Differential Equations and Linear Algebra
Supplementary Notes

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Contents

CHAPTER 1

Linear algebraic equations

1.1. Gaussian elimination in practice

1.1.1. Operation counts. Which method should we use, Gaussian elimination or the Gauss-Jordan method? The answer lies in the *efficiency* of the respective methods when solving large systems.

Gaussian elimination. Gaussian elimination with back-substitution applied to an $n \times n$ system requires

$$\left.\begin{aligned} &\frac{n^3}{3} + n^2 - \frac{n}{3} \quad \text{multiplications/divisions} \\ &\frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6} \quad \text{additions/subtractions} \end{aligned}\right\} \quad \Rightarrow \quad \frac{2n^3}{3} + \cdots \quad \text{flops.}$$

For large values of n, i.e. large systems, the $2n^3/3$ term dominates and we would normally quote that Gaussian elimination with back substitution for an $n \times n$ system requires $\sim 2n^3/3$ flops (flops $\equiv$ floating point operations).

Gaussian-Jordan. Gaussian-Jordan applied to an $n \times n$ system requires

$$\left.\begin{aligned} &\frac{n^3}{2} + \frac{n^2}{2} \quad \text{multiplications/divisions} \\ &\frac{n^3}{2} - \frac{n}{2} \quad \text{additions/subtractions} \end{aligned}\right\} \quad \Rightarrow \quad n^3 + \cdots \quad \text{flops.}$$

For large values of n Gaussian-Jordan requires $\sim n^3$ flops.

Hence Gauss-Jordan requires about 50% more effort than Gaussian elimination and this difference becomes significant when n is large. For example if $n = 100$, then $2n^3/3 \approx 666,666$ while $n^3 = 1,000,000$ which amounts to $333,333$ extra flops—see Meyer [**10**].

Thus Gauss-Jordan is not recommended for solving linear systems of equations that arise in practical situations, though it does have theoretical advantages—for example for finding the inverse of a matrix.

1.1.2. Finite accuracy computer arithmetic. When we implement Gaussian elimination on a computer, in practice for large systems of equations, we need to contend with the fact that we can only perform calculations to a given finite accuracy (modern algebraic computing packages can perform exact integer arithmetic but become impractical for large systems). Here we will simply present the main issues associated with finite accuracy arithmetic, providing some cursory examples (borrowed from Meyer [**10**]) demonstrating their consequences and give practical solutions for avoiding them. For more details and illustrative examples see the excellent book by Meyer [**10**].

1.1.3. Example. Consider the following system:

$$\begin{aligned} 47x + 28y &= 19\,, \\ 89x + 53y &= 36\,. \end{aligned}$$

The exact solution is

$$x = 1 \quad \text{and} \quad y = -1\,.$$

However using three digit arithmetic you get

$$x = -0.191 \quad \text{and} \quad y = 1.00\,.$$

1.1.4. Partial pivoting. In this last example, rounding errors are responsible for the anomalous answer, and though such errors can never be completely eliminated, there are some practical strategies to help minimize these errors.

Partial pivoting. The idea is to maximize the magnitude of the pivot at each step using only row interchanges. More precisely, at each step we interchange (if necessary) the pivot row with any row below it to ensure that the pivot has the maximum possible magnitude compared with the coefficients in the column below its position. The fourth step in a typical case would be:

$$\left(\begin{array}{cccccc|c} * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & \textcircled{\text{P}} & * & * & * \\ 0 & 0 & 0 & \text{P} & * & * & * \\ 0 & 0 & 0 & \text{P} & * & * & * \end{array}\right)$$

Search the positions in the fourth column and then interchange the rows so that the coefficient in the circled position is the largest in magnitude of all the terms labelled “P”.

Note. When partial pivoting is used, no multiplier ever exceeds 1 in magnitude—and the possibility that large numbers can "swamp" small ones in finite digit arithmetic is significantly reduced but not entirely eliminated.

1.1.5. Example (two-scale system). The exact solution to the system of equations

$$
\begin{aligned}
-10x + 10^5 y &= 10^5\,,\\
x + y &= 2\,,
\end{aligned}
$$

is

$$
x = \frac{1}{1.0001} \qquad \text{and} \qquad y = \frac{1.0002}{1.0001}\,.
$$

Using three digit accuracy and partial pivoting the approximate solution is

$$
x = 0 \qquad \text{and} \qquad y = 1\,.
$$

1.1.6. Scaling. In this last example the differing orders of magnitude of the coefficients means that the first equation applies on a different *scale* to the second, and this produced the anomalous approximate solution. A sensible strategy would be to rescale the system before we proceed to solve it.

The idea is to rescale the above system so that the coefficient of maximum possible magnitude is 1. For example if we multiply the first equation in the last example by 10^{-5}, we recover the example system we saw in the partial pivoting section, which we were very adequately able to solve using partial pivoting.

Hence the second refinement need for successful Gaussian elimination is a scaling strategy: which in fact combines both *row scaling*—multiplying selected rows by non-zero multipliers—with *column scaling*—multiplying selected columns of the coefficient matrix A by non-zero multipliers.

Note. Row scaling does not change the exact solution, but column scaling does! Column scaling the kth column is equivalent to changing the units of the kth unknown.

Practical scaling strategy.

- Always choose units that are natural to the problem so that there is not a distorted relationship between the sizes of physical parameters within a given problem.
- Row scale the system of linear equations so that the maximum magnitude of the coefficient in each row is 1, i.e. divide each row by the coefficient (in that row) of maximum magnitude.

Partial pivoting and this scaling strategy makes Gaussian elimination with back substitution a proven extremely reliable and effective tool for practical systems of linear equations.

1.1.7. Ill–conditioned systems. This example demonstrates that there are some systems that are very sensitive to small perturbations and why care must always be taken when numerically solving equations. Consider the system

$$.835x + .667y = .168\,, \tag{1.1a}$$

$$.333x + .266y = b_2\,. \tag{1.1b}$$

First let's suppose that $b_2 = .067$, then the exact solution is

$$x = 1. \qquad \text{and} \qquad y = -1. \qquad \text{(exact values)}.$$

Let's perturb b_2 slightly to $b_2 = .066$. Now the exact solution is

$$x = -666. \qquad \text{and} \qquad y = 834. \qquad \text{(exact values)}.$$

This is an example of an ill–conditioned system. It's sensitivity to small perturbations is intrinsic to the system itself, rather than the result of any numerical procedure.

> *Ill-conditioned systems:* A system of equations is said to be *ill–conditioned* when some small perturbation in the system can produce relatively large changes in the exact solution. Otherwise the system is said to be *well–conditioned*.

Let's examine what's happening here from a geometric perspective. The equations (1.1) above represent two straight lines (with $b_2 = .067$):

$$y = -1.25187\,x + .251874\,, \tag{1.2a}$$

$$y = -1.25188\,x + .25188\,. \tag{1.2b}$$

Notice that these two lines have very similar slopes and intercepts—they are extremely close to each other. They do intersect at the point $x = 1.$ and $y = -1.$ when $b_2 = .067$. However it is easy to imagine how if we were to perturb one of the lines slightly, the point of intersection (solution) could change quite dramatically (as we have seen).

1.2. Exercises

1.1. Consider the following system:

$$10^{-3}x - y = 1\,,$$

$$x + y = 0\,.$$

(a) Use 3-digit arithmetic without pivoting to solve this system.

(b) Find a system that is exactly satisfied by your solution from part (a), and note how close the system is to the original system.

(c) Now use partial pivoting and 3-digit arithmetic to solve the original system.

(d) Find a system that is exactly satisfied by your solution from part (c), and note how close this system is to the original system.

(e) Use exact arithmetic to obtain the solution to the original system, and compare the exact solution with the results of parts (a) and (c).

(f) Round the exact solution to three significant digits, and compare the result with those of parts (a) and (c).

1.2. Determine the exact solution of the following system:

$$\begin{aligned} 8x + \ 5y + \ 2z &= 15\,, \\ 21x + 19y + 16z &= 56\,, \\ 39x + 48y + 53z &= 140\,. \end{aligned}$$

Now change 15 to 14 in the first equation and again solve the system with exact arithmetic. Is the system ill-conditioned?

1.3. Using geometric considerations, rank the following three systems according to their condition.

(a)

$$\begin{aligned} 1.001x - y &= .235\,, \\ x + .0001y &= .765\,; \end{aligned}$$

(b)

$$\begin{aligned} 1.001x - y &= .235\,, \\ x + .9999y &= .765\,; \end{aligned}$$

(c)

$$\begin{aligned} 1.001x + y &= .235\,, \\ x + .9999y &= .765\,. \end{aligned}$$

1.4. Use Gaussian elimination, with partial pivoting, to solve the following systems of equations. Work to five decimal places.

$$\text{(a)}\quad \begin{aligned} x - \ y + 2z &= -2\,, \\ 3x - 2y + 4z &= -5\,, \\ 2y - 3z &= 2\,, \end{aligned} \qquad \text{(b)}\quad \begin{aligned} 33x + 16y + 72z &= 359\,, \\ -24x - 10y - 57z &= 281\,, \\ -8x - \ 4y - 17z &= \ 85\,. \end{aligned}$$

APPENDIX A

A.1. Euler's formula

For any real angle θ measured in radians,

$$e^{i\theta} = \cos\theta + i\sin\theta\,.$$

To prove this we use the series expansion definition for e^z:

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\,. \end{aligned}$$

This series is known to converge for all real and complex z. Let $z = i\theta$, with θ real, and noting that $i^2 = -1$, $i^3 = -i$,$i^4 = 1$, $i^5 = i$, …etc. , then for every real θ:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \\ &= \cos\theta + i\sin\theta\,. \end{aligned}$$

We have used that the Taylor series expansion for a given function $f(\theta)$ about $\theta = a$ is unique (in this case $\cos\theta$ and $\sin\theta$ about $\theta = 0$).

A.2. Electrical circuits and networks

A.2.1. Simple electrical circuit. Consider the flow of an electrical current $I(t)$ in a simple series circuit as shown in Figure A.1. The resistor has resistance R *Ohms*, the capacitor has capacitance C *Farads* and the inductor has inductance ℓ *Henrys* which are positive constants. A battery or power source provides an impressed voltage of $V(t)$ *Volts* at any given time. We have the following relations:

- The rate of change of total charge $Q(t)$ *Coulombs* in the capacitor at time t, is the current,
$$I(t) = \frac{\mathrm{d}Q}{\mathrm{d}t}\,.$$
- The voltage drop across the capacitor is Q/C, whilst the voltage drop across the inductor is $\ell\dfrac{\mathrm{d}I}{\mathrm{d}t}$.
- *Ohm's law*: The voltage drop V across any resistor of resistance R Ohms is given by $V = IR$, where I (measured in Amps) is the current passing through the resistor.
- *Kirchhoff's Voltage law:* in a closed circuit, the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit.

Combining all these relations, we see that for the circuit in Figure A.1,
$$\ell\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{C}Q = V(t)\,. \tag{A.1}$$
'Feedback-squeals' in electric circuits at concerts are an example of resonance effects in such equations. Note also that, from Kirchhoff's law, there is a nontrivial relationship between the initial values: $I'(0) = (V_0 - RI_0 - Q_0/C)/\ell$.

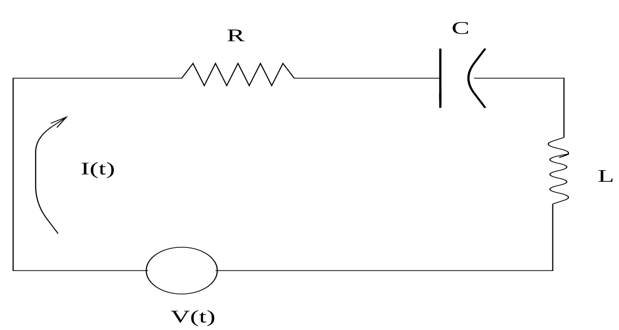

FIGURE A.1. Simple electrical circuit.

A.2.2. Electrical networks. Consider the following laws for electrical circuits.

- *Kirchhoff's node rule*: At any point of a circuit (including the battery), the sum of the inflowing currents equals the sum of the outgoing currents.
- *Kirchhoff's loop rule*: In any closed loop, the sum of the voltage drops equals the impressed electromotive force.

Combining these laws together with Ohm's Law from section A.2.1, we can write down the system of linear algebraic equations for the currents in various parts of the circuit shown in the Figure A.2:

$$\begin{aligned}
&\text{Node P:} && I_1 - I_2 + I_3 &&= 0\,,\\
&\text{Node Q:} && -I_1 + I_2 - I_3 &&= 0\,,\\
&\text{Right loop:} && 10I_2 + 25I_3 &&= 90\,,\\
&\text{Left loop:} && 20I_1 + 10I_2 &&= 80\,.
\end{aligned} \tag{A.2}$$

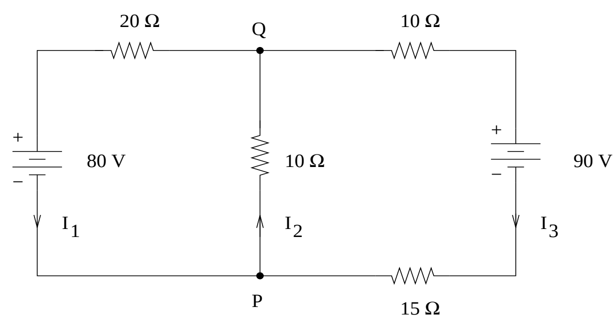

Figure A.2. Simple electrical network.

A.3. Determinants and inverse matrices

A.3.1. Identity matrix. A *square matrix* is one in which the number of rows and columns are equal. An *identity matrix* is a square matrix whose elements are all zero except those on the *leading diagonal* which are unity $(= 1)$. The leading diagonal is the one from the top left to the bottom right. The identity matrix is denoted by I (or I_n to show that it has size $n \times n$). For example

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

The identity matrix has the property that

$$AI = IA = A$$

for any matrix A.

A.3.2. Inverse matrices. If A is a square matrix, then provided it exists, the inverse matrix of A, denoted A^{-1}, has the property that

$$A^{-1}A = AA^{-1} = I \,.$$

It is important to note that not every square matrix has an inverse. If we can find A^{-1} then we can solve the system

$$A\boldsymbol{x} = \boldsymbol{b}$$

because

$$\begin{aligned} & & A\boldsymbol{x} &= \boldsymbol{b} \\ \Leftrightarrow & & A^{-1}A\boldsymbol{x} &= A^{-1}\boldsymbol{b} \\ \Leftrightarrow & & I\boldsymbol{x} &= A^{-1}\boldsymbol{b} \\ \Leftrightarrow & & \boldsymbol{x} &= A^{-1}\boldsymbol{b} \,. \end{aligned}$$

Thus the solution is simply $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

A.3.3. Determinants. Recall our discussion of 2×2 systems. We found that the determinant $D = a_{11}a_{22} - a_{12}a_{21}$ was significant. We found that there was a unique solution provided $D \neq 0$. If $D \neq 0$ the matrix A has an inverse A^{-1} given by

$$A^{-1} = \frac{1}{D} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} .$$

If $D = 0$, the matrix A has no inverse.

A.3.4. Example.

$$\begin{aligned} & A = \begin{pmatrix} -1 & 5 \\ 2 & 3 \end{pmatrix} \\ \Leftrightarrow \quad & \det A = -13 \neq 0 \\ \Leftrightarrow \quad & A^{-1} = -\frac{1}{13}\begin{pmatrix} 3 & -5 \\ -2 & -1 \end{pmatrix} . \end{aligned}$$

We can check that this is indeed the inverse by testing for the properties of an inverse matrix:

$$\begin{aligned} AA^{-1} &= -\frac{1}{13}\begin{pmatrix} -1 & 5 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -2 & -1 \end{pmatrix} \\ &= -\frac{1}{13}\begin{pmatrix} -3-10 & 5-5 \\ 6-6 & -10-3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \end{aligned}$$

These ideas can be extended from 2×2 matrices to 3×3, $\dots$, $n \times n$ etc. Again the determinant will dictate whether the matrix is *singular* (has no inverse) or *nonsingular* (has an inverse). Given a 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} ,$$

we define its determinant to be

$$\det A = a_{11} \cdot \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \cdot \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \cdot \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} .$$

A.3.5. Example.

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix} = 1 \cdot \det \begin{pmatrix} 3 & 5 \\ 5 & 12 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 1 & 5 \\ 1 & 12 \end{pmatrix} + 3 \cdot \det \begin{pmatrix} 1 & 3 \\ 1 & 5 \end{pmatrix} = 3 .$$

The value of $\det A$ may be obtained by expanding about *any* row or column (not just the first row as we have done) using the *chessboard* pattern of signs and the “cover up” method.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

A.3.6. Example. Let $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{pmatrix}$. Here are three of the possible ways to calculate $\det A$:

- Expand about the first row to give

$$\det A = +1 \det \begin{pmatrix} 6 & 4 \\ 0 & 2 \end{pmatrix} - 3 \det \begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix} + 0 \det \begin{pmatrix} 2 & 6 \\ -1 & 0 \end{pmatrix} = -12\,.$$

- Expand about the middle row to give

$$\det A = -2 \det \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} + 6 \det \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} - 4 \det \begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix} = -12\,.$$

- Expand about the last column to give

$$\det A = 0 \det \begin{pmatrix} 2 & 6 \\ -1 & 0 \end{pmatrix} - 4 \det \begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix} + 2 \det \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} = -12\,.$$

A.4. Matrix transpose

The *transpose* A^T of the $m \times n$ matrix A is the $n \times m$ matrix obtained by interchanging rows and columns.

A.4.1. Example. If

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

then their transposes are

$$A^T = \begin{pmatrix} 3 & 4 & 7 \\ 2 & 5 & 8 \\ 1 & 6 & 9 \end{pmatrix}, \quad B^T = \begin{pmatrix} 3 & 4 \\ 2 & 5 \\ 1 & 6 \end{pmatrix} \quad \text{and} \quad C^T = \begin{pmatrix} 1 & 4 \end{pmatrix}.$$

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