EXAMINATION QUESTIONS/SOLUTIONS SESSION 1999/2000

COURSE M2A2

SETTER MALHAM

QUESTION

SOLUTION 1

(a) $\delta J(y(x),\eta(x)) = \frac{d}{d\varepsilon} J(\underbrace{y(x)+\varepsilon\eta(x)}_{y(x;\varepsilon)})\Big|_{\varepsilon=0} = 0, \quad \forall \eta$

$\Rightarrow y(x)$ is the extremising curve.

$$\delta J = \int_a^b \frac{\partial}{\partial\varepsilon} F(x, y(x;\varepsilon), y_x(x;\varepsilon))\,dx \Big|_{\varepsilon=0}$$

$$= \int_a^b \frac{\partial F}{\partial y(x;\varepsilon)}\cdot\eta(x) + \frac{\partial F}{\partial y_x(x;\varepsilon)}\cdot\eta_x(x)\,dx\Big|_{\varepsilon=0}$$

$$= \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_x}\right)\right)\eta(x)\,dx$$

using integration by parts & that $\eta(a)=\eta(b)=0$.

$\delta J = 0$ for all twice continuously differentiable $\eta(x)$

$$\Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_x}\right) = 0. \qquad (+)$$

$$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'} - F\right) = y''\frac{\partial F}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{dF}{dx}$$

$$= y'' F_{y'} + y'\frac{d}{dx}F_{y'} - F_x - F_y y' - F_{y'} y''$$

$$= -y'\left(F_y - \frac{d}{dx}F_{y'}\right) - F_x$$

Hence $\frac{\partial F}{\partial x} + \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'} - F\right) = -y'\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)$

$= 0$ using $(+)$.

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24/8/2000

EXAMINATION QUESTIONS/SOLUTIONS	SESSION 1999/2000	COURSE M2A2
		SETTER MALHAM
		QUESTION 1
		SOLUTION 1

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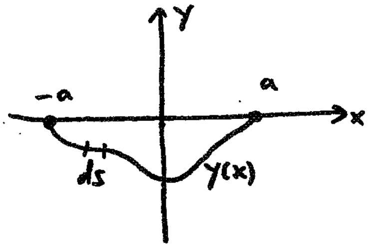
(b) mass of chain element $= \rho g \, ds$

potential energy $= \rho g \, ds \cdot y$



Total PE $= J = \int_{-a}^{a} \rho g \, y \, ds$

$$= \int_{-a}^{a} \rho g \, y \sqrt{1+y'^2} \, dx.$$

Constraint: $\int_{-a}^{a} ds = \int_{-a}^{a} \sqrt{1+y'^2} \, dx = \ell.$

Method of Lagrange multipliers $\Rightarrow$ minimise $\bar{J} = \int_{-a}^{a} F(y, y'; \lambda) \, dx$

where $F = (y+\lambda)(1+y'^2)^{1/2}$. Last part of (a) $\Rightarrow$

$$y' F_{y'} - F = \text{const} \quad \text{(no explicit } x \text{ in } F\text{)}.$$

$$\Rightarrow \left[y'^2 (1+y'^2)^{-1/2} - (1+y'^2)^{1/2} \right](y+\lambda) = \text{const} = A^{-1}$$

$$\Rightarrow \quad y'^2 = A^2 (y+\lambda)^2 - 1$$

Subs$^{\text{stn}}$ $a(y+\lambda) = \cosh\theta$ and integration $\Rightarrow$

$$y = A^{-1} \cosh[A(x+B)] - \lambda$$

Symmetry about origin $\Rightarrow B \equiv 0$

$x = a$, $y = 0 \Rightarrow A\lambda = \cosh[Aa]$ $\quad (\dagger)$

Chain length $\ell = \int_{-a}^{a} (1 + \sinh^2(Ax))^{1/2} \, dx$

$$\Rightarrow A\ell = 2\sinh[Aa] \quad (\ddagger)$$

$(\dagger)$ & $(\ddagger)$ determine A & λ.

$\ell > 2a$ for solution, otherwise chain not long enough!

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EXAMINATION QUESTIONS/SOLUTIONS SESSION 1998/99

COURSE M2A2

SETTER MALHAM

QUESTION

SOLUTION 2

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Let $x(t), y(t)$ be coordinates of mass m.

$$x(t) = \mu(t) + l\sin\theta(t)$$
$$y(t) = -l\cos\theta(t)$$

$$L = T - V = \tfrac{1}{2}M\dot{\mu}^2 + \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy$$

$$\Rightarrow L = \tfrac{1}{2}M\dot{\mu}^2 + \tfrac{1}{2}m(\dot{\mu}^2 + l^2\dot{\theta}^2 + 2l\dot{\mu}\dot{\theta}\cos\theta) + mgl\cos\theta. \quad 4$$

$$p_\mu = \frac{\partial L}{\partial\dot{\mu}} = (M+m)\dot{\mu} + ml\dot{\theta}\cos\theta \quad 2$$

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} + 2l\dot{\mu}\cos\theta \quad 2$$

L independent of explicit t and $\mu \Rightarrow$ H and p_μ are constants of the motion. 2

$p_\mu \equiv 0 \Rightarrow$ after integration $(M+m)\mu = -ml\sin\theta + A$ 2

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \Rightarrow$$

$$ml^2\ddot{\theta} + \tfrac{1}{2}m\frac{d}{dt}(2l\dot{\mu}\cos\theta) + ml\dot{\mu}\dot{\theta}\sin\theta + mgl\sin\theta = 0 \quad 2$$

where $\dot{\mu} = \dfrac{-ml\dot{\theta}}{M+m}\cos\theta.$

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DATE 5th March 1999

EXAMINATION QUESTIONS/SOLUTIONS SESSION 1998/99

COURSE M2A2

SETTER MALHAM

QUESTION 1

SOLUTION 2

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$$\mu = \frac{-ml}{M+m}\sin\theta + \frac{A}{M+m}, \quad \text{Let } \tilde{A} = \frac{A}{M+m}$$

$$\Rightarrow \quad x(t) - \tilde{A} = \left(l - \frac{ml}{M+m}\right)\sin\theta \quad \Rightarrow \text{result.}$$ 2+2

If $\tilde{m} = \dfrac{Ml}{M+m}$ then $\dfrac{(x-\tilde{A})^2}{\tilde{m}^2} + \dfrac{y^2}{l^2} = 1,$ 2

Ellipse. Arc of an Ellipse!

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DATE .... 8th March 1999 ....

Couette flow problem (Chorin & Marsden, pg. 31)

In cylindrical coordinates (r, θ, z), $\underline{u} = (u_r, u_\theta, u_z)$ the Euler eqns are

$$\frac{\partial u_r}{\partial t} + (\underline{u}\cdot\nabla)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho_0}\frac{\partial p}{\partial r} + f_r$$

$$\frac{\partial u_\theta}{\partial t} + (\underline{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho_0 \cdot r}\cdot\frac{\partial p}{\partial \theta} + f_\theta$$

$$\frac{\partial u_z}{\partial t} + (\underline{u}\cdot\nabla)u_z = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + f_z$$

where $p = p(r, \theta, z, t)$ is the pressure, ρ_0 is the uniform density & $\underline{f} = (f_r, f_\theta, f_z)$ is the body force per unit mass.

Here, $$\underline{u}\cdot\nabla \equiv u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$$

Further, the incompressibility condition is

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0.$$

We will also need the identities:

$$\nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right)$$

$$\nabla\times\underline{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial\theta} - \frac{\partial u_\theta}{\partial z}, \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}, \frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)$$

(a) Required to show that flow field given is a stationary solution of Euler's eqn's with $\rho_0 \equiv 1$.

Using $u_r = 0$, $u_z = 0$, $u_\theta = \frac{A}{r} + Br$ (all three components also independent of θ & z) $\Rightarrow$

$$-\frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r} \qquad \text{(no body force)}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial\theta}$$

$$0 = -\frac{\partial p}{\partial z}$$

and note that the incompressibility condition is identically satisfied. Hence p indept of θ & z

$$\Rightarrow \frac{\partial p}{\partial r} = \frac{1}{r}\left(\frac{A}{r} + Br\right)^2 .$$

$$\Rightarrow \frac{\partial p}{\partial r} = \frac{A^2}{r^3} + \frac{2AB}{r} + B^2 r$$

$$\Leftrightarrow \quad p = -\frac{1}{2}\frac{A^2}{r^2} + 2AB\log(r) + \frac{1}{2}B^2r^2 + K$$

where K is an arbitrary constant. Hence we have shown there exists a pressure field so that (u_r, u_θ, u_z) and p satisfy Euler's equations and the incompressibility condition.

(b) $$\underline{\omega} = \nabla \times \underline{u} = \left(0,\ 0,\ \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right)$$

$$= (0, 0, 2B)$$

(c) $$D := \frac{1}{2}\left((\nabla \underline{u}) + (\nabla \underline{u})^T\right)$$

$$= \frac{1}{2}\begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{r}\frac{\partial u_r}{\partial \theta} & \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\theta}{\partial r} & \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} & \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} & \frac{1}{r}\frac{\partial u_z}{\partial r} & \frac{\partial u_z}{\partial z} \end{pmatrix} + \frac{1}{2}(\nabla \underline{u})^T$$

$$\Rightarrow \quad D = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ -\frac{A}{r^2}+B & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -\frac{A}{r^2}+B & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad D = \frac{1}{2}\left(-\frac{A}{r^2}+B\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{(ignoring the z-coord)}$$

Deformation ~~tensors~~ tensor describes the expansion or contraction along characteristic directions of a fluid parcel.

(d) u_θ = angular velocity $= u_\theta(r)$.

Two cylinders are $r = R_1$ and $r = R_2$.

$$u_\theta(R_1) = \frac{A}{R_1} + BR_1 = -\frac{R_1R_2^2(\omega_2-\omega_1)}{R_2^2-R_1^2} - R_1\cdot\frac{(R_1^2\omega_1 - R_2^2\omega_2)}{R_2^2-R_1^2}$$

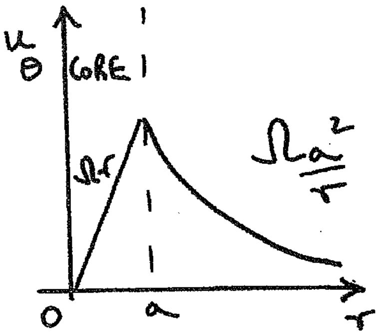
$$= +\frac{R_1R_2^2\omega_1 - R_1^3\omega_1}{R_2^2-R_1^2} = R_1\omega_1$$

$$u_\theta(R_2) = \frac{A}{R_2} + BR_2 = -\frac{R_1^2R_2(\omega_2-\omega_1)}{R_2^2-R_1^2} - \frac{R_2(R_1^2\omega_1 - R_2^2\omega_2)}{R_2^2-R_1^2}$$

$$= \frac{-R_1^2R_2\omega_2 + R_2^3\omega_2}{R_2^2-R_1^2} = R_2\omega_2$$

4.

(*)



(i) acceleration

EULER.

$$\underline{acc} = -\frac{u_\theta^2}{r}\hat{\underline{r}} = -\frac{1}{\rho}\frac{\partial p}{\partial r}\hat{\underline{r}} - \frac{1}{\rho}\frac{\partial p}{\partial z}\underline{k} - g\underline{k}$$

per unit mass

2

so $\dfrac{\partial p}{\partial z} = -g\rho$

$$\frac{\partial p}{\partial r} = \begin{cases} \rho\Omega^2 r & (r \le a) \\ \rho\Omega^2 \dfrac{a^4}{r^3} & (r > a) \end{cases}$$

$$\Rightarrow p = \begin{cases} p_0 - g\rho z + \frac{1}{2}\rho\Omega^2 r^2 & (r \le a) \\ p_0 - g\rho z - \frac{1}{2}\rho\Omega^2 \dfrac{a^4}{r^2} + \rho\Omega^2 a^2 & (r > a) \end{cases}$$

2

2

p continuous at $r=a$

$r=0$, $z=0$ on axis at free surface

Hence $p = p_0 \Rightarrow$
$$z = \begin{cases} \dfrac{\Omega^2}{2g} r^2 & (r \le a) \\ \dfrac{\Omega^2 a^2}{g} - \dfrac{\Omega^2}{2g}\dfrac{a^4}{r^2} & (r > a) \end{cases}$$

2

i.e. $\alpha = \dfrac{\Omega^2}{2g}$ and $F(r) = \dfrac{\Omega^2 a^2}{g} - \dfrac{\Omega^2 a^4}{2g r^2}$.

(ii) $H = p/\rho + \frac{1}{2}\underline{u}^2 + gz$

$$= \begin{cases} \dfrac{p_0}{\rho} + \Omega^2 r^2 & (r \le a) \\ \dfrac{p_0}{\rho} + \Omega^2 a^2 & (r > a) \end{cases}$$

2

Bernoulli theorem: H constant along streamlines and along vortex lines since flow is steady and ρ is uniform. Furthermore H is constant throughout a region where $\underline{\omega} = \nabla \times \underline{u} = \underline{0}$. (irrotational). 3

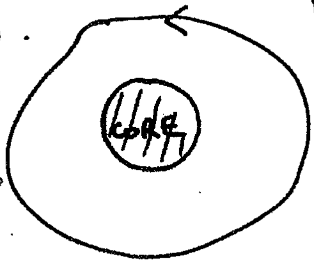
For this example: streamlines are horizontal circles $r = \text{const}$, $z = \text{const}$

for $r \le a$ vortex lines are vertical straight lines $r = \text{const}$, $\theta = \text{const}$

for $r > a$ $\nabla \times \underline{u} \equiv \underline{0}$. 3

TA So we expect H to be a function of r (only) when $r \le a$ and constant for $r > a$.

(iii).



[illegible]

$$K = \oint_{\mathcal{C}} \underline{u} \cdot d\underline{r} = 2\pi r_0 \frac{\Omega a^2}{r_0} = 2\pi(\Omega a^2).$$

$$\left[\equiv \int_S \underline{\omega} \cdot d\underline{S} = \int_{\substack{\text{CIRCULAR} \\ \text{CORE}}} (2\Omega \underline{k}) \cdot d\underline{S} = (2\Omega)(\pi a^2) \right]$$

So K constant 2

$$\Rightarrow \Omega a^2 \text{ constant}.$$

From part (i) $D = \dfrac{\Omega^2 a^2}{g}$ i.e. $z(r=\infty) - z(r=0)$

(Using this in part (i) to get D)

So that $D = \left(\dfrac{K^2}{4\pi^2 g}\right) \dfrac{1}{a^2}$ i.e. $D \propto \dfrac{1}{a^2}$ 2

5) $u_{tt} = u_{xx} \quad 0 < x < 1, \ t > 0$

We seek solutions of the form $u(x,t) = X(x)\,T(t)$

Since ends are fixed at same horizontal level we require $X(0) = 0 = X(1)$

Substituting into the equation we require

$$T''(t)\,X(x) = T(t)\,X''(x)$$

Thus it suffices to find X and T such that

$$\frac{X''}{X} = \frac{T''}{T} = -k \quad \text{where } k \text{ is a constant}$$

i.e

$$-X'' = kX \qquad X(0) = 0 = X(1) \qquad (1)$$

5. $$-T'' = kT \qquad (2)$$

(1) has non zero solutions iff $k = n^2\pi^2$ with corresponding solutions

1 $\sin n\pi x$ for $n = 1, 2, \ldots$

1 If $k = n^2\pi^2$, (2) has general solution $T(t) = A_n \cos n\pi t + B_n \sin n\pi t$

Hence any function of the form

1 $$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos n\pi t + B_n \sin n\pi t\right) \sin n\pi x$$

(4) [illegible] boundary conditions.

If $u(x,0) = \sin 2\pi x$ we require $\sin 2\pi x = \sum_{n=1}^{\infty} A_n \sin n\pi x$

2 Thus [illegible] $A_2 = 1$ and $A_n = 0$ for $n \neq 2$.

Now

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(-n\pi A_n \sin n\pi t + n\pi B_n \cos n\pi t\right) \sin n\pi x$$

Since $u_t(x,0) = x$ we require

$$x = \sum_{n=1}^{\infty} n\pi B_n \sin n\pi x$$

i.e

$$n\pi B_n = \frac{2}{1}\int_0^1 x \sin n\pi x \, dx$$

$$= 2\left[-\frac{1}{n\pi} x \cos n\pi x \Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi x \, dx \right]$$

$$= -\frac{2}{n\pi}\cos n\pi = (-1)^{n+1}\frac{2}{n\pi}$$

4 i.e $B_n = (-1)^{n+1}\dfrac{2}{n^2\pi^2}$.

(b) $$E(t) = \frac{1}{2}\int_0^1 \left[u_x^2(x,t) + u_t^2(x,t)\right] dx$$

Bookwork Hence $\dfrac{dE}{dt} = \displaystyle\int_0^1 \left[u_x(x,t)\,u_{xt}(x,t) + u_t(x,t)\,u_{tt}(x,t)\right] dx$

5 ctd $= u_x(x,t)\,u_t(x,t)\Big|_{x=0}^{x=1} - \int_0^1 u_{xx}(x,t)\,u_t(x,t)\,dx + \int_0^1 u_t(x,t)\,u_{tt}(x,t)\,dx$

Since $u(0,t) \equiv 0$, $u_t(0,t) = 0$; similarly $u_t(1,t) \equiv 0$.

Hence $\frac{dE}{dt} = \int_0^1 [u_{tt}(x,t) - u_{xx}(x,t)]\,u_t(x,t)\,dx \equiv 0$ as $u_{tt} \equiv u_{xx}$

Thus $E(t) \equiv E(0) = 0$ as $u(x,0) \equiv u_t(x,0) \equiv 0$

Hence $\int_0^1 [u_x^2(x,t) + u_t^2(x,t)]\,dx \equiv 0$

Thus $u_x(x,t) \equiv u_t(x,t) \equiv 0$ and so $u(x,t) \equiv$ constant

6 Hence as $u(x,0) \equiv 0$, $u(x,t) \equiv 0$

a) $u_t = u_{xx} \quad 0 < x < \pi \quad t > 0 \; ; \; u(0,t) = u(\pi,t) = 0$

We seek solutions of the form $u(x,t) = X(x)\,T(t)$

To ensure that $u(0,t) = u(\pi,t) = 0$ we require $X(0) = 0 = X(\pi)$.

$u_t = u_{xx} \Rightarrow X(x)\,T'(t) = X''(x)\,T(t)$

$\Rightarrow \dfrac{X''(x)}{X(x)} = \dfrac{T'(t)}{T(t)} = -k$ where k is a constant.

ie $-X'' = kX \quad X(0) = 0 = X(\pi) \quad (1)$

5 $T' = -kT \quad (2)$

(1) has non-zero solutions iff $k = n^2$ for $n = 1, 2, \ldots$ with solutions $\sin nx$.

1 If $k = n^2$, (2) has general solution $T(t) = A_n e^{-n^2 t}$

Hence any function of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin nx \, e^{-n^2 t}$$

2 is a solution of the PDE satisfying the boundary conditions

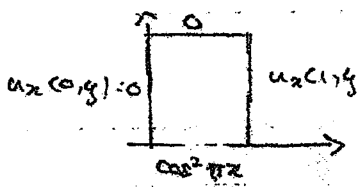
Since $u(x,0) = \sin x \cos x = \frac{1}{2}\sin 2x$ we require

$$\tfrac{1}{2}\sin 2x = \sum_{n=1}^{\infty} A_n \sin nx$$

and so we choose $A_2 = \frac{1}{2}$ and $A_n = 0$ for $n \neq 2$

2 Hence solution is $u(x,t) = \frac{1}{2} e^{-4t} \sin 2x$

(b).



$u_{xx} + u_{yy} = 0$

We seek solutions of the form $u(x,y) = X(x)\,Y(y)$.

Since $u_x(0,y) = 0 = u_x(1,y)$, we require $X'(0) = 0 = X'(1)$

Since $u(x,2) = 0$ we require $Y(2) = 0$

$u_{xx} + u_{yy} = 0 \Leftrightarrow X''(x)\,Y(y) + X(x)\,Y''(y) = 0$

$\Rightarrow \dfrac{X''}{X} = -\dfrac{Y''}{Y} = -k$ where k is a constant

i.e

$X'' = -kX \quad X'(0) = 0 = X'(1) \quad (1)$

5 $Y'' = kY \quad Y(2) = 0 \quad (2)$

6(c)(d) ① has non zero solutions iff $k = 0, \pi^2, 4\pi^2, \dots$ and the corresponding

1 solutions are $1, \cos\pi x, \cos 2\pi x, \dots$

If $k = 0$, ② has general solution $Y(y) = A(2-y)$

1 If $k = 1, 2, \dots$ ② has general solution $Y(y) = A \sinh n\pi(2-y)$

Thus we have the general solution

3 $$u(x,y) = A_0(2-y) + \sum_{n=1}^{\infty} A_n \cos n\pi x \sinh n\pi(2-y)$$

1) If $u(x,0) = \cos^2 \pi x = \frac{1}{2}(\cos 2x + 1)$ we require

$$\frac{1}{2}(\cos 2\pi x + 1) = A_0 2 + \sum_{n=1}^{\infty} A_n \sinh 2n\pi \cos n\pi x$$

Thus we choose $A_0 = \frac{1}{4}$, $A_2 = \frac{1}{2\sinh 4\pi}$, $A_n = 0$, $n \neq 0, 2$

Thus we have solution

3 $$u(x,y) = \frac{1}{4}(2-y) + \frac{1}{2\sinh 4\pi} \cos 2\pi x \sinh 2\pi(2-y)$$