Final exam 2006: solutions

Solution 1 (10 marks)

(a)

$$\omega_0 = +\sqrt{7}.$$

((3 marks))

(b)

$$y_{PI} = A\sin(3t) + B\cos(3t).$$

((3 marks))

(c) This is a resonant case; the frequency of the oscillatory forcing matches the natural frequency of unforced oscillations of the system ($\omega_0 = \sqrt{7}$), hence try

$$y_{PI} = A\,t\sin\left(\sqrt{7}\,t\right) + B\,t\cos\left(\sqrt{7}\,t\right).$$

((4 marks))

Solution 2 (15 marks)

The auxillary equation is

$$\lambda^2 - 5\lambda + 4 = 0$$

$$\Leftrightarrow \lambda = 4, \lambda = 1$$

$$\Longrightarrow y_{CF} = C_1 e^{4t} + C_2 e^{t}.$$

((3 marks))

The inhomogeneity in the ODE suggests we try

$$y_{PI} = A e^{-2t}$$

$$\Longrightarrow y_{PI}{}' = -2 A e^{-2t}$$

$$\Longrightarrow y_{PI}{}'' = 4 A e^{-2t}.$$

Substituting our guess for the PI into the ODE

$$\Longrightarrow 4 A e^{-2t} - 5(-2 A e^{-2t}) + 4(A e^{-2t}) = 3 e^{-2t}$$

$$\Leftrightarrow 4A + 10A + 4A = 3$$

$$\Leftrightarrow A = \frac{1}{6}.$$

Hence the general solution to the ODE is

$$y = y_{CF} + y_{PI}$$

$$\Longrightarrow \; y = C_1 e^{4t} + C_2 e^{t} + \frac{1}{6} e^{-2t}.$$

((6 marks))

Now we substitute in the initial conditions. First using that

$$y(0) = 1,$$

$$\Longrightarrow \; 1 = C_1 e^{0} + C_2 e^{0} + \frac{1}{6} e^{0}$$

$$\Longrightarrow \; 1 = C_1 + C_2 + \frac{1}{6}$$

$$\Longleftrightarrow \; C_1 + C_2 = \frac{5}{6}.$$

Second, after differentiating the general solution with respect to t:

$$\Longrightarrow \; y' = 4 C_1 e^{4t} + C_2 e^{t} - \frac{1}{3} e^{-2t},$$

we can use the second initial condition

$$y'(0) = 0,$$

$$\Longrightarrow \; 0 = 4 C_1 e^{0} + C_2 e^{0} - \frac{1}{3} e^{0}$$

$$\Longleftrightarrow \; 0 = 4 C_1 + C_2 - \frac{1}{3}$$

$$\Longleftrightarrow \; 4 C_1 + C_2 = \frac{1}{3}.$$

Solving the two simultaneous equations for C_1 and C_2 we see that

$$C_1 = -\frac{1}{6}, \quad C_2 = 1.$$

Hence the solution to the initial value problem is

$$y = -\frac{1}{6} e^{4t} + e^{t} + \frac{1}{6} e^{-2t}.$$

((6 marks))

■ Solution 3 (10 marks)

The auxillary equation is

$$\lambda^2 + 6\lambda + 9 = 0$$

$$\lambda = -3,\ \lambda = -3$$

$$\Longrightarrow\ y_{CF} = C_1\, e^{-3t} + C_2\, t\, e^{-3t}.$$

((**3 marks**))

The inhomogeneity in the ODE suggests we try

$$y_{PI} = A\, t^2\, e^{-3t}$$

$$\Longrightarrow\ y_{PI}{}' = 2\, A\, t\, e^{-3t} - 3\, A\, t^2\, e^{-3t}$$

$$\Longrightarrow\ y_{PI}{}'' = 2\, A\, e^{-3t} - 12\, A\, t\, e^{-3t} + 9\, A\, t^2\, e^{-3t}.$$

Substituting our guess for the PI into the ODE and dividing through by the exponential term

$$\Longrightarrow\ 2\,A - 12\,A\,t + 9\,A\,t^2 + 6\,(2\,A\,t - 3\,A\,t^2) + 9\,(A\,t^2) = 5$$

$$\Longleftrightarrow\ A = \frac{5}{2}.$$

Hence the general solution to the ODE is

$$y = y_{CF} + y_{PI}$$

$$\Longrightarrow\ y = C_1\, e^{-3t} + C_2\, t\, e^{-3t} + \frac{5}{2}\, t^2\, e^{-3t}.$$

((**7 marks**))

■ Solution 4 (10 marks)

(a) Since from the table

$$\mathbb{L}[\sin(3\,t)] = \frac{3}{s^2 + 9},$$

the shift theorem (also in the table)

$$\Longrightarrow\ \mathbb{L}[e^{-8t} \sin(3\,t)] = \frac{3}{(s+8)^2 + 9}.$$

((**4 marks**))

(b) Factorize the denominator

$$\frac{1}{s^2 + 8\,s + 16} = \frac{1}{(s+4)^2}.$$

Since from the table

$$\mathbb{L}[t] = \frac{1}{s^2}$$

and then using the Shift Theorem

$$\mathbb{L}[e^{-4t}\, t] = \frac{1}{(s+4)^2}.$$

Hence

$$\mathbb{L}^{-1}\left[\frac{1}{s^2+8s+16}\right] = e^{-4t}\, t.$$

((**6 marks**))

■ Solution 5 (15 marks)

Taking the Laplace transform of both sides of the ODE

$$\Longrightarrow \quad \mathbb{L}[y''(t) + 3y'(t) + 2y(t)] = \mathbb{L}[\delta(t-4)]$$

$$\Longleftrightarrow \quad s^2\,\bar{y}(s) - s\,y(0) - y'(0) + 3(s\,\bar{y}(s) - y(0)) + 2\,\bar{y}(s) = e^{-4s}$$

$$\Longleftrightarrow \quad (s^2 + 3s + 2)\,\bar{y}(s) = e^{-4s}$$

$$\Longleftrightarrow \quad \bar{y}(s) = \frac{e^{-4s}}{s^2+3s+2}$$

$$\Longleftrightarrow \quad \bar{y}(s) = \frac{e^{-4s}}{(s+1)(s+2)}.$$

((**5 marks**))

Using partial fractions, we can expand

$$\frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$\Longleftrightarrow \quad 1 = A(s+2) + B(s+1)$$

$$\Longleftrightarrow \quad 1 = (A+B)\,s + 2A + B.$$

Equating coefficients

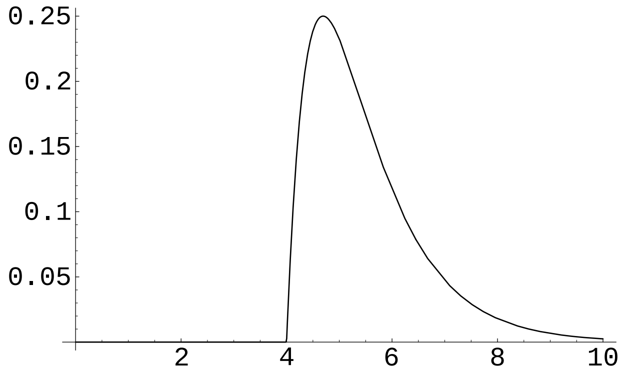
$$\begin{cases} s^0: & 1 = 2A + B, \\ s^1: & 0 = A + B. \end{cases} \quad \Longrightarrow \quad A = 1, \;\; B = -1.$$

((**5 marks**))

$$\Longrightarrow \quad \bar{y}(s) = e^{-4s}\left(\frac{1}{(s+1)} - \frac{1}{(s+2)}\right)$$

$$\Longleftrightarrow \quad y(t) = \begin{cases} e^{-(t-4)} - e^{-2(t-4)}, & t > 4, \\ 0, & t \le 4. \end{cases}$$

((**4 marks**))



- Graphics -

((1 mark))

■ Solution 6 (10 marks)

The augmented matrix is

$$H := \begin{pmatrix} 2 & 0 & 1/2 & 8 \\ 0 & 2 & 1/2 & 16 \\ 1 & 1 & -1 & 0 \end{pmatrix};$$

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H = H /. H[[3]] → 2 H[[3]] - H[[1]];
H // MatrixForm
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$$\begin{pmatrix} 2 & 0 & \frac{1}{2} & 8 \\ 0 & 2 & \frac{1}{2} & 16 \\ 0 & 2 & -\frac{5}{2} & -8 \end{pmatrix}$$

((4 marks))

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H = H /. H[[3]] → H[[3]] - H[[2]];
H // MatrixForm
```

$$\begin{pmatrix} 2 & 0 & \frac{1}{2} & 8 \\ 0 & 2 & \frac{1}{2} & 16 \\ 0 & 0 & -3 & -24 \end{pmatrix}$$

((3 marks))

Hence using back−substitution, the solution is

$I_1 = 2$, $I_2 = 6$, $I_3 = 8$.

((3 marks))

■ Solution 7 (30 marks)

(a) The eigenvalues are the solutions to the characteristic equation

$$\det\begin{pmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{pmatrix} = 0$$

$$\Longleftrightarrow\ -(2+\lambda)\,((2+\lambda)\,(2+\lambda)-1) - 1\,(-1\,(2+\lambda)) = 0$$

$$\Longleftrightarrow\ -(2+\lambda)\,((2+\lambda)\,(2+\lambda)-1-1) = 0$$

$$\Longleftrightarrow\ -(2+\lambda)\,(\lambda^2+4\,\lambda+2) = 0$$

$$\Longleftrightarrow\ (2+\lambda)\,(\lambda+4\,\lambda+2) = 0.$$

Hence the eigenvalues are

$$\lambda = -2,\ \lambda = -2-\sqrt{2},\ \lambda = -2+\sqrt{2}.$$

((**3 marks**))

For $\lambda=-2$, the eigenvector solves

$$(A-\lambda\,I)\,x = 0$$

$$\Longleftrightarrow\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first and third equations immediately imply

$$x_2 = 0,$$

while the second equation implies

$$x_1 = -x_3$$

Hence the eigenvector corresponding to $\lambda=-2$ is, for any $\alpha\neq 0$:

((**4 marks**))

$$x = \alpha\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda=-2-\sqrt{2}$, the eigenvector solves

$$(A-\lambda\,I)\,x = 0$$

$$\Longleftrightarrow\ \begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix is

$$H := \begin{pmatrix} \sqrt{2} & 1 & 0 & 0 \\ 1 & \sqrt{2} & 1 & 0 \\ 0 & 1 & \sqrt{2} & 0 \end{pmatrix}$$

```
H = H /. H[[2]] → √2 H[[2]] - H[[1]];
H // MatrixForm
```

$$\begin{pmatrix} \sqrt{2} & 1 & 0 & 0 \\ 0 & 1 & \sqrt{2} & 0 \\ 0 & 1 & \sqrt{2} & 0 \end{pmatrix}$$

```
H = H /. H[[3]] → H[[3]] - H[[2]];
H // MatrixForm
```

$$\begin{pmatrix} \sqrt{2} & 1 & 0 & 0 \\ 0 & 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence the eigenvector corresponding to $\lambda = -2 - \sqrt{2}$ is, for any $\beta \neq 0$:

$$x = \beta \begin{pmatrix} -1/\sqrt{2} \\ 1 \\ -1/\sqrt{2} \end{pmatrix}.$$

((**4 marks**))

For $\lambda = -2 + \sqrt{2}$, the eigenvector solves

$$(A - \lambda I)\, x = 0$$

$$\Longleftrightarrow \begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix is

$$H := \begin{pmatrix} -\sqrt{2} & 1 & 0 & 0 \\ 1 & -\sqrt{2} & 1 & 0 \\ 0 & 1 & -\sqrt{2} & 0 \end{pmatrix}$$

```
H = H /. H[[2]] → √2 H[[2]] + H[[1]];
H // MatrixForm
```

$$\begin{pmatrix} -\sqrt{2} & 1 & 0 & 0 \\ 0 & -1 & \sqrt{2} & 0 \\ 0 & 1 & -\sqrt{2} & 0 \end{pmatrix}$$

```
H = H /. H[[3]] → H[[3]] + H[[2]];
H // MatrixForm
```

$$\begin{pmatrix} -\sqrt{2} & 1 & 0 & 0 \\ 0 & -1 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence the eigenvector corresponding to $\lambda=-2+\sqrt{2}$ is, for any $\gamma\neq 0$:

$$\mathbf{X} = \gamma \begin{pmatrix} 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{pmatrix}.$$

((**4 marks**))

(b) Look for a solution of the form

$$\mathbf{Y}(t) = \mathbf{C}\, e^{\lambda t}$$

$$\Longrightarrow \ (\lambda)^2\, \mathbf{C}\, e^{\lambda t} = \mathbf{A}\, \mathbf{C}\, e^{\lambda t}$$

$$\Longleftrightarrow \ (\mathbf{A} - \lambda^2\, \mathbf{I})\, \mathbf{C} = \mathbf{O}.$$

((**4 marks**))

(c) Since

$$\lambda^2 = -\omega^2$$

then for $\lambda_1^2 = -2$ we have

$$\omega_1 = \pm\sqrt{2},$$

which corresponds to an oscillation of frequency $\omega_1 = +\sqrt{2}$.

For $\lambda_2^2 = -2 - \sqrt{2}$ we have

$$\omega_2 = \pm\sqrt{2+\sqrt{2}},$$

and so another natural frequency of oscillation is $\omega_2 = +\sqrt{2+\sqrt{2}}$.

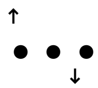
For $\lambda_3^2 = -2 + \sqrt{2}$ we have

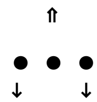
$$\omega_3 = \pm\sqrt{2-\sqrt{2}},$$

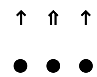
and so the last natural frequency of oscillation is $\omega_3 = +\sqrt{2-\sqrt{2}}$.

((**3 marks**))

(d)

$\omega_1 = +\sqrt{2}$: 

$\omega_2 = +\sqrt{2+\sqrt{2}}$: 

$\omega_3 = +\sqrt{2-\sqrt{2}}$: 

((**6 marks**))

(e) The general solution is

$$\mathbf{Y}(t) = \left(a_1 \cos\left(\sqrt{2}\, t\right) + b_1 \sin\left(\sqrt{2}\, t\right)\right)\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$+ \left(a_2 \cos\left(\sqrt{2+\sqrt{2}}\, t\right) + b_2 \sin\left(\sqrt{2+\sqrt{2}}\, t\right)\right)\begin{pmatrix} -1/\sqrt{2} \\ 1 \\ -1/\sqrt{2} \end{pmatrix}$$

$$+ \left(a_3 \cos\left(\sqrt{2-\sqrt{2}}\, t\right) + b_3 \sin\left(\sqrt{2-\sqrt{2}}\, t\right)\right)\begin{pmatrix} 1/\sqrt{2} \\ 1 \\ 1/\sqrt{2} \end{pmatrix},$$

where a_1, b_1, a_2, b_2, a_3 and b_3 are arbitrary constants (fixed by the initial data).

((**2 marks**))