Question 1 (20 Marks)
A cone of semi-angle α has its axis vertical and vertex downwards, as in Figure 1 (overleaf). A point mass m slides without friction on the inside of the cone under the influence of gravity which acts along the negative z direction. The Lagrangian for the particle is

$$L(r, \theta, \dot{r}, \dot{\theta}) = \tfrac{1}{2}m\left(r^2\dot{\theta}^2 + \frac{\dot{r}^2}{\sin^2\alpha}\right) - \frac{mgr}{\tan\alpha},$$

where (r, θ) are plane polar coordinates as shown in Figure 1 (overleaf).

(a) Show that the generalized momenta p_r and p_θ corresponding to the coordinates r and θ, respectively, are given by

$$p_r = \frac{m\dot{r}}{\sin^2\alpha} \qquad \text{and} \qquad p_\theta = mr^2\dot{\theta}.$$

(b) Show that the Hamiltonian for this system is given by

$$H(r, \theta, p_r, p_\theta) = \frac{\sin^2\alpha}{2m}\,p_r^2 + \frac{p_\theta^2}{2mr^2} + \frac{mgr}{\tan\alpha}.$$

(c) Explain why the Hamiltonian H and generalized momentum p_θ are constants of the motion.

(d) In light of the information in part (c) above, we can express the Hamiltonian in the form

$$H(r, \theta, p_r, p_\theta) = \frac{\sin^2\alpha}{2m}\,p_r^2 + V(r),$$

where

$$V(r) = \frac{p_\theta^2}{2mr^2} + \frac{mgr}{\tan\alpha}.$$

In other words, we can now think of the system as a particle moving in a potential given by $V(r)$.

Sketch V as a function of r. Describe qualitatively the different dynamics for the particle you might expect to see.

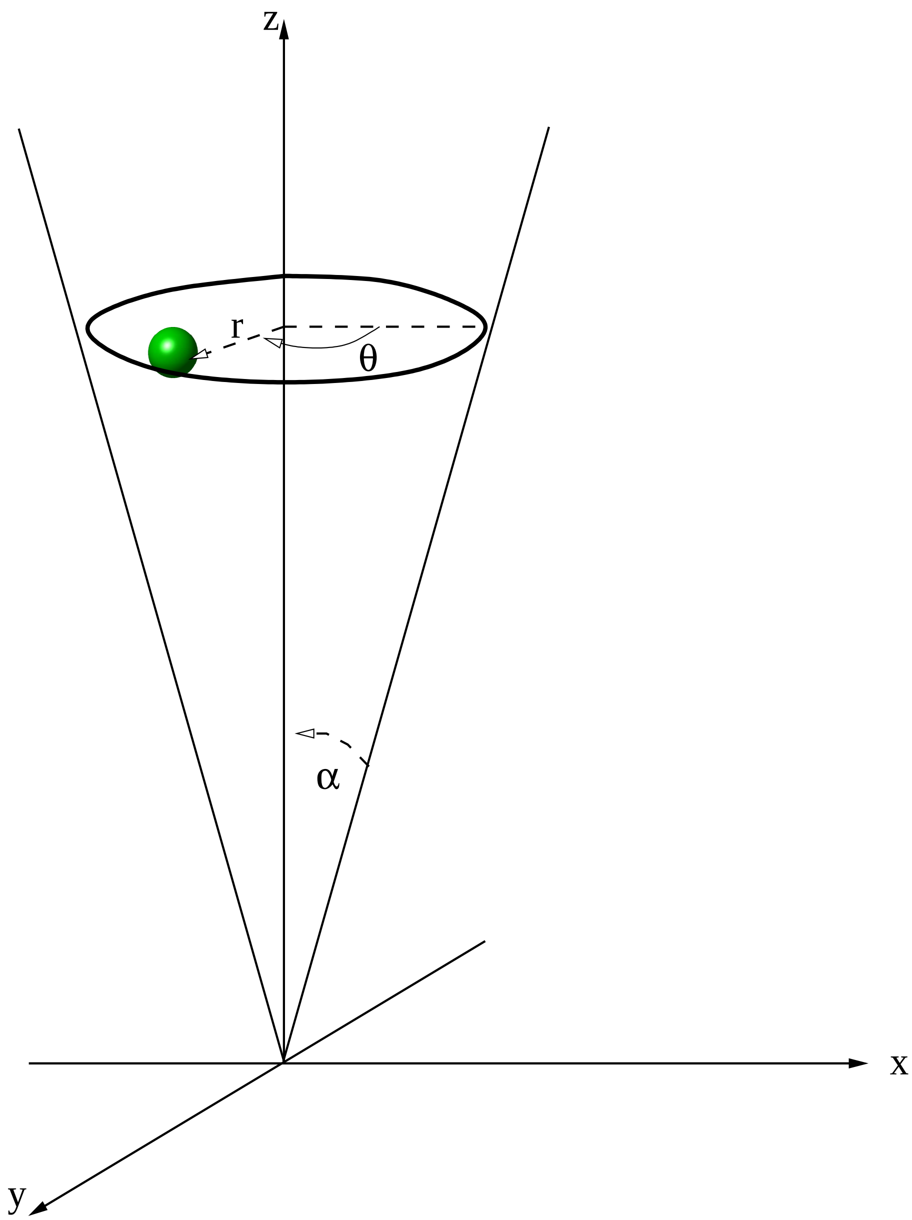

Figure 1: Particle sliding without friction inside a cone of semi-angle α, axis vertical and vertex downwards.

Question 2 (20 Marks)

(a) For a given velocity flow field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ prescribed at position $\boldsymbol{x}$ and time t, the particle trajectories are given by the solutions to the system of ordinary differential equations

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}(t), t).$$

Explain what particle trajectories are. Streamlines are given by the solutions to the system of ordinary differential equations (where t is fixed)

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = \boldsymbol{u}\big(\boldsymbol{x}(s), t\big).$$

Explain what streamlines are. For the two-dimensional flow in Cartesian coordinates given by

$$\begin{aligned} u &= u_0, \\ v &= v_0 \cos(kx - \alpha t), \end{aligned}$$

where u_0, v_0, k and α are constants, find the general equation for a streamline. Show that the streamline passing through $(x, y) = (0, 0)$ at $t = 0$ is

$$y = \frac{v_0}{k u_0} \sin(kx).$$

Find the equation for the path of a particle which is at $(x, y) = (0, 0)$ at $t = 0$.

(b) Euler's equations of motion for an ideal homogeneous incompressible fluid are

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= -\frac{1}{\rho_0} \nabla p + \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} &= 0, \end{aligned}$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ is the fluid velocity at position $\boldsymbol{x}$ and time t, ρ_0 is the uniform constant density, $p = p(\boldsymbol{x}, t)$ is the pressure, and $\boldsymbol{f}$ is the body force per unit mass.

Suppose that the flow is stationary so that

$$\frac{\partial \boldsymbol{u}}{\partial t} = 0,$$

and that the body force is conservative so that $\boldsymbol{f} = -\nabla \phi$ for some potential function $\phi = \phi(\boldsymbol{x})$. Using the identity

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \tfrac{1}{2} \nabla(|\boldsymbol{u}|^2) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}),$$

show from Euler's equations of motion that the Bernoulli quantity

$$H := \tfrac{1}{2} |\boldsymbol{u}|^2 + \frac{p}{\rho_0} + \phi$$

is constant along streamlines.

Question 3 (20 Marks)

(a) Using the Euler equations for an ideal incompressible flow in cylindrical coordinates (see formula sheet) show that at position (r, θ, z), for a flow which is independent of θ with $u_r = u_z = 0$, we have

$$\frac{u_\theta^2}{r} = \frac{1}{\rho_0}\frac{\partial p}{\partial r},$$
$$0 = \frac{1}{\rho_0}\frac{\partial p}{\partial z} + g,$$

where $p = p(r, z)$ is the pressure and g is the acceleration due to gravity (assume this to be the body force per unit mass). Verify that any such flow is indeed incompressible.

(b) In a *simple* model for a hurricane the air is taken to have uniform constant density ρ_0 and each fluid particle traverses a horizontal circle whose centre is on the fixed vertical z-axis. The (angular) speed u_θ at a distance r from the axis is

$$u_\theta = \begin{cases} \Omega r, & \text{for } 0 \le r \le a, \\ \Omega \frac{a^{3/2}}{r^{1/2}}, & \text{for } r > a, \end{cases}$$

where Ω and a are known constants.

(i) Now consider the flow given above in the inner region $0 \le r \le a$. Using the equations in part (a) above, show that the pressure in this region is given by

$$p = P_0 + \tfrac{1}{2}\rho_0\Omega^2 r^2 - g\rho_0 z,$$

where P_0 is a constant. A free surface of the fluid is one for which the pressure is constant. Show that the shape of a free surface for $0 \le r \le a$ is a paraboloid of revolution, i.e. it has the form

$$z = Ar^2 + B,$$

for some constants A and B. Specify the exact form of A and B.

(ii) Now consider the flow given above in the outer region $r > a$. Again using the equations in part (a) above, and that the pressure must be continuous at $r = a$, show that the pressure in this region is given by

$$p = P_0 - \frac{\rho_0}{r}\Omega^2 a^3 - g\rho_0 z + \tfrac{3}{2}\rho_0\Omega^2 a^2,$$

where P_0 is the same constant (reference pressure) as that in part (i) above.

Question 4 (20 Marks)

(a) Find the solution of the heat equation

$$u_t = k u_{xx}$$

where k is a known diffusion parameter, for $0 < x < L$, $t > 0$ subject to the boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0$$

for $t > 0$ and an initial condition

$$u(x,0) = \begin{cases} T_0, & \text{for } 0 \le x \le \frac{1}{2}L, \\ 0, & \text{for } \frac{1}{2}L < x \le L. \end{cases}$$

Explain briefly the physical situation represented by the equation above.

(b) Suppose $u = u(x,t)$ satisfies the heat equation

$$u_t = u_{xx}$$

for $0 \le x \le 1$ and $t > 0$, the initial condition

$$u(x,0) = 0$$

for $0 \le x \le 1$, and the boundary conditions

$$u(0,t) = u(1,t) = 0$$

for $t > 0$. Show, by considering the function

$$E(t) := \int_0^1 u^2(x,t)\,\mathrm{d}x,$$

that $u(x,t) \equiv 0$.

Question 5 (20 Marks)

(a) Solve Laplace's equation, $\nabla^2 u = 0$ for $u = u(x, y)$ on the rectangle $0 \leq x \leq 2,\ 0 \leq y \leq 1$ subject to the boundary conditions

$$u(x, 0) = 0 \qquad \text{and} \qquad u(x, 1) = \sin x$$

for $0 \leq x \leq 2$, and

$$u(0, y) = u(2, y) = 0$$

for $0 \leq y \leq 1$.

(b) The steady state temperature $u(r, \theta)$ of a plate in the shape of the circle, with centre the origin and radius equal to 1, satisfies the differential equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

where r and θ denote plane polar coordinates. If the steady state solution satisfies the boundary condition that

$$u(1, \theta) = \cos^2 \theta,$$

find the solution to the differential equation $u = u(r, \theta)$.

Formulae

Euler's equations for an ideal homogeneous incompressible fluid in cylindrical coordinates (r, θ, z) with the velocity field expressed as $\boldsymbol{u} = (u_r, u_\theta, u_z)$ are

$$\begin{aligned}
\frac{\partial u_r}{\partial t} + (\boldsymbol{u} \cdot \nabla) u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho_0} \frac{\partial p}{\partial r} + f_r, \\
\frac{\partial u_\theta}{\partial t} + (\boldsymbol{u} \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho_0 r} \frac{\partial p}{\partial \theta} + f_\theta, \\
\frac{\partial u_z}{\partial t} + (\boldsymbol{u} \cdot \nabla) u_z &= -\frac{1}{\rho_0} \frac{\partial p}{\partial z} + f_z,
\end{aligned}$$

where $p = p(r, \theta, z, t)$ is the pressure, ρ_0 is the uniform constant density and $\boldsymbol{f} = (f_r, f_\theta, f_z)$ is the body force per unit mass. Here we also have

$$\boldsymbol{u} \cdot \nabla = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}.$$

Further the incompressibility condition $\nabla \cdot \boldsymbol{u} = 0$ is given in cylindrical coordinates by

$$\frac{1}{r} \frac{\partial (r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0.$$