin{matrix} \Lambda^3 \ni L & \longmapsto & \Lambda^1_* \ni \sum_\alpha E^\alpha(L)\,\mathrm{d}u^\alpha = \hat{\mathrm{d}}(L) + \left\{\begin{matrix}\mathrm{Div}\\ \Delta\end{matrix}\right\}\eta_L \end{matrix}$$

$$\overleftarrow{\left\{\begin{matrix}v_Q\\ X_Q\end{matrix}\right\}}\lrcorner$$

$$Q\cdot E(L) = \left\{\begin{matrix}v_Q\\ X_Q\end{matrix}\right\}\lrcorner\,\hat{\mathrm{d}}L + \left\{\begin{matrix}\mathrm{Div}\\ \Delta\end{matrix}\right\}\left\{\begin{matrix}v_Q\\ X_Q\end{matrix}\right\}\lrcorner\,\eta_L$$

- the formula for η_L is explicit, exact, symbolic
- the first summand is a total derivative or difference by the symmetry condition

OK let's try for a Neother's Theorem for Finite Element!

D. Arnold, Beijing ICM Plenary talk

Given a system of moments and sundry other data, aka degrees of freedom, that yield projection operators such that the diagram commutes:

$$\begin{array}{ccccccccccccc} 0 & \to & \mathbb{R} & \to & \Lambda^0 & \to & \Lambda^1 & \to & \Lambda^2 & \to & \Lambda^3 & \to & 0 \\ & & & & \Pi_0 \downarrow & & \Pi_1 \downarrow & & \Pi_2 \downarrow & & \Pi_3 \downarrow & & \\ 0 & \to & \mathbb{R} & \to & \mathcal{F}^0 & \to & \mathcal{F}^1 & \to & \mathcal{F}^2 & \to & \mathcal{F}^3 & \to & 0 \end{array}$$

all relative to some triangulation.

Yields stability!! A Lagrangian is composed of wedge products of 1-, 2- and 3- forms. Choose the discretisation of each to be in the relevant $\mathcal{F}_i$. Then commutativity implies conditions for Brezzi's theorem to hold.

In one dimension: with $e_n = (x_n, x_{n+1})$, Π_0 to piecewise linear, Π_1 to piecewise constant with moment

$$\alpha_n = \int_{x_n}^{x_{n+1}} u(x)\psi_n(x)\,\mathrm{d}x$$

Commutativity of the diagram

$$\begin{array}{ccc} u & \overset{\mathrm{d}}{\mapsto} & u_x \mathrm{d}x \\ \Pi_0 \downarrow & & \downarrow \Pi_1 \\ u|_{e_n} = A_n x + B_n & \mapsto & A_n = \int_{x_n}^{x_{n+1}} u'(x)\psi_n(x)\,\mathrm{d}x \end{array}$$

implies

$$A_n = u(x)\psi_n(x)]_{x_n}^{x_{n+1}} - \int_{x_n}^{x_{n+1}} u(x)\psi_n'(x)\,\mathrm{d}x$$

Note that

$$\int_{x_n}^{x_{n+1}} \psi_n(x)\,\mathrm{d}x = 1.$$

is required by the projection property.

A finite element Lagrangian is built up of wedge products of forms in $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$. Call this resulting space $\widetilde{\mathcal{F}}_3$. In each top-dimensional simplex, denoted τ, integrate to get

$$L = \sum_{\tau} L_{\tau}(\alpha_{\tau}^{1}, \cdots \alpha_{\tau}^{p})$$

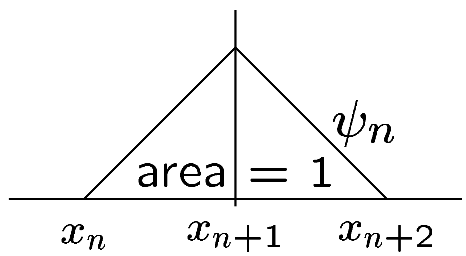
where α_{τ}^{j} is the j^{th} degree of freedom in τ. L can also depend on mesh data.

Can now take $\hat{\mathrm{d}}$ which is the variation with respect to the α_{τ}^{j}.

EXAMPLE In one dimension,

$$\begin{array}{ccccccccc} 0 \to \mathbb{R} & \to \Lambda^0 & \overset{\mathsf{d}}{\to} & \Lambda^1 \to 0 \\ & \Pi_0 \downarrow & & \Pi_1 \downarrow \\ 0 \to \mathbb{R} & \to \mathcal{F}_0 & \overset{\mathsf{d}}{\to} & \mathcal{F}_1 \to 0 \end{array}$$

Π_1 is to piecewise constant functions with moment $\bar{u}(n) = \int_{x_n}^{x_{n+2}} u(x)\psi_n(x)\,\mathrm{d}x$ where



on (x_n, x_{n+2}), while Π_0 is to piecewise linear functions with moments

$$\alpha_n = \frac{1}{x_n - x_{n+2}} \int_{x_n}^{x_{n+1}} u(x)\,\mathrm{d}x$$

that is, α_n, α_{n+1} are used in (x_n, x_{n+2});

$$u \;\mapsto\; 2\frac{\alpha_{n+1}-\alpha_n}{x_{n+2}-x_n}x + \left(\frac{x_{n+1}+x_{n+2}}{x_{n+2}-x_n}\right)\alpha_n - \left(\frac{x_{n+1}+x_n}{x_{n+2}-x_n}\right)\alpha_{n+1}$$

Very simple example

$\int \frac{1}{2} u_x^2 \, \mathrm{d}x$ projects to

$$\sum_n \int_{x_{2n}}^{x_{2n+2}} \frac{1}{2} \Pi(u)_x^2 \, \mathrm{d}x = \sum_n 2 \left(\frac{(\alpha_{2n} - \alpha_{2n+1})^2}{x_{2n+2} - x_{2n}}. \right)$$

Then

$$\begin{aligned} \hat{\mathrm{d}} L_{2n} &= 4 \tfrac{\alpha_{2n} - \alpha_{2n+1}}{x_{2n+2} - x_{2n}} (\mathrm{d}\alpha_{2n} - \mathrm{d}\alpha_{2n+1}) \\ &= 4 \left(\tfrac{\alpha_{2n} - \alpha_{2n+1}}{x_{2n+2} - x_{2n}} - \tfrac{\alpha_{2n-1} - \alpha_{2n}}{x_{2n+1} - x_{2n}} \right) \mathrm{d}\alpha_{2n} \\ & \quad + (S - \mathrm{id})(\text{something}) \end{aligned}$$

The discrete Euler Lagrange equation is then, after "integration",

$$\frac{\alpha_{2n} - \alpha_{2n+1}}{x_{2n+2} - x_{2n}} = c$$

Look now at the "Noether pattern" for the Finite Element variational complex

$$
\begin{array}{ccccc}
\to \widetilde{\mathcal{F}}_2 \to \widetilde{\mathcal{F}}_3 & \overset{\pi\circ\widehat{\mathsf{d}}\circ\int}{\to} & \widetilde{\mathcal{F}}^1_* \to \widetilde{\mathcal{F}}^2_* \to \\
\Cup & & \Cup \\
L_\tau & & E(L_\tau) + \delta(\eta_L) \\
 & & = \widehat{\mathsf{d}} L_\tau \\
 & \overleftarrow{v_{Q_\tau} \lrcorner} &
\end{array}
$$

where δ is the mesh dependent coboundary operator (recall $\delta(f)(\tau) = f(\partial\tau)$).

Step 1: find η_L Step 2: find v_Q

If then $v_{Q_\tau}\lrcorner\widehat{\mathsf{d}}(L_\tau) = \delta(\text{something})$ we will have that

$$0 = Q_\tau \cdot E(L_\tau) + \delta(\text{something}).$$

Group actions on moments

The clue is the variational symmetry group action on $\int_\Omega L(x,u,\cdots)\,\mathrm{d}x$

Define

$$\epsilon\cdot\int_\tau u(x)\psi_\tau(x)\,\mathrm{d}x$$

$$=\int_\tau \epsilon\cdot u(x)\psi_\tau(\epsilon\cdot x)\tfrac{\mathrm{d}\epsilon\cdot x}{\mathrm{d}x}\,\mathrm{d}x$$

Example Recall the projective action

$$\epsilon\cdot x=\frac{x}{1-\epsilon x},\qquad \epsilon\cdot u(x)=\frac{u(x)}{1-\epsilon x}$$

Then the induced action on the moments

$$\alpha_n=\int_{x_n}^{x_{n+1}}\frac{u(x)}{x^3}\,\mathrm{d}x,\qquad \beta_n=\int_{x_n}^{x_{n+1}}\frac{u(x)}{x^4}\,\mathrm{d}x$$

is

$$\epsilon\cdot\alpha_n=\alpha_n,\qquad \epsilon\cdot\beta_n=\beta_n-\epsilon\alpha_n$$

In general for this action,

$$\epsilon \cdot \int_{x_n}^{x_{n+1}} x^m u(x)\,\mathrm{d}x$$

$$= \int_{x_n}^{x_{n+1}} \frac{x^m}{(1-\epsilon x)^m} \frac{u(x)}{1-\epsilon x} \frac{\mathrm{d}x}{(1-\epsilon x)^2}$$

$$= \int_{x_n}^{x_{n+1}} \frac{x^m u(x)}{(1-\epsilon x)^{m+3}}\,\mathrm{d}x$$

THINK: if you want a coherent scheme which maps to itself under this projective action, and involves only a finite amount of data, then take your moments to be

$$u(x) \mapsto \int_{x_n}^{x_{n+1}} \frac{u(x)}{x^m}\,\mathrm{d}x, \qquad m = 3, 4, \ldots N.$$

CONCLUSIONS

- The underlying algebraic pattern of the exact variational complexes provide a framework for generalisations of Noether's Theorem and conservation laws in general.

- Symmetry-adapted moments would appear to be necessary.

- Next: formulae for η_{L_τ} where

$$\widehat{\mathsf{d}}(L_\tau) = E(L_\tau) + \delta(\eta_{L_\tau})$$

  in terms of the mesh dependent coboundary operator.