Chapter 7 (Systems of differential equations): Solutions

Solution 7.1.

The eigenvalues are the solutions to the characteristic equation

$$\det(A - \lambda I) = 0$$

$$\Longleftrightarrow \det\begin{pmatrix} -4-\lambda & 3 \\ 3 & -4-\lambda \end{pmatrix} = 0$$

$$\Longleftrightarrow (\lambda+4)^2 - 9 = 0$$

$$\Longleftrightarrow \lambda^2 + 8\lambda + 7 = 0$$

$$\Longleftrightarrow (\lambda+1)(\lambda+7) = 0.$$

Hence the eigenvalues are

$$\lambda_1 = -1,\ \lambda_2 = -7.$$

For $\lambda_1 = -1$, the eigenvector solves

$$\begin{pmatrix} -4+1 & 3 \\ 3 & -4+1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow c_2 = c_1.$$

Hence the eigenvector corresponding to $\lambda_1 = -1$ is, for any $\alpha \neq 0$:

$$C_1 = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = -7$, the eigenvector solves

$$\begin{pmatrix} -4+7 & 3 \\ 3 & -4+7 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow c_2 = -c_1.$$

Hence the eigenvector corresponding to $\lambda_2 = -7$ is, for any $\beta \neq 0$:

$$C_2 = \beta\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Looking for a solution of the form

$$Y(t) = C e^{\lambda t}$$

to

$$Y''(t) = A\,Y$$

gives

$$\Longrightarrow\quad (\lambda)^2\, C\, e^{\lambda t} = A\, C\, e^{\lambda t}$$

$$\Longleftrightarrow\quad (A - \lambda^2 I)\, C = O.$$

This is the eigenvalue problem but λ^2 with instead of λ, i.e.

$$\lambda_1^2 = -1,\ \lambda_2^2 = -7.$$

If we set

$$\lambda = i\,\omega \quad\Longleftrightarrow\quad \lambda^2 = -\omega^2$$

then for $\lambda_1^2 = -1$ we have

$$\omega_1 = \pm 1,$$

and so one natural frequency of oscillation is $\omega_1 = +1$.

For $\lambda_2^2 = -7$ we have

$$\omega_2 = \pm\sqrt{7},$$

and so another natural frequency of oscillation is $\omega_2 = +\sqrt{7}$.

The corresponding modes of oscillation are

$\omega_1 = +1$: ●→ ●→

$\omega_2 = +\sqrt{7}$: ←● ●→

The general solution is

$$Y(t) = (a_1 \cos(t) + b_1 \sin(t)) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$+ \left(a_2 \cos\left(\sqrt{7}\, t\right) + b_2 \sin\left(\sqrt{7}\, t\right)\right) \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

where a_1, b_1, a_2 and b_2 are arbitrary constants (fixed by the initial data).

■ Solution 7.2.

(a) The eigenvalues are the solutions to the characteristic equation

$$\det\begin{pmatrix} -1-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{pmatrix} = 0$$

$$\Longleftrightarrow\ -(1+\lambda)\,((2+\lambda)(1+\lambda)-1) - 1\,(-1\,(1+\lambda)) = 0$$

$$\Longleftrightarrow\ -(1+\lambda)\,((2+\lambda)(1+\lambda)-1-1) = 0$$

$$\Longleftrightarrow\ -(1+\lambda)\,(\lambda^2+3\lambda) = 0$$

$$\Longleftrightarrow\ \lambda\,(1+\lambda)\,(\lambda+3) = 0.$$

Hence the eigenvalues are

$$\lambda = 0,\ \lambda = -1,\ \lambda = -3.$$

For $\lambda=0$, the eigenvector solves

$$(A-\lambda I)\,X = 0 \ \Longleftrightarrow\ \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix is

$$H := \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

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H = H /. H[[2]] → H[[2]] + H[[1]];
H // MatrixForm
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$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

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H = H /. H[[3]] → H[[3]] + H[[2]];
H // MatrixForm
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$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence

$$x_2 = x_3,\ x_1 = x_3$$

and the eigenvector corresponding to $\lambda=0$ is, for any $\alpha\neq0$:

$$X = \alpha\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda=-1$, the eigenvector solves

$$(A-\lambda I)\,X = 0 \ \Longleftrightarrow\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix is

$$H := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The first and third equations immediately imply

$$x_2 = 0,$$

while the second equation gives

$$x_1 = -x_3$$

Hence the eigenvector corresponding to $\lambda=-1$ is, for any $\beta\neq 0$:

$$X = \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda=-3$, the eigenvector solves

$$(A - \lambda I)\, X = 0 \iff \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix is

$$H := \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

```
H = H /. H[[2]] → 2 H[[2]] - H[[1]];
H // MatrixForm
```

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

```
H = H /. H[[3]] → H[[3]] - H[[2]];
H // MatrixForm
```

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence

$$x_2 = -2\, x_3, \quad x_1 = x_3$$

and the eigenvector corresponding to $\lambda=-3$ is, for any $\gamma\neq 0$:

$$X = \gamma \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

(b) Looking for a solution of the form

$$Y(t) = C\, e^{\lambda t}$$

to

$$\mathbf{Y}''(t) = \mathbf{A}\,\mathbf{Y}$$

gives

$$\Longrightarrow \; (\lambda)^2\, \mathbf{C}\, e^{\lambda t} = \mathbf{A}\,\mathbf{C}\, e^{\lambda t}$$

$$\Longleftrightarrow \; (\mathbf{A} - \lambda^2\, \mathbf{I})\, \mathbf{C} = \mathbf{0}.$$

(c) From part (a) we know

$$\lambda_1^2 = 0, \; \lambda_2^2 = -1, \; \lambda_3^2 = -3.$$

If we set

$$\lambda = i\,\omega \; \Longleftrightarrow \; \lambda^2 = -\omega^2$$

then for $\lambda_1^2 = 0$ we have

$$\omega_1 = 0 \quad (\text{twice}),$$

which doesn't correspond to any oscillation at all but only translation and uniform velocity invariance.

For $\lambda_2^2 = -1$ we have

$$\omega_2 = \pm 1,$$

and so one natural frequency of oscillation is $\omega_2 = +1$.

For $\lambda_3^2 = -3$ we have

$$\omega_3 = \pm\sqrt{3},$$

and so the other natural frequency of oscillation is $\omega_3 = +\sqrt{3}$.

(d)

$\omega_1 = 0$: ● → ● → ● →

$\omega_2 = 1$: ← ● ● ● →

$\omega_3 = \sqrt{3}$: ● → ←— ● ● →

(e) The general solution is

$$\mathbf{Y}(t) = (a_1 + b_1\, t) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (a_2 \cos(t) + b_2 \sin(t)) \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$+ \left(a_3 \cos\left(\sqrt{3}\, t\right) + b_3 \sin\left(\sqrt{3}\, t\right)\right) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

where a_1, b_1, a_2, b_2, a_3 and b_3 are arbitrary constants (fixed by the initial data).