Question 1 (17 Marks)

(a) Suppose we wish to derive the Euler–Lagrange equation that results from extremizing the quantity

$$J(y) := \int_a^b F(x, y, y_x)\,\mathrm{d}x,$$

for twice continuously differentiable curves $y = y(x)$ defined for $a \leq x \leq b$ where $b > a$. By considering variations of the form

$$y(x;\epsilon) := u(x) + \epsilon\eta(x),$$

for some small real parameter ϵ and for twice continuously differentiable functions $\eta = \eta(x)$ that vanish at $x = a$ and $x = b$, show that the curve $y = y(x)$ that extremizes J necessarily satisfies the Euler–Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y_x}\right) = 0.$$

Show that an alternative form for the Euler–Lagrange equation is

$$\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(F - y_x\frac{\partial F}{\partial y_x}\right) = 0.$$

(b) A uniform heavy chain of length ℓ, hangs between two fixed supports at the points $(-a, 0)$ and $(a, 0)$. Show that the potential energy for this chain is given by

$$J = \rho g \int_{-a}^{+a} y\sqrt{1 + (y')^2}\,\mathrm{d}x$$

where $y = y(x)$ represents the shape of the chain, ρ is its mass density per unit length and g is the acceleration due to gravity. By using the results of part (a) above or otherwise, find the shape of the curve $y = y(x)$ that minimizes the potential energy of the chain.

Under what condition on a and ℓ is there no solution curve $y = y(x)$?

Question 2 (17 Marks)

A pendulum system consists of a light rod, of length ℓ, with a mass M connected at one end that can slide freely along the x-axis, and a mass m at the other end that swings freely in the vertical plane containing the x-axis.

If $\mu(t)$ represents the position of the mass M along the x-axis at time t, and $\theta(t)$ is the angle the rod makes with the vertical, show that the Lagrangian for this system is

$$L(\mu, \theta, \dot{\mu}, \dot{\theta}) = \tfrac{1}{2}M\dot{\mu}^2 + \tfrac{1}{2}m(\ell^2\dot{\theta}^2 + \dot{\mu}^2 + 2\ell\dot{\mu}\dot{\theta}\cos\theta) + mg\ell\cos\theta.$$

Derive explicit expressions for the generalized momenta p_μ and p_θ conjugate to μ and θ, respectively.

Explain why the Hamiltonian (no need to derive it) and p_μ are constants of the motion.

Assume $p_\mu = 0$ (this corresponds to assuming that the centre of mass of the system is not uniformly translating in the x-direction) and show that

$$(M+m)\mu = -m\ell\sin\theta + A,$$

where A is an arbitrary constant.

Hence write down the Euler–Lagrange equation of motion for the angle $\theta = \theta(t)$.

Use your result/condition for $\mu = \mu(t)$ above to show that the position of the mass m at time t in Cartesian x and y coordinates is given by

$$\begin{aligned} x &= \left(\frac{M\ell}{M+m}\right)\sin\theta + \frac{A}{M+m}, \\ y &= -\ell\cos\theta. \end{aligned}$$

What is the shape of this curve with respect to the x and y coordinates?

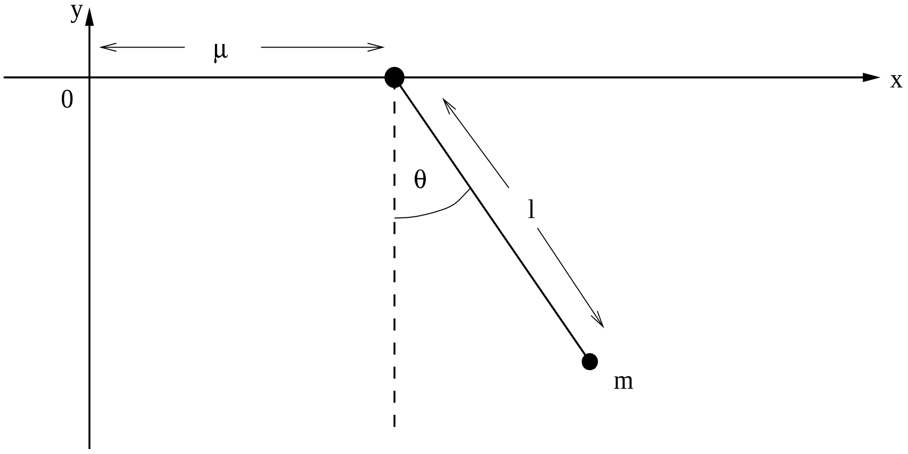

Figure 1: Pendulum with moving frictionless support.

Question 3 (17 Marks)
[Chorin and Marsden, Page 31] Let Ω be the region between two concentric cylinders of radii R_1 and R_2, where $R_1 < R_2$. Suppose the velocity field $\boldsymbol{u}$ of the fluid flow inside Ω, in cylindrical coordinates, is given by

$$u_r = 0, \qquad u_z = 0, \qquad \text{and} \qquad u_\theta = \frac{A}{r} + Br,$$

where

$$A = -\frac{R_1^2 R_2^2(\omega_2 - \omega_1)}{R_2^2 - R_1^2} \qquad \text{and} \qquad B = -\frac{R_1^2\omega_1 - R_2^2\omega_2}{R_2^2 - R_1^2}.$$

Show that:

(a) $\boldsymbol{u}$ is a stationary solution of Euler's equations of motion for an ideal fluid with $\rho = 1$;

(b) the vorticity field $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = (0, 0, 2B)$;

(c) the deformation tensor is

$$D = -\frac{A}{r^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and discuss its physical meaning;

(d) the angular velocity of the flow on the two cylinders is ω_1 and ω_2.

Question 4 (17 Marks)

As the water (of uniform density ρ) flows out through a hole at the bottom of a bath the residual rotation is confined to a core of radius a, so that the water particles may be taken to move on horizontal circles with

$$u_\theta = \begin{cases} \Omega r, & r \leq a, \\ \frac{\Omega a^2}{r}, & r > a. \end{cases}$$

(a) If the pressure at the free surface is p_0 (uniform), show that the free surface has the form $z = \alpha r^2$ for $r \leq a$, where α is to be found and find the form $z = F(r)$ when $r > a$.

(b) Evaluate the Bernoulli function

$$H := \frac{P}{\rho} + \tfrac{1}{2}|\boldsymbol{u}|^2 + gz,$$

as a function of position (r, θ, z) and explain why this result is consistent with Bernoulli's theorem (here P is the pressure and g is the acceleration due to gravity).

(c) If the radius a of the uniformly rotating core shrinks whilst the circulation around it remains constant show, using part (a) above, that the depth D of the depression in the water surface below the general level of the bath water is such that D is proportional to $1/a^2$.

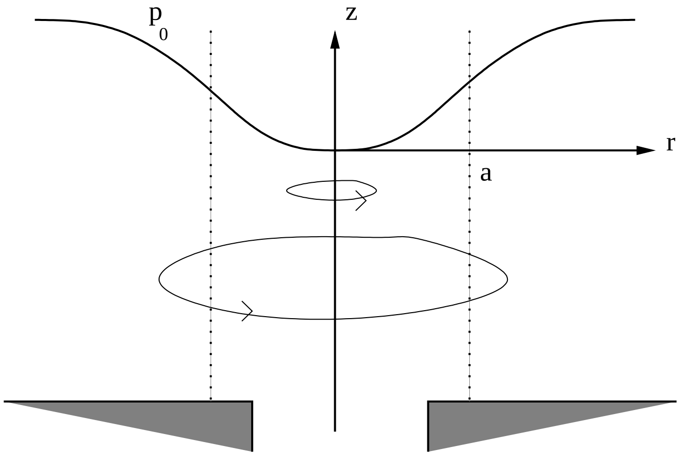

Figure 2: Water draining from a bath.

Question 5 (16 Marks)

(a) A string of unit length is fixed at its ends on the same horizontal level at $x = 0$ and $x = 1$. When the string vibrates, its displacement $u(x,t)$ satisfies the equation

$$u_{tt} = u_{xx},$$

for $0 < x < 1$ and $t > 0$. If the string has initial displacement $u(x,0) = \sin(2\pi x)$ and initial velocity $u_t(x,0) = x$ for $0 < x < 1$, determine $u(x,t)$.

(b) By considering the function

$$E(t) = \tfrac{1}{2}\int_0^1 \Big(\big(u_x(x,t)\big)^2 + \big(u_t(x,t)\big)^2\Big)\,\mathrm{d}x,$$

show that the only solution of

$$u_{tt} = u_{xx},$$

for $0 < x < 1$ and $t > 0$, with boundary conditions

$$u(0,t) = u(1,t) = 0$$

for all $t > 0$ and initial conditions

$$u(x,0) = u_t(x,0) = 0$$

for $0 < x < 1$, is $u \equiv 0$.

Question 6 (16 Marks)

(a) Solve the heat equation

$$u_t = u_{xx},$$

for $0 < x < \pi$ and $t > 0$, subject to the boundary conditions

$$u(0,t) = u(\pi,t) = 0$$

for all $t > 0$ and the initial condition

$$u(x,0) = \sin x \cos x$$

for $0 < x < \pi$.

(b) Solve Laplace's equation

$$\nabla^2 u = 0,$$

for for $0 < x < 1$, $0 < y < 2$, subject to the boundary conditions

$$\begin{aligned} u(x,0) = \cos^2(\pi x),\ u(x,2) = 0, &\qquad \text{for } 0 < x < 1, \\ u_x(0,y) = u_x(1,y) = 0, &\qquad \text{for } 0 < y < 2. \end{aligned}$$