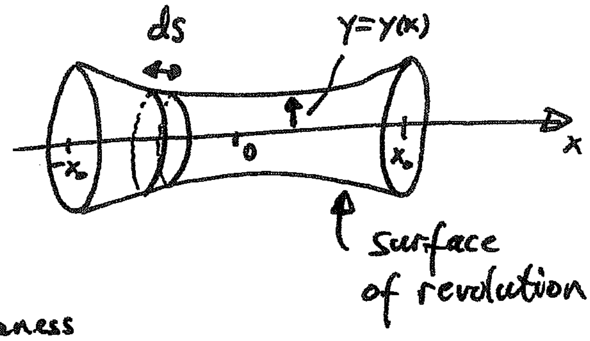
Solution (Soap film)

SEEN — WIS

(a) Surface area of element shown is

$\approx 2\pi y \cdot ds$

radius, thickness



3

Hence surface area between $-x_0$ and x_0 is

$$J(y) = \int_{-x_0}^{x_0} 2\pi y\, ds = 2\pi \int_{-x_0}^{x} y \cdot \sqrt{1+(y_x)^2}\, dx$$

(b) Since integrand $F = F(y, y_x)$ and independent of explicit x, use alternative form of the Euler-Lagrange eqn $\Rightarrow$

7

$$F - y_x \frac{\partial F}{\partial y_x} = C \qquad \text{where } C = \text{constant.}$$

$$\Leftrightarrow \quad y\cdot(1+(y_x)^2)^{1/2} - y_x \frac{\partial}{\partial y_x}\left(y\cdot(1+(y_x)^2)^{1/2}\right) = C$$

$$\Leftrightarrow \quad y\cdot(1+(y_x)^2)^{1/2} - y_x \cdot y \cdot \tfrac{1}{2}(1+(y_x)^2)^{-1/2}\cdot 2y_x = C$$

Solution (Soap film cont'd)

$$\Leftrightarrow \quad y\cdot(1+(y_x)^2)^{1/2} - \frac{y\cdot(y_x)^2}{(1+(y_x)^2)^{1/2}} = C$$

$$\Leftrightarrow \quad \frac{y\cdot(1+(y_x)^2) - y\cdot(y_x)^2}{(1+(y_x)^2)^{1/2}} = C$$

$$\Leftrightarrow \quad y = C\cdot(1+(y_x)^2)^{1/2}$$

$$\Leftrightarrow \quad (y_x)^2 = C^{-2}y^2 - 1$$

(c) Using subst $y = C\cosh\theta \Rightarrow$

$$\frac{dy}{dx} = \sqrt{C^{-2}y^2 - 1}$$

$$\Leftrightarrow \quad C\cdot\sinh\theta\cdot\frac{d\theta}{dx} = \sqrt{\cosh^2\theta - 1}$$

$$\Leftrightarrow \quad \frac{d\theta}{dx} = C^{-1} \qquad \text{since } \sinh^2\theta = \cosh^2\theta - 1$$

$$\Leftrightarrow \quad \theta = C^{-1}(x + b) \qquad \text{arb const}$$

$$\Leftrightarrow \quad y = C\cosh(C^{-1}(x+b)).$$

5

Solution (Soap film cont'd)

(c) (cont'd) Solution should be symmetric about $x=0$ $\Rightarrow$ $b=0$. } 2

(use boundary condition that (rings concentric and same size)
$y(x_0) = C\cosh(C^{-1}(x_0+b))$
must equal $y(-x_0) = C\cosh(C^{-1}(-x_0+b))$

$$\Leftrightarrow \quad C\cosh(C^{-1}(x_0+b)) = C\cosh(C^{-1}(-x_0+b))$$

$$\Leftrightarrow \quad C\cosh(C^{-1}(-x_0-b)) = C\cosh(C^{-1}(-x_0+b))$$

(since cosh even function)

$$\Leftrightarrow \quad -x_0-b = -x_0+b$$

$$\Leftrightarrow \quad 2b=0$$

$$\Leftrightarrow \quad b=0 \quad)$$

(d) Use that $y=a$ at $x=+x_0$ (case $x=-x_0$ automatic since solution even with $b=0$) } 3

$$\Rightarrow \quad a = C\cosh(C^{-1}(x_0))$$

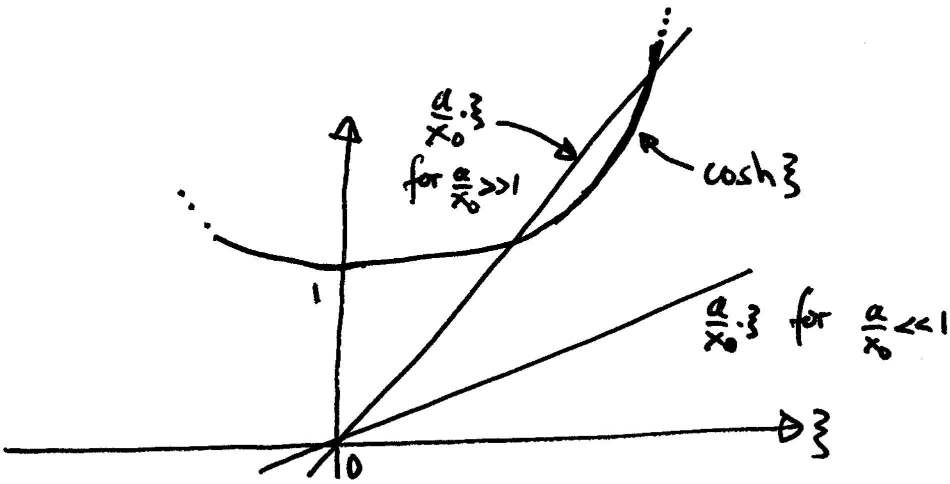
$$\Leftrightarrow \quad a/C = \cosh(x_0/C)$$

Solution (Soap film cont'd)

(d) (cont'd)

$$\Longleftrightarrow \quad \frac{a}{x_0} \cdot \frac{x_0}{c} = \cosh\left(\frac{x_0}{c}\right)$$

$$\Longleftrightarrow \quad \frac{a}{x_0} \cdot \xi = \cosh(\xi) \quad \text{with } \xi = \frac{x_0}{c}.$$



From graph, two solutions when $\frac{a}{x_0}$ suff large and no solution when $\frac{a}{x_0}$ too small.

NOT SEEN

Solution (Swinging Atwood's machine)

$$L(r,\theta,\dot r,\dot\theta) = \tfrac{1}{2}(M+m)\dot r^2 + \tfrac{1}{2}mr^2\dot\theta^2 - gr(M-m\cos\theta)$$

(a)

$$p_r = \frac{\partial L}{\partial \dot r} = \tfrac{1}{2}(M+m)2\dot r = (M+m)\dot r$$

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = \tfrac{1}{2}mr^2\cdot 2\dot\theta = mr^2\dot\theta$$

4

(b)

$$H = \dot r p_r + \dot\theta p_\theta - L$$

$$= \frac{p_r}{(M+m)}\cdot p_r + \frac{p_\theta}{mr^2}\cdot p_\theta - \tfrac{1}{2}(M+m)\left(\frac{p_r}{M+m}\right)^2 - \tfrac{1}{2}mr^2\left(\frac{p_\theta}{mr^2}\right)^2 + gr(M-m\cos\theta)$$

$$= \frac{1}{2}\frac{p_r^2}{M+m} + \frac{1}{2}\frac{p_\theta^2}{mr^2} + gr(M-m\cos\theta)$$

6

Solution (Swinging Atwood's machine, cont'd)

(c) Lagrangian and Hamiltonian depend explicitly on t or not together, and in fact

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} = \frac{dH}{dt}$$

L indept of explicit t $\Rightarrow$ H is a constant of motion. 2

Total kinetic energy $= \frac{1}{2}(M+m)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$ is a homogeneous quadratic function of generalized velocities $\dot{r}$ and $\dot{\theta}$. Hence Hamiltonian H is the total energy. 2

<u>Solution</u> (Swinging Atwood's machine, cont'd)

(d) Using Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot \theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Longleftrightarrow \quad \frac{d}{dt}\left((M+m)\dot r\right) - mr\dot\theta^2 + g(M - m\cos\theta) = 0$$

$$\frac{d}{dt}\left(mr^2\dot\theta\right) - \left(-g(-m)r(-\sin\theta)\right) = 0$$

$$\Longleftrightarrow \quad \ddot r = \frac{1}{M+m}\left(mr\dot\theta^2 - g(M - m\cos\theta)\right) \quad ①$$

$$2r\dot r\dot\theta + r^2\ddot\theta + gr\sin\theta = 0 \quad ②$$

$$② \Longleftrightarrow \quad \ddot\theta = -2\frac{\dot r}{r}\dot\theta - \frac{g}{r}\sin\theta.$$

6

SEEN — WES —

<u>Solution</u> (Couette flow)

In cylindrical coordinates (r, θ, z), $\underline{u} = (u_r, u_\theta, u_z)$
the Euler eqns are

$$\frac{\partial u_r}{\partial t} + (\underline{u}\cdot\nabla)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + f_r$$

$$\frac{\partial u_\theta}{\partial t} + (\underline{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{\rho\cdot r}\cdot\frac{\partial p}{\partial \theta} + f_\theta$$

$$\frac{\partial u_z}{\partial t} + (\underline{u}\cdot\nabla)u_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + f_z$$

where $p = p(r, \theta, z, t)$ is the pressure, ρ is the uniform density & $\underline{f} = (f_r, f_\theta, f_z)$ is the body force per unit mass.

Here, $$\underline{u}\cdot\nabla \equiv u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$$

Further, the incompressibility condition is

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0.$$

Solution (Couette flow cont'd)

Since flow stationary $\Rightarrow \partial/\partial t = 0$

Flow independent of $\theta \Rightarrow \partial/\partial\theta = 0$ for all components of $\underline{u}$.

Since $\left\{\begin{matrix} u_r = u_z = 0 \\ \& \ \partial/\partial\theta = 0 \end{matrix}\right\} \Rightarrow \underline{u}\cdot\nabla = 0$

6

The only body force per unit mass is $-g\underline{k}$ so $f_r = f_\theta = 0$ and $f_z = -g$. Hence from Euler's equations:

$$-\frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \qquad (1)$$

$$0 = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta}, \qquad (2)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \qquad (3)$$

(2) $\Rightarrow p$ independent of θ and thus $p = p(r,z)$ satisfies

$$\frac{u_\theta^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r},$$

$$\& \quad 0 = \frac{1}{\rho}\frac{\partial p}{\partial z} + g.$$

Solution (Couette flow cont'd)

(b)(i) Using that $u_\theta = \frac{A}{r} + Br$ and $\rho \equiv 1$

$$\Rightarrow \quad \frac{\partial p}{\partial r} = \frac{1}{r}\left(\frac{A}{r} + Br\right)^2$$

$$\Leftrightarrow \quad \frac{\partial p}{\partial r} = \frac{A^2}{r^3} + \frac{2AB}{r} + B^2 r$$

$$\Leftrightarrow \quad p = -\frac{1}{2}\frac{A^2}{r^2} + 2AB\log r + \frac{1}{2}B^2r^2 + F(z)$$

where F is an arbitrary function. Sub$^{\text{stg}}$ this expression for p into the partial differential equation $\frac{\partial p}{\partial z} = -g$ implies

$$F'(z) = -g \quad \Leftrightarrow \quad F = -gz + C$$

where C is an arbitrary constant.

Hence

$$p(r,z) = -\frac{1}{2}\frac{A^2}{r^2} + 2AB\log r + \frac{1}{2}B^2r^2 - gz + C.$$

8

Solution (Couette flow cont'd)

(b)

(ii) $$\frac{u_\theta}{r} = \frac{A}{r^2} + B$$

Hence
$$\frac{u_\theta(R_1)}{R_1} = \frac{A}{R_1^2} + B$$
$$= -\frac{R_2^2(\omega_2-\omega_1)}{R_2^2-R_1^2} - \frac{(R_1^2\omega_1 - R_2^2\omega_2)}{R_2^2-R_1^2}$$
$$= \frac{R_2^2\omega_1 - R_1^2\omega_1}{R_2^2-R_1^2}$$
$$= \omega_1,$$

and
$$\frac{u_\theta(R_2)}{R_2} = -\frac{R_1^2(\omega_2-\omega_1)}{R_2^2-R_1^2} - \frac{(R_1^2\omega_1 - R_2^2\omega_2)}{R_2^2-R_1^2}$$
$$= \frac{-R_1^2\omega_2 + R_2^2\omega_2}{R_2^2-R_1^2}$$
$$= \omega_2 .$$

4

NOT SEEN

(iii) Using cylindrical coord form for $\underline{\omega} = \nabla \times \underline{u}$

$$\underline{\omega} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r}\frac{\partial}{\partial r}(A + Br^2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2B \end{pmatrix}.$$

2

<u>Solution</u> (Clepsydra / Water clock)

(a) Euler's equations of motion with $\frac{\partial \underline{u}}{\partial t} = 0$ and identity given for $\underline{u}\cdot\nabla\underline{u}$ $\Rightarrow$

$$\tfrac{1}{2}\nabla(|\underline{u}|^2) - \underline{u}\times(\nabla\times\underline{u}) = -\tfrac{1}{\rho}\nabla p - \nabla\phi$$

$$\Leftrightarrow \quad \nabla\Big(\underbrace{\tfrac{1}{2}|\underline{u}|^2 + \tfrac{p}{\rho} + \phi}_{H}\Big) = \underline{u}\times(\nabla\times\underline{u})$$

8

Let $\underline{x}(s)$ denote a streamline, i.e. $\underline{x}'(s) = \underline{u}(\underline{x}(s))$

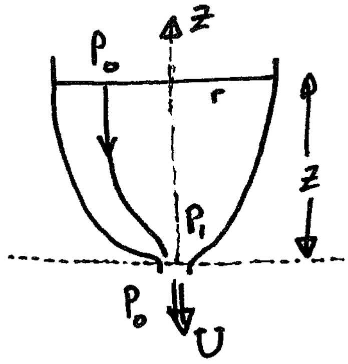
For arbitrary s_1 and s_2:

$$\begin{aligned} H(\underline{x}(s_2)) - H(\underline{x}(s_1)) &= \int_{s_1}^{s_2} dH(\underline{x}(s)) \\ &= \int_{s_1}^{s_2} \nabla H\cdot\underline{x}'(s)\,ds \\ &= \int_{s_1}^{s_2} \nabla H\cdot\underline{u}(\underline{x}(s))\,ds \\ &= \int_{s_1}^{s_2} \underbrace{(\underline{u}\times\underline{\omega})\cdot\underline{u}}_{\equiv 0}\,ds \\ &= 0 \end{aligned}$$

where we have used that the triple product with two arg's same is always zero. Since s_1 & s_2 arb $\Rightarrow$ result.

Solution (Clepsydra/Water clock, cont'd)

(b) Want to find z as a function of r such that the water level falls at a constant rate.



~~Note that inside the container at the level at which a water particle is pushed through the orifice, the pressure is~~

$$\cancel{P_1 = P_0 + \rho g z.}$$

~~(The difference $P_1 - P_0$ accelerates the water through the orifice.)~~

(1) Using Bernoulli's Theorem for a streamline starting at the free surface and going through the orifice

$$\frac{1}{2}\cdot\left(\frac{dz}{dt}\right)^2 + \frac{P_0}{\rho} = \frac{1}{2}U^2 + \frac{P_0}{\rho} - gz$$

$$\Leftrightarrow \qquad \frac{1}{2}\left(\frac{dz}{dt}\right)^2 + \cancel{\frac{P_0}{\rho}} = \frac{1}{2}U^2 + \cancel{\frac{P_0}{\rho}} + gz$$

$$\Leftrightarrow \qquad \frac{1}{2}\left(\frac{dz}{dt}\right)^2 = \frac{1}{2}U^2 + gz$$

4

(ii)

First we assume that the water level falls at a constant rate which is small, i.e. $\frac{dz}{dt}$ is small and so $\left(\frac{dz}{dt}\right)^2$ is very small

$$\Rightarrow \quad \tfrac{1}{2}U^2 \simeq gz$$

$$\Leftrightarrow \quad U \simeq \sqrt{2gz}.$$

2

(iii)

Volume flux $= \pi r^2 \frac{dz}{dt} = S \cdot U$

where $S =$ cross-sectional area of orifice.

3

(iv)

Hence $\quad \pi r^2 \frac{dz}{dt} = S \cdot \sqrt{2g} \cdot z^{1/2}$

Since $\frac{dz}{dt}$ must be constant

$$\Rightarrow \quad z \propto r^4.$$

3

Solution (heat eqn)

(a) $u_t = k\, u_{xx}$

BCs: $u(0,t) = 0, \quad u(L,t) = 0$

IC: $u(x,0) = T_0$.

Seek a solution of the form $u(x,t) = X(x)\cdot T(t)$

Subst into heat eqn $\Rightarrow$

$$X \cdot T' = k \cdot X'' \cdot T$$

$$\Leftrightarrow \quad \frac{X''(x)}{X(x)} = \frac{1}{k}\cdot\frac{T'(t)}{T(t)} = -\lambda$$

where λ is a constant.

Hence

① $X''(x) = -\lambda X(x), \quad X(0) = X(L) = 0$

② $T'(t) = -k\lambda T(t)$.

4

① has non-zero solns iff $\lambda = \frac{n^2\pi^2}{L^2}$ for $n = 1, 2, \dots$ and corresponding solns are $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$.

3

Solution (heat eqn, cont'd)

If $\lambda = \frac{n^2\pi^2}{L^2} \Rightarrow T' = -\frac{kn^2\pi^2}{L^2} T$

$$\Leftrightarrow \quad T(t) = A_n \exp\left(-\frac{kn^2\pi^2}{L^2} t\right).$$ 3

Hence any function of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cdot e^{-\frac{kn^2\pi^2}{L^2} \cdot t}$$

satisfies the heat eqn and BCs.

Now choose A_n to ensure $u(x,0) = T_0 \sin\left(\frac{2\pi x}{L}\right)$
i.e.

$$T_0 \sin\left(\frac{2\pi x}{L}\right) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cdot 1 .$$

Choice $A_2 = T_0$ and $A_1 = A_3 = A_4 = \cdots = 0$ ensures equality above.

Hence $u(x,t) = T_0 \sin\left(\frac{2\pi x}{L}\right) \cdot e^{-\frac{4k\pi^2}{L^2} \cdot t}$ 3

Eqn describes temp of metal bar, thermal diffusivity k, lying between $x=0$ and $x=L$, ends of bar maintained at $0°c$, and initial temp profile sinusoidal as given. 1

Solution (heat eqn, cont'd)

(b) $u_t = u_{xx}$, BCs: $u(0,t) = u(L,t) = 0$,

IC: $u(x,0) = 0$.

$$\frac{d}{dt} E(t) = \int_0^L \frac{\partial}{\partial t}(u^2)\, dx$$

$$= \int_0^L 2u \frac{\partial u}{\partial t}\, dx$$

$$= 2\int_0^L u\, u_{xx}\, dx$$

$$= \cancel{\Big[2 u u_x\Big]_{x=0}^{x=L}}^{\,0} - 2\int_0^L (u_x)^2 dx$$

$$= -2\int_0^L (u_x)^2 dx.$$

Hence $\frac{d}{dt} E(t) \leq 0$ and since $E(0) = 0$

using the IC and $E(t) \geq 0 \Rightarrow E(t) \equiv 0$.

Hence $u(x,t) \equiv 0$.

6