Final exam 2005: solutions

Solution 1 (10 marks)

(a)

$$\omega_0 = +\sqrt{4} = 2.$$

((**3 marks**))

(b)

$$y_{PI} = A\sin(3t) + B\cos(3t).$$

((**3 marks**))

(c) This is a resonant case; the frequency of the oscillatory forcing matches the natural frequency of unforced oscillations of the system ($\omega_0 = 2$), hence try

$$y_{PI} = A\,t\sin(2t) + B\,t\cos(2t).$$

((**4 marks**))

Solution 2 (15 marks)

The auxillary equation is

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\Leftrightarrow \lambda = 2, \lambda = 1$$

$$\Rightarrow y_{CF} = C_1 e^{2t} + C_2 e^{t}.$$

((**3 marks**))

The inhomogeneity in the ODE suggests we try

$$y_{PI} = A e^{-t}$$

$$\Rightarrow y_{PI}' = -A e^{-t}$$

$$\Rightarrow y_{PI}'' = A e^{-t}.$$

Substituting our guess for the PI into the ODE

$$\Rightarrow A e^{-t} - 3(-A e^{-t}) + 2(A e^{-t}) = 3 e^{-t}$$

$$\Leftrightarrow A + 3A + 2A = 3$$

$$\Leftrightarrow A = \frac{1}{2}.$$

Hence the general solution to the ODE is

$$y = y_{CF} + y_{PI}$$

$$\Longrightarrow \; y = C_1 e^{2t} + C_2 e^{t} + \frac{1}{2} e^{-t}.$$

((**6 marks**))

Now we substitute in the initial conditions. First using that

$$y(0) = 1,$$

$$\Longrightarrow \; 1 = C_1 e^{2\cdot 0} + C_2 e^{0} + \frac{1}{2} e^{-0}$$

$$\Longrightarrow \; 1 = C_1 + C_2 + \frac{1}{2}$$

$$\Longleftrightarrow \; C_1 + C_2 = \frac{1}{2}.$$

Second, after differentiating the general solution with respect to t:

$$\Longrightarrow \; y' = 2 C_1 e^{2t} + C_2 e^{t} - \frac{1}{2} e^{-t},$$

we can use the second initial condition

$$y'(0) = 2,$$

$$\Longrightarrow \; 2 = 2 C_1 e^{2\cdot 0} + C_2 e^{0} - \frac{1}{2} e^{-0}$$

$$\Longleftrightarrow \; 2 = 2 C_1 + C_2 - \frac{1}{2}$$

$$\Longleftrightarrow \; 2 C_1 + C_2 = \frac{5}{2}.$$

Solving the two simultaneous equations for C_1 and C_2 we see that

$$C_1 = 2, \quad C_2 = -\frac{3}{2}.$$

Hence the solution to the initial value problem is

$$y = 2 e^{2t} - \frac{3}{2} e^{t} + \frac{1}{2} e^{-t}.$$

((**6 marks**))

■ Solution 3 (10 marks)

The auxillary equation is

$$\lambda^2 + 2\lambda + 1 = 0$$

$$\lambda = -1,\ \lambda = -1$$

$$\Longrightarrow\ y_{CF} = C_1 e^{-t} + C_2 t e^{-t}.$$

((**3 marks**))

The inhomogeneity in the ODE suggests we try

$$y_{PI} = A t^2 e^{-t}$$

$$\Longrightarrow\ y_{PI}' = 2 A t e^{-t} - A t^2 e^{-t}$$

$$\Longrightarrow\ y_{PI}'' = 2 A e^{-t} - 4 A t e^{-t} + A t^2 e^{-t}.$$

Substituting our guess for the PI into the ODE and dividing through by the exponential term

$$\Longrightarrow\ 2A - 4At + At^2 + 2(2At - At^2) + (At^2) = 2$$

$$\Longleftrightarrow\ A = 1.$$

Hence the general solution to the ODE is

$$y = y_{CF} + y_{PI}$$

$$\Longrightarrow\ y = C_1 e^{-t} + C_2 t e^{-t} + t^2 e^{-t}.$$

((**7 marks**))

■ Solution 4 (10 marks)

(a) Since from the table

$$\mathbb{L}[t^3] = \frac{6}{s^4},$$

the shift theorem (also in the table)

$$\Longrightarrow\ \mathbb{L}[e^{7t} t^3] = \frac{6}{(s-7)^4}.$$

((**4 marks**))

(b) Complete the square in the denominator

$$\frac{1}{s^2 + 4s + 7} = \frac{1}{(s+2)^2 + 3}.$$

Since from the table

$$\mathbb{L}\left[\sin\left(\sqrt{3}\, t\right)\right] = \frac{\sqrt{3}}{s^2 + 3} \quad \Longrightarrow \quad \mathbb{L}\left[\frac{1}{\sqrt{3}} \sin\left(\sqrt{3}\, t\right)\right] = \frac{1}{s^2 + 3}$$

and then using the Shift Theorem

$$\mathbb{L}\left[e^{-2t} \frac{1}{\sqrt{3}} \sin\left(\sqrt{3}\, t\right)\right] = \frac{1}{(s+2)^2+3}.$$

Hence

$$\mathbb{L}^{-1}\left[\frac{1}{s^2+4s+7}\right] = \frac{1}{\sqrt{3}} e^{-2t} \sin\left(\sqrt{3}\, t\right).$$

((**6 marks**))

■ Solution 5 (15 marks)

Taking the Laplace transform of both sides of the ODE

$$\Longrightarrow \mathbb{L}[y''(t) + 4y'(t) + 3y(t)] = \mathbb{L}[\delta(t-1)]$$

$$\Longleftrightarrow s^2 \bar{y}(s) - s\,y(0) - y'(0) + 4(s\,\bar{y}(s) - y(0)) + 3\bar{y}(s) = e^{-s}$$

$$\Longleftrightarrow (s^2 + 4s + 3)\,\bar{y}(s) = e^{-s} + 1$$

$$\Longleftrightarrow \bar{y}(s) = \frac{e^{-s}}{s^2+4s+3} + \frac{1}{s^2+4s+3}$$

$$\Longleftrightarrow \bar{y}(s) = \frac{e^{-s}}{(s+1)(s+3)} + \frac{1}{(s+1)(s+3)}.$$

((**5 marks**))

Using partial fractions, we can expand

$$\frac{1}{(s+1)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+3)}$$

$$\Longleftrightarrow 1 = A(s+3) + B(s+1)$$

$$\Longleftrightarrow 1 = (A+B)\,s + 3A + B.$$

Equating coefficients

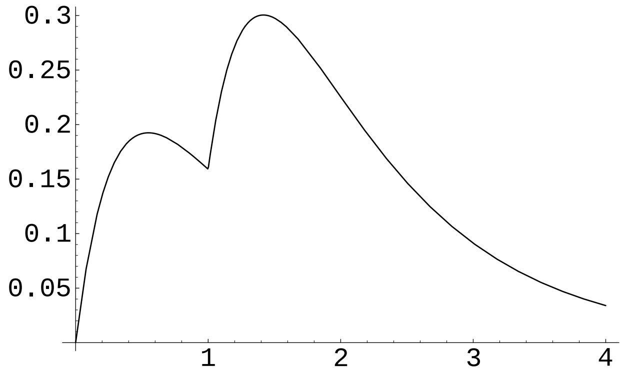
$$\begin{cases} s^0: & 1 = 3A + B, \\ s^1: & 0 = A + B. \end{cases} \Longrightarrow A = \frac{1}{2},\ B = -\frac{1}{2}.$$

((**5 marks**))

$$\Longrightarrow \bar{y}(s) = \frac{1}{2} e^{-s}\left(\frac{1}{(s+1)} - \frac{1}{(s+3)}\right) + \frac{1}{2}\left(\frac{1}{(s+1)} - \frac{1}{(s+3)}\right)$$

$$\Longleftrightarrow y(t) = \begin{cases} \frac{1}{2} e^{-(t-1)} - \frac{1}{2} e^{-3(t-1)} + \frac{1}{2} e^{-t} - \frac{1}{2} e^{-3t}, & t > 1, \\ \frac{1}{2} e^{-t} - \frac{1}{2} e^{-3t}, & t \le 1. \end{cases}$$

((**4 marks**))



((1 mark))

■ Solution 6 (15 marks)

The augmented matrix is

```
H := (-1  1  10   0   0   3
       1  0   0   8   0   9
       0  1   0   0   5  12
       0  1  -1   0  -1   0
       1  0   1  -1   0   0);
```

```
H = H /. H[[2]] → H[[2]] + H[[1]];
H = H /. H[[5]] → H[[5]] + H[[1]];
H // MatrixForm
```

−1	1	10	0	0	3
0	1	10	8	0	12
0	1	0	0	5	12
0	1	−1	0	−1	0
0	1	11	−1	0	3

((3 marks))

```
H = H /. H[[3]] → H[[3]] - H[[2]];
H = H /. H[[4]] → H[[4]] - H[[2]];
H = H /. H[[5]] → H[[5]] - H[[2]];
H // MatrixForm
```

−1	1	10	0	0	3
0	1	10	8	0	12
0	0	−10	−8	5	0
0	0	−11	−8	−1	−12
0	0	1	−9	0	−9

((3 marks))

```
H = H /. H[[4]] → 10 H[[4]] - 11 H[[3]];
H = H /. H[[5]] → 10 H[[5]] + H[[3]];
H // MatrixForm
```

$$\begin{pmatrix} -1 & 1 & 10 & 0 & 0 & 3 \\ 0 & 1 & 10 & 8 & 0 & 12 \\ 0 & 0 & -10 & -8 & 5 & 0 \\ 0 & 0 & 0 & 8 & -65 & -120 \\ 0 & 0 & 0 & -98 & 5 & -90 \end{pmatrix}$$

((**3 marks**))

```
H = H /. H[[5]] → 8 H[[5]] + 98 H[[4]];
H // MatrixForm
```

$$\begin{pmatrix} -1 & 1 & 10 & 0 & 0 & 3 \\ 0 & 1 & 10 & 8 & 0 & 12 \\ 0 & 0 & -10 & -8 & 5 & 0 \\ 0 & 0 & 0 & 8 & -65 & -120 \\ 0 & 0 & 0 & 0 & -6330 & -12480 \end{pmatrix}$$

((**3 marks**))

Hence using back−substitution, the solution is

$$I_1 = \frac{179}{211},\ I_2 = \frac{452}{211},\ I_3 = \frac{36}{211},\ I_4 = \frac{215}{211},\ I_5 = \frac{416}{211}.$$

((**3 marks**))

■ Solution 7 (25 marks)

(a) The eigenvalues are the solutions to the characteristic equation

$$\det\begin{pmatrix} -4-\lambda & 4 & 0 \\ 6 & -12-\lambda & 6 \\ 0 & 4 & -4-\lambda \end{pmatrix} = 0$$

$$\Longleftrightarrow\ -(4+\lambda)\,((12+\lambda)\,(4+\lambda) - 24) - 4\,(-6\,(4+\lambda)) = 0$$

$$\Longleftrightarrow\ -(4+\lambda)\,((12+\lambda)\,(4+\lambda) - 24 - 24) = 0$$

$$\Longleftrightarrow\ -(4+\lambda)\,(\lambda^2 + 16\,\lambda) = 0$$

$$\Longleftrightarrow\ \lambda\,(4+\lambda)\,(\lambda+16) = 0.$$

Hence the eigenvalues are

$$\lambda = 0,\ \lambda = -4,\ \lambda = -16.$$

((**3 marks**))

For λ=0, the eigenvector solves

$$(\mathbf{A}-\lambda\,\mathbf{I})\,\mathbf{x}=\mathbf{0}$$

$$\Longleftrightarrow \begin{pmatrix} -4 & 4 & 0 \\ 6 & -12 & 6 \\ 0 & 4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix is

$$H := \begin{pmatrix} -4 & 4 & 0 & 0 \\ 6 & -12 & 6 & 0 \\ 0 & 4 & -4 & 0 \end{pmatrix}$$

```
H = H /. H[[2]] → 2 H[[2]] + 3 H[[1]];
H // MatrixForm
```

$$\begin{pmatrix} -4 & 4 & 0 & 0 \\ 0 & -12 & 12 & 0 \\ 0 & 4 & -4 & 0 \end{pmatrix}$$

```
H = H /. H[[3]] → 3 H[[3]] + H[[2]];
H // MatrixForm
```

$$\begin{pmatrix} -4 & 4 & 0 & 0 \\ 0 & -12 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence

$$x_2 = x_3, \; x_1 = x_3$$

and the eigenvector corresponding to λ=0 is, for any $\alpha\neq$0:

((**4 marks**))

$$\mathbf{x} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For λ=−4, the eigenvector solves

$$(\mathbf{A}-\lambda\,\mathbf{I})\,\mathbf{x}=\mathbf{0}$$

$$\Longleftrightarrow \begin{pmatrix} 0 & 4 & 0 \\ 6 & -8 & 6 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix is

$$H := \begin{pmatrix} 0 & 4 & 0 & 0 \\ 6 & -8 & 6 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}$$

The first and third equations immediately imply

$$x_2 = 0,$$

while the second equation gives

$$x_1 = -x_3$$

Hence the eigenvector corresponding to $\lambda=-4$ is, for any $\beta\neq0$:

$$\mathbf{x} = \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

((**4 marks**))

For $\lambda=-16$, the eigenvector solves

$$(\mathbf{A} - \lambda\, \mathbf{I})\, \mathbf{x} = \mathbf{0}$$

$$\Longleftrightarrow \begin{pmatrix} 12 & 4 & 0 \\ 6 & 4 & 6 \\ 0 & 4 & 12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix is

$$\mathbf{H} := \begin{pmatrix} 12 & 4 & 0 & 0 \\ 6 & 4 & 6 & 0 \\ 0 & 4 & 12 & 0 \end{pmatrix}$$

```
H = H /. H[[2]] → 2 H[[2]] - H[[1]];
H // MatrixForm
```

$$\begin{pmatrix} 12 & 4 & 0 & 0 \\ 0 & 4 & 12 & 0 \\ 0 & 4 & 12 & 0 \end{pmatrix}$$

```
H = H /. H[[3]] → H[[3]] - H[[2]];
H // MatrixForm
```

$$\begin{pmatrix} 12 & 4 & 0 & 0 \\ 0 & 4 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence

$$x_2 = -3\, x_3, \quad x_1 = x_3$$

and the eigenvector corresponding to $\lambda=-16$ is, for any $\gamma\neq0$:

$$\mathbf{x} = \gamma \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}.$$

((**4 marks**))

(b) Look for a solution of the form

$$\mathbf{Y}(t) = \mathbf{C}\, e^{i\,\omega\,t}$$

$$\Longrightarrow (i\,\omega)^2\, \mathbf{C}\, e^{i\,\omega\,t} = \mathbf{A}\,\mathbf{C}\, e^{i\,\omega\,t}$$

$$\Longleftrightarrow (\mathbf{A} + \omega^2\, \mathbf{I})\, \mathbf{C} = \mathbf{O}.$$

((**3 marks**))

(c) Since

$$\lambda = -\omega^2$$

then for $\lambda_1 = 0$ we have

$$\omega_1 = 0 \quad (\text{twice}),$$

which doesn't correspond to any oscillation at all but only uniform translation.

For $\lambda_2 = -4$ we have

$$\omega_2 = \pm 2,$$

and so one natural frequency of oscillation is $\omega_2 = +2$.

For $\lambda_3 = -16$ we have

$$\omega_3 = \pm 4,$$

and so the other natural frequency of oscillation is $\omega_3 = +4$.

((**2 marks**))

(d)

$\lambda_1 = 0$: ●→ ●→ ●→

$\lambda_2 = -4$: ←● ● ●→

$\lambda_3 = -16$: ●→ ⟵● ●→

((**3 marks**))

(e) The general solution is

$$Y(t) = (a_1 + b_1 t)\begin{pmatrix}1\\1\\1\end{pmatrix} + (a_2 \cos(2t) + b_2 \sin(2t))\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$+ (a_3 \cos(4t) + b_3 \sin(4t))\begin{pmatrix}1\\-3\\1\end{pmatrix},$$

where a_1, b_1, a_2, b_2, a_3 and b_3 are arbitrary constants (fixed by the initial data).

((**2 marks**))