Final exam 2004: solutions

Solution 1(a)

$$\omega = +\sqrt{9} = 3.$$

Solution 1(b)

$$y_{PI} = A\sin(2t) + B\cos(2t).$$

Solution 1(c)

This is a resonant case; the frequency of the oscillatory forcing matches the natural frequency of unforced oscillations of the system ($\omega = 3$), hence try

$$y_{PI} = A\,t\sin(3t) + B\,t\cos(3t).$$

Solution 2(a)

The auxillary equation is

$$\lambda^2 + 5\lambda + 4 = 0$$

$$\Longleftrightarrow \quad \lambda = -4,\ \lambda = -1$$

$$\Longrightarrow \quad y_{CF} = C_1 e^{-4t} + C_2 e^{-t}.$$

The inhomogeneity in the ODE suggests we try

$$y_{PI} = A e^{2t}$$

$$\Longrightarrow \quad y_{PI}' = 2A e^{2t}$$

$$\Longrightarrow \quad y_{PI}'' = 4A e^{2t}.$$

Substituting our guess for the PI into the ODE

$$\Longrightarrow \quad 4A e^{2t} + 5(2A e^{2t}) + 4(A e^{2t}) = 2e^{2t}$$

$$\Longleftrightarrow \quad 4A + 10A + 4A = 2$$

$$\Longleftrightarrow \quad A = \frac{1}{9}.$$

Hence the general solution to the ODE is

$$y = y_{CF} + y_{PI}$$

$$\Longrightarrow \quad y = C_1 e^{-4t} + C_2 e^{-t} + \frac{1}{9} e^{2t}.$$

■ Solution 2(b)

The auxillary equation is

$$4\lambda^2 + 12\lambda + 25 = 0$$

$$\Longleftrightarrow \quad \lambda = -\frac{3}{2} - 2i, \lambda = -\frac{3}{2} + 2i$$

$$\Longrightarrow \quad y_{CF} = e^{-\frac{3}{2}t} (C_1 \sin(2t) + C_2 \cos(2t)).$$

The inhomogeneity in the ODE suggests we try

$$y_{PI} = A \sin(t) + B \cos(t)$$

$$\Longrightarrow \quad y_{PI}' = A \cos(t) - B \sin(t)$$

$$\Longrightarrow \quad y_{PI}'' = -A \sin(t) - B \cos(t).$$

Substituting our guess for the PI into the ODE

$$\Longrightarrow \quad 4(-A \sin(t) - B \cos(t)) + 12(A \cos(t) - B \sin(t)) + 25(A \sin(t) + B \cos(t)) = 9 \sin(t)$$

Hence equating coefficients:

$$\sin(t): \quad -4A - 12B + 25A = 9 \quad \Longleftrightarrow \quad 7A - 4B = 3,$$

$$\cos(t): \quad -4B + 12A + 25B = 0 \quad \Longleftrightarrow \quad 4A + 7B = 0.$$

Solving this pair of simultaneous equations

$$\Longrightarrow \quad A = \frac{21}{65}, B = -\frac{12}{65}.$$

Hence the general solution to the ODE is

$$y = y_{CF} + y_{PI}$$

$$\Longrightarrow \quad y = e^{-\frac{3}{2}t} (C_1 \sin(2t) + C_2 \cos(2t)) + \frac{21}{65} \sin(t) - \frac{12}{65} \cos(t).$$

■ Solution 3

The auxillary equation is

$$\lambda^2 - 8\lambda + 16 = 0$$

$$\lambda = 4, \lambda = 4$$

$$\Longrightarrow \quad y_{CF} = C_1 e^{4t} + C_2 t e^{4t}.$$

The inhomogeneity in the ODE suggests we try

$$y_{PI} = A\,t^2\,e^{4t}$$

$$\Longrightarrow\ y_{PI}' = 2\,A\,t\,e^{4t} + 4\,A\,t^2\,e^{4t}$$

$$\Longrightarrow\ y_{PI}'' = 2\,A\,e^{4t} + 16\,A\,t\,e^{4t} + 16\,A\,t^2\,e^{4t}.$$

Substituting our guess for the PI into the ODE and dividing through by the exponential term

$$\Longrightarrow\ 2\,A + 16\,A\,t + 16\,A\,t^2 - 8\,(2\,A\,t + 4\,A\,t^2) + 16\,(A\,t^2) = 2$$

$$\Longleftrightarrow\ A = 1.$$

Hence the general solution to the ODE is

$$y = y_{CF} + y_{PI}$$

$$\Longrightarrow\ y = C_1\,e^{4t} + C_2\,t\,e^{4t} + t^2\,e^{4t}.$$

■ Solution 4(a)

Since from the table

$$\mathbb{L}[\sin(2\,t)] = \frac{2}{s^2+4},$$

the shift theorem (also in the table)

$$\Longrightarrow\ \mathbb{L}[e^{5t}\sin(2\,t)] = \frac{2}{(s-5)^2+4}.$$

■ Solution 4(b)

Since the denominator factorizes

$$\frac{1}{s^2+s-2} = \frac{1}{(s-1)\,(s+2)}$$

we can use partial fractions to split this fraction into the sum of two simpler fractions, try

$$\frac{1}{(s-1)\,(s+2)} = \frac{A}{(s-1)} + \frac{B}{(s+2)}$$

$$\Longleftrightarrow\ 1 = A\,(s+2) + B\,(s-1)$$

$$\Longleftrightarrow\ 1 = (A+B)\,s + 2\,A - B.$$

Equating coefficients

$$s^0:\quad 1 = 2A - B,\quad \begin{cases} s^0: & 1 = 2A - B, \\ s^1: & 0 = A + B. \end{cases} \implies A = \frac{1}{3},\ B = -\frac{1}{3}.$$

$$\implies \frac{1}{(s-1)(s+2)} = \frac{1/3}{(s-1)} - \frac{1/3}{(s+2)}.$$

Using the table of Laplace transforms

$$\mathbb{L}\left[\frac{1}{3}e^{t} - \frac{1}{3}e^{-2t}\right] = \frac{1/3}{(s-1)} - \frac{1/3}{(s+2)}.$$

Hence

$$\mathbb{L}^{-1}\left[\frac{1}{s^2+s-2}\right] = \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t}.$$

■ Solution 5

Taking the Laplace transform of both sides of the ODE

$$\implies \mathbb{L}[y''(t) + 7y'(t) + 6y(t)] = \mathbb{L}[\delta(t - 2\pi)]$$

$$\iff s^2\bar{y}(s) - s\,y(0) - y'(0) + 7(s\bar{y}(s) - y(0)) + 6\bar{y}(s) = e^{-2\pi s}$$

$$\iff (s^2 + 7s + 6)\bar{y}(s) = e^{-2\pi s}$$

$$\iff \bar{y}(s) = \frac{e^{-2\pi s}}{s^2 + 7s + 6}$$

$$\iff \bar{y}(s) = \frac{e^{-2\pi s}}{(s+1)(s+6)}.$$

Using partial fractions, we can expand

$$\frac{1}{(s+1)(s+6)} = \frac{A}{(s+1)} + \frac{B}{(s+6)}$$

$$\iff 1 = A(s+6) + B(s+1)$$

$$\iff 1 = (A+B)s + 6A + B.$$

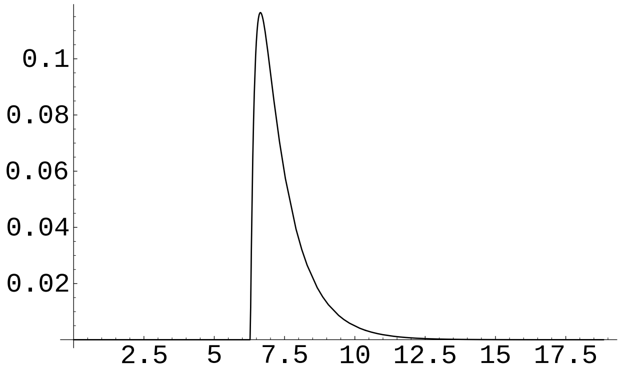
Equating coefficients

$$\begin{cases} s^0: & 1 = 6A + B, \\ s^1: & 0 = A + B. \end{cases} \implies A = \frac{1}{5},\ B = -\frac{1}{5}.$$

$$\implies \bar{y}(s) = \frac{1}{5}\frac{e^{-2\pi s}}{(s+1)} - \frac{1}{5}\frac{e^{-2\pi s}}{(s+6)}$$

$$\iff y(t) = \begin{cases} \frac{1}{5}e^{-(t-2\pi)} - \frac{1}{5}e^{-6(t-2\pi)}, & t > 2\pi, \\ 0, & t \le 2\pi. \end{cases}$$

$$\text{Plot}\left[\text{UnitStep}[t-2\pi]\left(\frac{1}{5}e^{-(t-2\pi)}-\frac{1}{5}e^{-6(t-2\pi)}\right),\{t,0,6\pi\}\right]$$



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- Graphics -
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■ Solution 6

The augmented matrix is

$$H := \begin{pmatrix} 2 & 0 & 3 & 8 \\ 0 & 2 & 3 & 16 \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

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H = H /. H[[3]] → 2 H[[3]] - H[[1]];
H // MatrixForm
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$$\begin{pmatrix} 2 & 0 & 3 & 8 \\ 0 & 2 & 3 & 16 \\ 0 & 2 & -5 & -8 \end{pmatrix}$$

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H = H /. H[[3]] → H[[3]] - H[[2]];
H // MatrixForm
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$$\begin{pmatrix} 2 & 0 & 3 & 8 \\ 0 & 2 & 3 & 16 \\ 0 & 0 & -8 & -24 \end{pmatrix}$$

Hence using back−substitution, the solution is

$$I_1 = -\frac{1}{2},\ I_2 = \frac{7}{2},\ I_3 = 3.$$

That I_1 is negative simply means that the current actually flows in the direction opposite to that assumed in the figure.

■ Solution 7

The eigenvalues are the solutions to the characteristic equation

$$\det\begin{pmatrix} -1-\lambda & 6 & 0 \\ 2 & 1-\lambda & 1 \\ -2 & -1 & -1-\lambda \end{pmatrix} = 0$$

$$\Longleftrightarrow\ -(1+\lambda)\,(-(1-\lambda)\,(1+\lambda)+1)-6\,(-2\,(1+\lambda)+2)=0$$

$$\Longleftrightarrow\ -(1+\lambda)\,(\lambda^2)-6\,(-2\,\lambda)=0$$

$$\Longleftrightarrow\ \lambda\,(\lambda^2+\lambda-12)=0.$$

Hence the eigenvalues are

$$\lambda=0,\ \lambda=3,\ \lambda=-4.$$

For λ=3, the eigenvector solves

$$(\mathbf{A}-\lambda\,\mathbf{I})\,\mathbf{x}=\mathbf{0}$$

$$\Longleftrightarrow\ \begin{pmatrix} -1-3 & 6 & 0 \\ 2 & 1-3 & 1 \\ -2 & -1 & -1-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix is

$$H := \begin{pmatrix} -4 & 6 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ -2 & -1 & -4 & 0 \end{pmatrix}$$

```
H = H /. H[[2]] → 2 H[[2]] + H[[1]];
H = H /. H[[3]] → 2 H[[3]] - H[[1]];
H // MatrixForm
```

$$\begin{pmatrix} -4 & 6 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -8 & -8 & 0 \end{pmatrix}$$

```
H = H /. H[[3]] → H[[3]] + 4 H[[2]];
H // MatrixForm
```

$$\begin{pmatrix} -4 & 6 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence

$$x_3=-x_2,\ x_1=\frac{3}{2}\,x_2$$

and the eigenvector corresponding to λ=3 is, for any $\alpha\neq 0$:

$$\mathbf{x}=\alpha\begin{pmatrix} \frac{3}{2} \\ 1 \\ -1 \end{pmatrix}.$$

■ Solution 8(a)

Look for a solution of the form

$$\mathbf{Y}\,(t)=\mathbf{C}\,e^{i\,\omega\,t}$$

$$\Longrightarrow\ (i\,\omega)^2\,\mathbf{C}\,e^{i\,\omega\,t}=\mathbf{A}\,\mathbf{C}\,e^{i\,\omega\,t}$$

$$\Longleftrightarrow\ (\mathbf{A} + \omega^2\,\mathbf{I})\,\mathbf{C} = \mathbf{O}.$$

Solution 8(b)

The eigenvalues are the solutions to the characteristic equation

$$\det(\mathbf{A} + \omega^2\,\mathbf{I}) = \mathbf{O}$$

$$\Longleftrightarrow\ \det\begin{pmatrix} -5+\omega^2 & 3 \\ 3 & -5+\omega^2 \end{pmatrix} = 0$$

$$\Longleftrightarrow\ (\omega^2 - 5)^2 - 9 = 0$$

$$\Longleftrightarrow\ (\omega^2)^2 - 10\,\omega^2 + 16 = 0$$

$$\Longleftrightarrow\ (\omega^2 - 2)\,(\omega^2 - 8) = 0$$

$$\Longleftrightarrow\ \omega^2 = 2,\ \omega^2 = 8$$

$$\Longleftrightarrow\ \omega = \pm 2\sqrt{2},\ \omega = \pm\sqrt{2}.$$

Hence $\omega = 2\sqrt{2}$ and $\omega = \sqrt{2}$ are the fundamental frequencies of vibration.

Solution 8(c)

For $\omega^2 = 2$, the eigenvector solves

$$\begin{pmatrix} -5+2 & 3 \\ 3 & -5+2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow\ \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow\ C_2 = C_1.$$

Hence the eigenvector corresponding to $\omega^2 = 2$ is, for any $\alpha \neq 0$:

$$\mathbf{C}^{(1)} = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and the corresponding mode of oscillation is

$$\mathbf{Y}^{(1)}(t) = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\sqrt{2}\,t},$$

i.e. they oscillate exactly in phase.

For $\omega^2 = 8$, the eigenvector solves

$$\begin{pmatrix} -5+8 & 3 \\ 3 & -5+8 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow\ \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow \; C_2 = -C_1 .$$

Hence the eigenvector corresponding to $\omega^2 = 2$ is, for any $\beta \neq 0$:

$$C^{(2)} = \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and the corresponding mode of oscillation is

$$Y^{(2)}(t) = \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\,2\sqrt{2}\,t},$$

i.e. they oscillate exactly out of phase with each other.

■ Solution 8(d)

The general solution is

$$Y(t) = \mathrm{Re}[K_1 Y^{(1)}(t) + K_2 Y^{(2)}(t)]$$

$$\Longrightarrow \; Y(t) = \mathrm{Re}\left[\tilde{K}_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\sqrt{2}\,t} + \tilde{K}_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\,2\sqrt{2}\,t}\right],$$

where $\tilde{K}_1$ and $\tilde{K}_2$ are arbitrary complex constants (fixed by the initial data).