F19AB2 FINAL EXAM 2009 : SOLUTIONS

Question 1.

(a)

$$p_r = \frac{\partial L}{\partial \dot{r}} = \frac{m\dot{r}}{\sin^2\alpha}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

(b)

$$H(r,\theta,p_r,p_\theta) = p_r\cdot\dot{r} + p_\theta\cdot\dot{\theta} - L$$

$$= p_r\cdot\left(\frac{\sin^2\alpha}{m}\cdot p_r\right) + p_\theta\cdot\left(\frac{p_\theta}{mr^2}\right)$$

$$- \tfrac{1}{2}mr^2\left(\frac{p_\theta}{mr^2}\right)^2 - \tfrac{1}{2}\frac{m}{\sin^2\alpha}\cdot\left(\frac{\sin^2\alpha\cdot p_r}{m}\right)^2$$

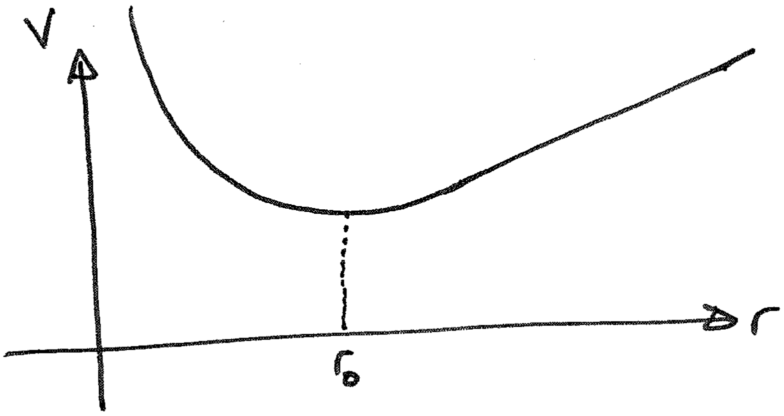
$$+ \frac{mgr}{\tan\alpha}$$

$$= \frac{1}{2m}\cdot\sin^2\alpha\, p_r^2 + \frac{1}{2m}\cdot\frac{p_\theta^2}{r^2} + \frac{mgr}{\tan\alpha}$$

(c) H is independent of explicit $t \Rightarrow$ it is a constant of motion

H independent of explicit $\theta \Rightarrow p_\theta$ is a constant of the motion.

(d)



Roughly, the particle can be viewed has moving in the potential well shown. The height in the cone corresponding to $r=r_0$ will represent an equilibrium solution → the particle started off at that height will slide around horizontally on the inside of the cone.

Started off at a different height the particle will prescribe trajectories that oscillate about this equilibrium height.

Question 2.

(a) The particle trajectories are the curves traced out by the fluid particles as time progresses.

For any fixed time t, streamlines would be the curves traced out by the fluid particles if their velocity was always the "frozen" velocity $\underline{u}(\underline{x}, t)$, frozen at time t.

For given velocity field, streamlines are integral curves of

$$\frac{dx}{ds} = u_0$$

$$\frac{dy}{ds} = V_0 \cos(kx - \alpha t)$$

Hence

$$\frac{dy}{dx} = \frac{V_0}{u_0} \cos(kx - \alpha t)$$

$$\iff \quad y = c + \frac{V_0}{u_0 k} \sin(kx - \alpha t)$$

where C is an arbitrary constant fixed by prescribing a point the streamline must pass through.

The streamline through $(0,0)$ at $t=0$ $\Rightarrow C=0$

$$\Rightarrow \quad y = \frac{V_0}{ku_0}\sin(kx)$$

For particle paths:

$$\frac{dx}{dt} = u_0 \quad \Leftrightarrow \quad x = x_0 + u_0 t \quad ①$$

$$\frac{dy}{dt} = V_0 \cos(kx - \alpha t)$$

$$\Leftrightarrow \quad \frac{dy}{dt} = V_0 \cos(ku_0 t - \alpha t + kx_0)$$

$$\Leftrightarrow \quad y = y_0 + \frac{V_0}{(ku_0-\alpha)}\sin\left((ku_0-\alpha)t + kx_0\right) - \frac{V_0}{(ku_0-\alpha)}\sin(kx_0) \quad ②$$

Particle starting at $(0,0)$ at $t=0$ $\Rightarrow$ subst $x_0 = y_0 = 0$ into ① & ② $\Rightarrow$

$$x = u_0 t$$

$$y = \frac{V_0}{ku_0-\alpha}\cdot\sin\left((ku_0-\alpha)t\right)$$

(b) Using the information provided, Euler's equation of motion reduces to

$$\tfrac{1}{2}\nabla(|\underline{u}|^2) - \underline{u}\times(\nabla\times\underline{u}) = -\nabla\left(\frac{p}{\rho_0}\right) - \nabla\phi$$

$$\Longleftrightarrow \qquad \nabla H = \underline{u}\times(\nabla\times\underline{u})$$

where $H := \frac{1}{2}|\underline{u}|^2 + \frac{p}{\rho_0} + \phi$.

Let $\underline{x}(s)$ be a streamline $\Rightarrow$

$$H(\underline{x}(s_2)) - H(\underline{x}(s_1)) = \int_{\underline{x}(s_1)}^{\underline{x}(s_2)} \nabla H\cdot\underline{x}'(s)\,ds$$

$$= \int_{\underline{x}(s_1)}^{\underline{x}(s_2)} \nabla H\cdot\underline{u}(\underline{x}(s))\,ds$$

$$= \int_{\underline{x}(s_1)}^{\underline{x}(s_2)} \underbrace{\underline{u}\times(\nabla\times\underline{u})\cdot\underline{u}(\underline{x}(s))}_{\equiv 0}\,ds$$

$$= 0.$$

Hence H is constant along streamlines.

Question 3.

(a) From form of Euler equations in cylindrical coords given on formulae sheet, setting $u_r = u_z = 0$ and any derivatives ∂_θ equal to zero we get

$$-\frac{u_\theta^2}{r} = -\frac{1}{\rho_0}\frac{\partial p}{\partial r} \qquad (f_r = 0) \qquad ①$$

$$0 = -\frac{1}{\rho_0 r}\frac{\partial p}{\partial \theta} \qquad (f_\theta = 0) \qquad ②$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g \qquad (f_z = -g) \qquad ③$$

Hence p is independent of θ (from ②).

Incompressibility condition is immediate as $u_r = u_z = 0$ and $\frac{\partial u_\theta}{\partial \theta} = 0$.

(b) (i) Using equations ① & ③ from part (a)

$$① \iff \frac{\partial p}{\partial r} = \frac{\rho_0}{r}\cdot(\Omega r)^2 \qquad \text{for } 0 \le r \le a$$

$$\iff p = \frac{1}{2}\rho_0\Omega^2 r^2 + \underbrace{f(z)}_{\text{arbitrary function of } z}$$

$$③ \iff \frac{\partial p}{\partial z} = -\rho_0 g$$

$$\iff f'(z) = -\rho_0 g$$

$$\iff f(z) = -\rho_0 g z + P_0$$

i.e. $p = P_0 + \frac{1}{2}\rho_0 \Omega^2 r^2 - \rho_0 g z$.

Free surface, suppose p constant, and solve for z $\Rightarrow$

$$z = \underbrace{\frac{1}{\cancel{\rho_0} g}\cdot\frac{1}{2}\cancel{\rho_0}\,\Omega^2}_{A} r^2 + \underbrace{\frac{P_0 - p}{\rho_0 g}}_{B}$$

(ii) Again using equations ① & ③ from part (a):

$$① \iff \frac{\partial p}{\partial r} = \frac{\rho_0}{r}\cdot\frac{\Omega^2 a^3}{r}$$

$$\iff p = -\rho_0 \Omega^2 a^3 \cdot \frac{1}{r} + \underbrace{g(z)}_{\text{arbitrary function}}$$

$$③ \iff g'(z) = -\rho_0 g$$

$$\iff g(z) = -\rho_0 g z + K \quad \nwarrow \text{arbitrary constant}$$

Hence for $r>a$ we have

$$p = -\rho_0 \Omega^2 a^3 \cdot \frac{1}{r} - \rho_0 g z + K$$

Pressure continuous across $r=a$ $\Rightarrow$

$$P_0 + \frac{1}{2}\rho_0 \Omega^2 a^2 - \cancel{\rho_0 g z} = -\rho_0 \Omega^2 a^3 \cdot \frac{1}{a} - \cancel{\rho_0 g z} + K$$

$$\Leftrightarrow \quad K = P_0 + \frac{3}{2}\rho_0 \Omega^2 a^2$$

Hence for $r>a$:

$$p = -\rho_0 \Omega^2 a^3 \cdot \frac{1}{r} - \rho_0 g z + P_0 + \frac{3}{2}\rho_0 \Omega^2 a^2.$$

4(a) $u_t = ku_{xx}\,;\qquad u(0,t) = 0 = u(L,t),\quad u(x,0) = \begin{cases} T_0 \text{ for } 0 \le x \le \frac{L}{2} \\ 0 \text{ for } \frac{L}{2} < x \le L \end{cases}.$

We seek solutions of the form $u(x,t) = X(x)\,T(t)$.
To ensure that $u(0,t) = 0 = u(L,t)$ we require that $X(0) = 0 = X(L)$.
Substituting into the equation we must have that

$$T'(t)X(x) = kX''(x)T(t)$$

Thus it is sufficient to have

$$\frac{X''(x)}{X(x)} = \frac{1}{k}\,\frac{T'(t)}{T(t)} = -\lambda$$

where λ is a constant,
i.e., we require

$$-X'' = \lambda X,\quad X(0) = 0 = X(L)\qquad (1)$$

$$T' = -k\lambda T\qquad (2)$$

(1) has nonzero solutions if and only if $\lambda = \frac{n^2\pi^2}{L^2}$ for $n = 1, 2, \ldots,, \ldots,$ and the corresponding eigenfunctions are $\sin(\frac{n\pi x}{L})$.

If $\lambda = \frac{n^2\pi^2}{L^2}$, (2) becomes $T' = -k\frac{n^2\pi^2}{L^2}T$ and so has solution $T = A_n e^{-k\frac{n^2\pi^2}{L^2}t}$

Hence any function of the form $u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L})e^{-k\frac{n^2\pi^2}{L^2}t}$ satisfies the PDE and the boundary conditions.

We now choose A_n to ensure that $u(x,0) = \begin{cases} T_0 \text{ for } 0 \le x \le \frac{L}{2} \\ 0 \text{ for } \frac{L}{2} < x \le L \end{cases}.$

Since $u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L})$, we choose A_n's as Fourier sine coefficients, i.e.,

$$A_n = \frac{2}{L}\int_0^{\frac{L}{2}} T_0 \sin(\frac{n\pi x}{L})\,dx = \frac{2T_0}{L}\cdot\frac{L}{n\pi} - \cos(\frac{n\pi x}{L})|_{x=0}^{x=L/2} = \frac{2T_0}{n\pi}[1 - \cos(\frac{n\pi}{2})].$$

Equation describes temperature $u(x,t)$ of a metal bar, thermal diffusivity k, lying between $x = 0$ and $x = L$ with the ends of the bar maintained at 0°, the left hand half of the bar initially at $T°$ and the right hand half initially at 0°.

(b) Suppose that u satisfies

$$u_t = u_{xx}\,;\qquad u(0,t) = 0 = u(1,t),\quad u(x,0) = 0\quad 0 \le x \le 1.$$

Let $E(t) = \int_0^1 u^2(x,t)\,dx$.
Clearly $E(t) \ge 0$ for all t and since $u(x,0) \equiv 0$, $E(0) = 0$.
Also

$$\begin{aligned} \tfrac{d}{dt}\,[E(t)\,] &= 2\int_0^1 u(x,t)u_t(x,t)\,dx = 2\int_0^1 u(x,t)u_{xx}(x,t)\,dx \\ &= 2u(x,t)u_x(x,t)|_{x=0}^{x=1} - 2\int_0^1 u_x^2(x,t)\,dx = -2\int_0^1 u_x^2(x,t)\,dx \le 0. \end{aligned}$$

It follows that $E(t) \equiv 0$ and so $u(x,t) \equiv 0$.

5. We seek solutions of the form $u(x,y) = X(x)Y(y)$.
Thus we require $X''Y + XY'' = 0$.
Hence it is sufficient to have

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -k$$

where k is a constant.

To ensure that $u(0,y) = u(2,y) = u(x,0) = 0$, we require that $X(0) = X(2) = 0$ and $Y(0) = 0$.
Thus it is sufficient to have

$$-X'' = kX; \qquad X(0) = 0 = X(2) \qquad (1)$$

$$Y'' = kY; \qquad Y(0) = 0 \qquad (2)$$

(1) has nonzero solutions if and only if $k = \frac{n^2\pi^2}{4}$ with corresponding solutions $X(x) = \sin(\frac{n\pi x}{2})$ for $n = 1, 2, \ldots$.
If $k = \frac{n^2\pi^2}{4}$, (2) has solution $Y(y) = A_n \sinh(\frac{n\pi y}{2})$.
Thus equation has solution

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{2}) \sinh(\frac{n\pi y}{2})$$

and we must choose the A_n's to ensure that $u(x,1) = \sin(\pi x)$.
Thus we require

$$\sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{2}) \sinh(\frac{n\pi}{2}) = \sin(\pi x)$$

and so we choose $A_n = 0$ for $n \neq 2$ and $A_2 \sinh\pi = 1$, i.e., $A_2 = \frac{1}{\sinh \pi}$.
Thus we have solution $u(x,y) = \frac{\sin(\pi x) \sinh(\pi y)}{\sinh(\pi)}$.

(b) $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$

We seek solutions of the form $u(r,\theta) = R(r)\Theta(\theta)$.

Clearly Θ must have period 2π.

Also we have a solution to the equation if

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

Thus it is sufficient to have that

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = k$$

where k is a constant.

Thus we require

$$r^2R'' + rR' - kR = 0 \qquad (1)$$

$$-\Theta'' = k\Theta; \quad \Theta \text{ has period } 2\pi. \qquad (2)$$

(2) has non-zero solutions if and only if $k = n^2$ for $n = 0, 1, 2, \ldots$; when $n = 0$ eigenfunction is a constant and when $n \geq 1$, eigenfunctions are $\sin(n\theta)$ and $\cos(n\theta)$.

When $k = n^2$, (1) becomes

$$r^2R'' + rR' - n^2R = 0 \; - \; \text{an Euler equation}$$

When $n = 0$, we have solutions 1 and $\ln r$.

When $n = 1$, we have solutions r^n and r^{-n}.

Since we require solutions to be bounded at $r = 0$, we do not make use of the solutions $\ln r$ or r^{-n}.

Thus we seek a solution of the form

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty}(\,A_n\cos(n\theta) + B_n\sin(n\theta)\,)r^n$$

such that $u(1,\theta) = \cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$.

Hence we choose $A_0 = \frac{1}{2}$, $A_2 = \frac{1}{2}$ and all the other coefficients $= 0$.

Thus we have solution

$$u(r,\theta) = \frac{1}{2}\,[\;1 + \cos(2\theta)r^2\;].$$