Question 1 (20 Marks)

A soap film is stretched between two rings of radius a which lie in parallel planes a distance $2x_0$ apart—the axis of symmetry of the two rings is coincident—see Figure 1.

(a) Explain why the surface area of the surface of revolution is given by

$$J(y) = 2\pi \int_{-x_0}^{x_0} y\sqrt{1+(y_x)^2}\,\mathrm{d}x,$$

where the radius of the surface of revolution is given by $y = y(x)$ for $x \in [-x_0, x_0]$.

(b) Show that extremizing the surface area $J(y)$ in part (a) leads to the following ordinary differential equation for $y = y(x)$ (*hint:* you may find the alternative form for the Euler–Lagrange equation useful here):

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = C^{-2}y^2 - 1$$

where C is an arbitrary constant.

(c) Use the substitution $y = C\cosh\theta$ and the identity $\cosh^2\theta - \sinh^2\theta = 1$ to show that the solution to the ordinary differential equation in part (b) is

$$y = C\cosh\bigl(C^{-1}(x+b)\bigr)$$

where b is another arbitrary constant. Explain why we can deduce that $b = 0$.

(d) Using the end-point conditions $y = a$ at $x = \pm x_0$, discuss the existence of solutions in relation to the ratio a/x_0.

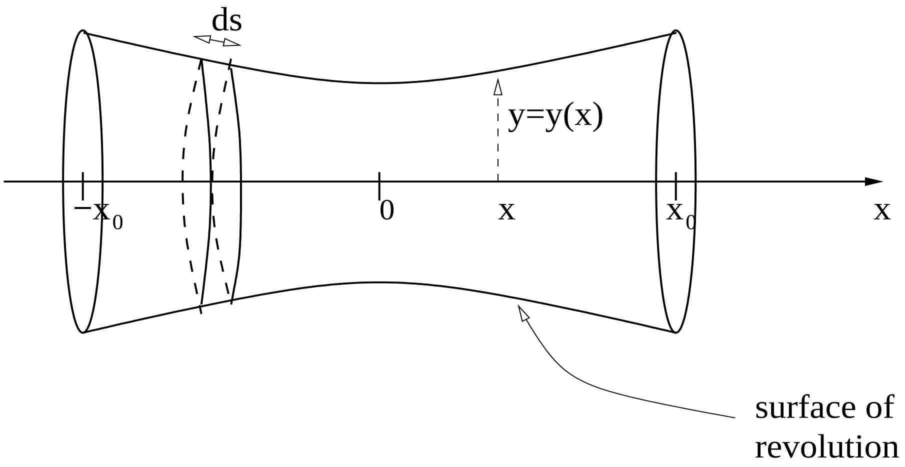

Figure 1: Soap film stretched between two concentric rings. The radius of the surface of revolution is given by $y = y(x)$ for $x \in [-x_0, x_0]$.

Question 2 (20 Marks)

The *swinging Atwood's machine* is a mechanism that resembles a simple Atwood's machine except that one of the masses is allowed to swing in a two-dimensional plane—see Figure 2. A string of length ℓ, with a mass M at one end and a mass m at the other, is stretched over two frictionless pulleys as shown in Figure 2. The mass M hangs vertically downwards; it only moves up and down. The mass m on the other hand is free to swing in a vertical plane as shown. The Lagrangian for this system has the form

$$L(r,\theta,\dot r,\dot\theta) = \tfrac{1}{2}(M+m)\dot r^2 + \tfrac{1}{2}mr^2\dot\theta^2 - gr(M - m\cos\theta),$$

where (r,θ) are the plane polar coordinates of the mass m that can swing in the vertical plane. Here g is the acceleration due to gravity.

(a) Show that the generalized momenta p_r and p_θ corresponding to the coordinates r and θ, respectively, are given by

$$p_r = (M+m)\dot r \qquad \text{and} \qquad p_\theta = mr^2\dot\theta.$$

(b) Using the results from part (a), show that the Hamiltonian for this system is given by

$$H(r,\theta,p_r,p_\theta) = \frac{p_r^2}{2(M+m)} + \frac{p_\theta^2}{2mr^2} + gr(M - m\cos\theta).$$

(c) Explain why the Hamiltonian H is a constant of the motion. Is the Hamiltonian H equal to the total energy?

(d) By either using Lagrange's equations of motion, or, using Hamilton's equations of motion, show that the swinging Atwood's machine evolves according to a pair of second order ordinary differential equations

$$\begin{aligned} \ddot r &= \frac{1}{M+m}\big(mr\dot\theta^2 - g(M - m\cos\theta)\big), \\ r\,\ddot\theta &= -2\dot r\dot\theta - g\sin\theta. \end{aligned}$$

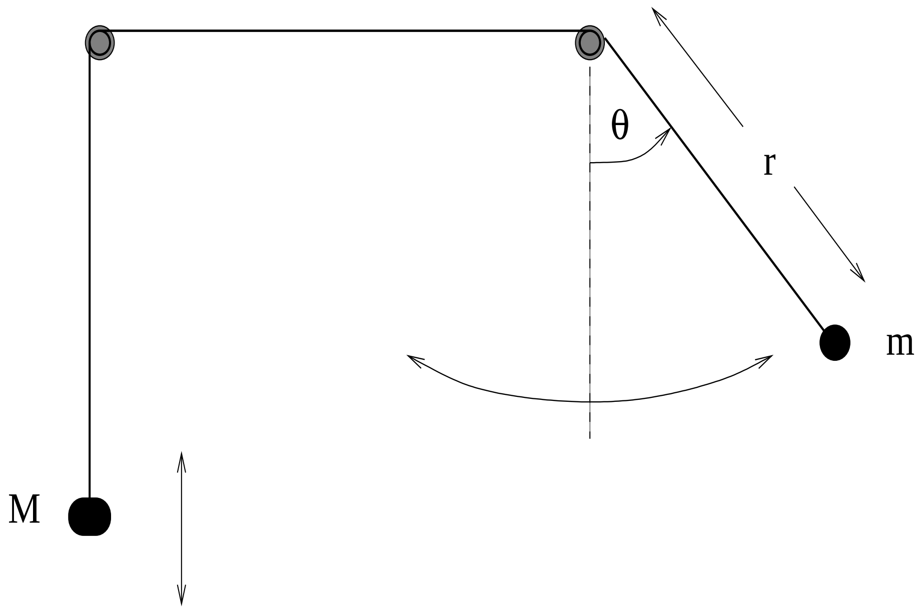

Figure 2: Swinging Atwood's machine: a string of length ℓ, with a mass M at one end and a mass m at the other, is stretched over two pulleys. The mass M hangs vertically downwards; it only moves up and down. The mass m is free to swing in a vertical plane.

Question 3 (20 Marks)

(a) Using the Euler equations for an ideal incompressible flow in cylindrical coordinates (see the formulae sheet at the end of the exam paper) show that at position (r, θ, z) with z the coordinate in the vertical upward direction, for a *stationary* flow which is independent of θ with $u_r = u_z = 0$, we have

$$\begin{aligned} \frac{u_\theta^2}{r} &= \frac{1}{\rho}\frac{\partial p}{\partial r}, \\ 0 &= \frac{1}{\rho}\frac{\partial p}{\partial z} + g, \end{aligned}$$

where $p = p(r, z)$ is the pressure, ρ is the constant uniform mass density and g is the acceleration due to gravity (assume this to be the body force per unit mass).

(b) Let Ω be the region between two concentric cylinders of radii R_1 and R_2, where $R_1 < R_2$. Suppose the velocity field in cylindrical coordinates $\boldsymbol{u} = (u_r, u_\theta, u_z)$ of the fluid flow inside Ω, is given by

$$u_r = 0, \qquad u_z = 0, \qquad \text{and} \qquad u_\theta = \frac{A}{r} + Br,$$

where

$$A = -\frac{R_1^2 R_2^2(\omega_2 - \omega_1)}{R_2^2 - R_1^2} \qquad \text{and} \qquad B = -\frac{R_1^2\omega_1 - R_2^2\omega_2}{R_2^2 - R_1^2}.$$

This is known as a *Couette flow*—see Figure 3. Show that the:

(i) velocity field $\boldsymbol{u} = (u_r, u_\theta, u_z)$ is a stationary solution of Euler's equations of motion for an ideal fluid with density $\rho \equiv 1$ (*hint:* you need to find a pressure field p that is consistent with the velocity field given);

(ii) angular velocity of the flow (i.e. the quantity u_θ/r) is ω_1 on the cylinder $r = R_1$ and ω_2 on the cylinder $r = R_2$.

(iii) the vorticity field $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = (0, 0, 2B)$.

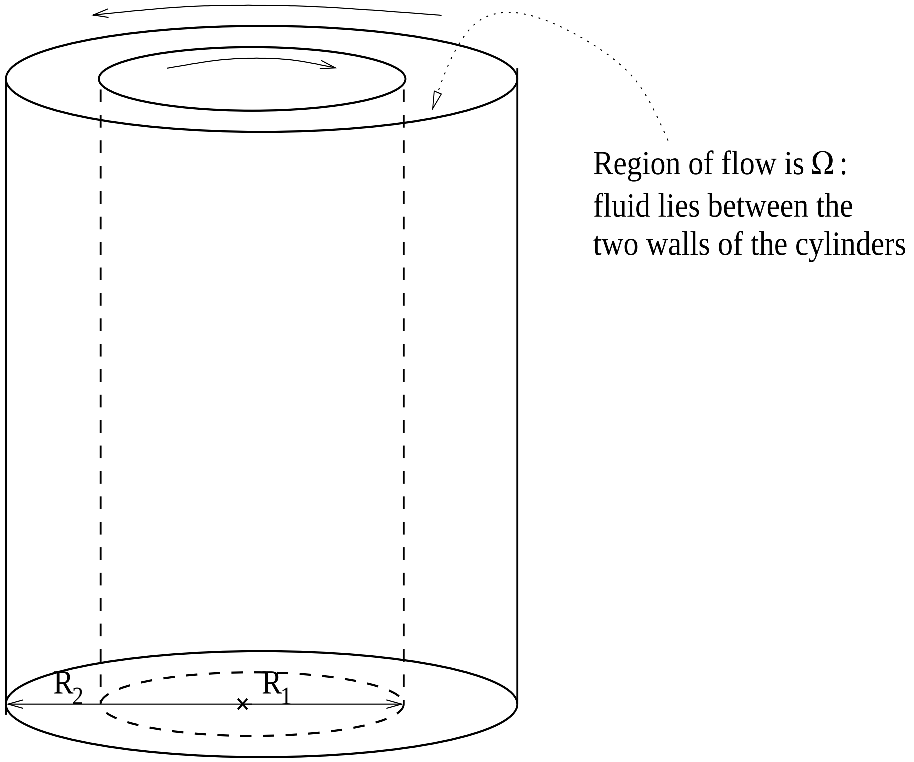

Figure 3: Couette flow between two concentric cylinders of radii $R_1 < R_2$.

Question 4 (20 Marks)

(a) Consider Euler's equations of motion for an ideal homogeneous incompressible fluid—these are given on the formulae sheet at the end of the exam paper. Let $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ denote the fluid velocity at position $\boldsymbol{x}$ and time t, ρ the uniform constant density, $p = p(\boldsymbol{x}, t)$ the pressure, and $\boldsymbol{f}$ the body force per unit mass. Suppose that the flow is stationary so that

$$\frac{\partial \boldsymbol{u}}{\partial t} = 0,$$

and that the body force is conservative so that $\boldsymbol{f} = -\nabla\phi$ for some potential function $\phi = \phi(\boldsymbol{x})$. Using the identity

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \tfrac{1}{2}\nabla(|\boldsymbol{u}|^2) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}),$$

show from Euler's equations of motion that the Bernoulli quantity

$$H := \tfrac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} + \phi$$

is constant along streamlines.

(b) A clepsydra has the form of a surface of revolution containing water and the level of the free surface of the water falls at a *constant* rate, as the water flows out through a small hole in the base. The basic setup is shown in Figure 4.

(i) Apply the result in part (a) for the Bernoulli quantity H to one of the typical streamlines shown in Figure 4 to show that

$$\frac{1}{2}\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 = \tfrac{1}{2}U^2 - gz$$

where z is the height of the free surface above the small hole in the base, U is the velocity of the water coming out of the small hole and g is the acceleration due to gravity.

(ii) Assuming that the constant rate at which the level surface is falling is very slow, explain why we can deduce that

$$U \approx \sqrt{2gz}.$$

(iii) If S is the cross-sectional area of the hole in the bottom, and A is the cross-sectional area of the free surface, explain why we must have

$$A\,\frac{\mathrm{d}z}{\mathrm{d}t} = S\,U.$$

(iv) Combine the results from (ii) and (iii) above, to find the shape of the container that guarantees that the free surface of the water drops at a constant rate.

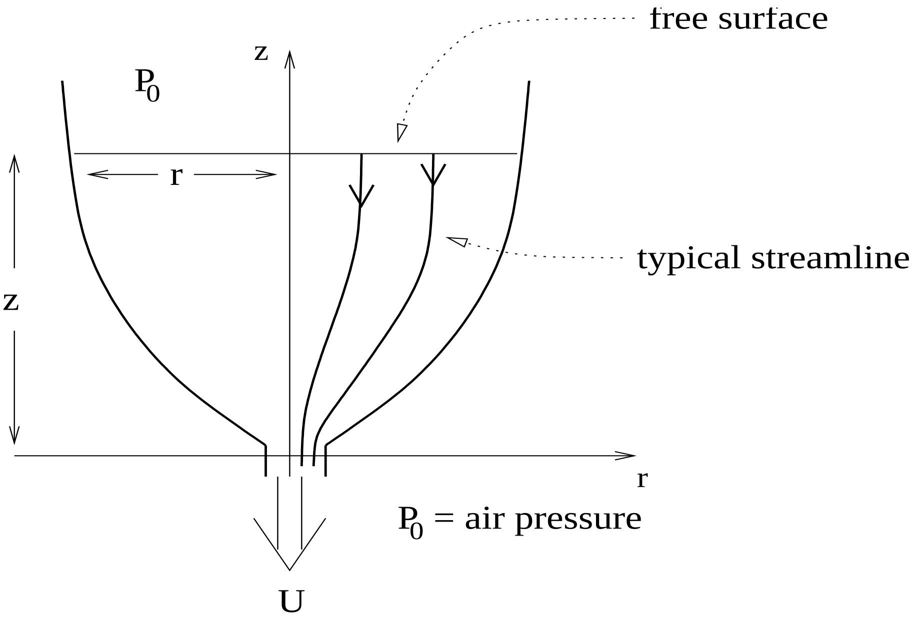

Figure 4: Clepsydra (water clock).

Question 5 (20 Marks)

(a) Find the solution of the heat equation

$$u_t = ku_{xx}$$

where k is a known diffusion parameter, for $0 < x < L$, $t > 0$ subject to the boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0$$

for $t > 0$ and an initial condition

$$u(x,0) = T_0 \sin\left(\frac{2\pi x}{L}\right),$$

for $0 \leq x \leq L$, where T_0 is a constant. Explain briefly the physical situation represented by the equation above.

(b) Suppose $u = u(x,t)$ satisfies the heat equation

$$u_t = u_{xx}$$

for $0 \leq x \leq L$ and $t > 0$, the initial condition

$$u(x,0) = 0$$

for $0 \leq x \leq L$, and the boundary conditions

$$u(0,t) = u(L,t) = 0$$

for $t > 0$. Show, by considering the function

$$E(t) := \int_0^L u^2(x,t)\,\mathrm{d}x,$$

that $u(x,t) \equiv 0$.

Formulae

(I) Euler's equations of motion for an ideal homogeneous incompressible fluid are

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \boldsymbol{f},$$
$$\nabla \cdot \boldsymbol{u} = 0,$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ is the fluid velocity at position $\boldsymbol{x}$ and time t, ρ is the uniform constant density, $p = p(\boldsymbol{x}, t)$ is the pressure, and $\boldsymbol{f}$ is the body force per unit mass.

(II) Euler's equations for an ideal homogeneous incompressible fluid in cylindrical coordinates (r, θ, z) with the velocity field expressed as $\boldsymbol{u} = (u_r, u_\theta, u_z)$ are

$$\begin{aligned} \frac{\partial u_r}{\partial t} + (\boldsymbol{u} \cdot \nabla) u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} + f_r, \\ \frac{\partial u_\theta}{\partial t} + (\boldsymbol{u} \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + f_\theta, \\ \frac{\partial u_z}{\partial t} + (\boldsymbol{u} \cdot \nabla) u_z &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + f_z, \end{aligned}$$

where $p = p(r, \theta, z, t)$ is the pressure, ρ is the uniform constant density and $\boldsymbol{f} = (f_r, f_\theta, f_z)$ is the body force per unit mass. Here we also have

$$\boldsymbol{u} \cdot \nabla = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}.$$

Further the incompressibility condition $\nabla \cdot \boldsymbol{u} = 0$ is given in cylindrical coordinates by

$$\frac{1}{r} \frac{\partial (r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0.$$

Lastly in cylindrical coordinates the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ is given by

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_r \\ \omega_\theta \\ \omega_z \end{pmatrix} = \nabla \times \boldsymbol{u} = \begin{pmatrix} \frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \\ \frac{1}{r} \frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \end{pmatrix}.$$