EXAMPLES : FOURIER SERIES

1. Find the Fourier series of each of the following functions

   (i) $f(x) = 1 - x^2, \quad -1 < x < 1.$

   (ii) $g(x) = |x|, \quad -\pi < x < \pi.$

   (iii) $h(x) = \begin{cases} 0 \text{ if } -2 < x < 0 \\ 1 \text{ if } 0 \leq x < 2. \end{cases}$

   In each case sketch the graph of the function to which the Fourier series converges over an x- range of three periods of the Fourier series.

2. Find the Fourier series for $f(x) = \frac{x^2}{4}$, $-\pi < x < \pi$. Hence deduce that

   (i) $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots$

   (ii) $\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots$

   (iii) $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \ldots$

3. Find the Fourier cosine series and the Fourier sine series for the function

$$f(x) = \begin{cases} 1 \text{ if } 0 < x < 1 \\ 0 \text{ if } 1 \leq x < 2. \end{cases}$$

4. Find the Fourier cosine series for the function $f(x) = \sin(x)$, $0 < x < \pi$. What is the Fourier sine series for f?

EXAMPLES : THE HEAT EQUATION

1. The ends of a metal bar of unit length are maintained at a temperature of $0°$. The temperature u satisfies the equation

$$u_t = u_{xx} \qquad 0 < x < 1, \;\; t > 0.$$

Find an expression for the temperature $u(x,t)$ when the initial temperature distribution in the bar is given by
$(i)\, f(x) = \sin(2\pi x)$ for $0 < x < 1$;
$(ii)\, f(x) = x(1-x)$ for $0 \leq x \leq 1$.

2. A metal bar 2 metres long and with thermal diffusivity $= 10^{-4}$ m^2 /sec is heated to a uniform temperature of $100°C$. At time $t = 0$ the ends of the bar are plunged into an ice bath at $0°C$ and are kept at that temperature but no heat is allowed to escape through the lateral surface. Find an expressior for the temperature at any point on the bar at any later time.

3. Consider a uniform bar of length l with an initial temperature given by $\sin(\frac{\pi x}{l}), 0 \leq x \leq l$. Suppose that both ends of the bar are insulated and the the thermal diffusivity of the bar is 1 . Find a series expansion for the temperature $u(x,t)$. What is the steady state temperature as $t \longrightarrow \infty$?

EXAMPLES : LAPLACE'S EQUATION

1. Solve $\nabla^2 u(x,y) = 0$ for $0 < x < 1, 0 < y < 1$, subject to the boundary conditions
(i) $u(x,0) = u(x,1) = u(0,y) = 0$; $u(1,y) = y(1-y)$;
(ii) $u(x,0) = u(0,y) = 0$; $u(x,1) = u(1,y) = 1$.

2. Find the solution $u(x,y)$ of Laplace's equation in the semi-infinite strip $0 < x < a$, $y > 0$, satisfying the boundary conditions

$$u(0,y) = 0, \quad u(a,y) = 0, y > 0; \qquad u(x,0) = f(x), 0 \leq x \leq a$$

and the additional condition that $u(x,y) \longrightarrow 0$ as $y \longrightarrow \infty$.

3. Find the solution $u(r,\theta)$ of Laplace's equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

(i) inside the circle $r = a$ which satisfies the boundary condition $u(a,\theta) = \sin^2(\theta)$, $0 \leq \theta \leq 2\pi$.

(ii) outside the circle $r = a$ which is bounded and which satisfies the boundary condition $u(a,\theta) = \sin^2(\theta)$, $0 \leq \theta \leq 2\pi$.

(iii) in the semi-circle $x^2 + y^2 \leq 1$ satisfying the boundary conditions $u(x,y) = 0$ on the x-axis and $u(r,\theta) = \sin(\theta)\cos(\theta)$ for $0 \leq \theta \leq \pi$.

(iv) in the annulus $1 \leq r \leq 2$ satisfying the boundary conditions $u(r,\theta) = 0$ when $r = 1$ and $u(r,\theta) = \sin(\theta)\cos(\theta)$ when $r = 2$.

(v) in the circular wedge $0 < r < a, 0 < \theta < \alpha$ satisfying the boundary conditions $u(r,0) = u(r,\alpha) = 0, 0 \leq r \leq a$ and $u(a,\theta) = 1, 0 \leq \theta \leq \alpha$.

EXAMPLES : THE WAVE EQUATION

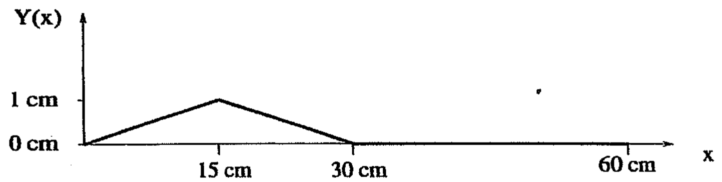
1. Find the solution of the wave equation $u_{xx} = \frac{1}{c^2} u_{tt}$ for $0 < x < 1$ and $t > 0$ satisfying the boundary conditions $u(0,t) = u(1,t) = 0$ and the initial conditions

(i) $u(x,0) = \begin{cases} x & \text{for } 0 \le x \le 1/2 \\ 1-x & \text{for } 1/2 \le x \le 1 \end{cases}$ and $u_t(x,0) = 0$ for $0 < x < 1$;

(ii) $u(x,0) = 0$ and $u_t(x,0) = x(1-x)$ for $0 < x < 1$;

(iii) $u(x,0) = \sin(\pi x)$ and $u_t(x,0) = x$ for $0 < x < 1$.

2. When Mstislav Rostropovich plays *pizzicato* on his 60cm long G string, he manages to give it an initial displacement $y(x,t=0) = Y(x)$ as shown



and an initial velocity of zero. What is $y(x,t>0)$ in this case (given that y obeys the wave equation $y_{xx} = \frac{1}{c^2} y_{tt}$)?

3. A vibrating string moving in an elastic medium satis£es the equation $a^2 u_{xx} - \alpha^2 u = u_{tt}$ where α is proportional to the coef£cient of elasticity of the medium. Suppose that the string is £xed at its ends and has initial zero displacement and initial velocity given by $u_t(x,0) = g(x)$. Determine $u(x,t)$.