Question 1 (15 Marks)
Find the general solution of the following differential equations:

(a)
$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 3\frac{\mathrm{d}y}{\mathrm{d}t} - 4y = 0;$$

(b)
$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t} + 5y = 3\mathrm{e}^{2t}.$$

Question 2 (15 Marks)
Consider the flow of an electrical current $I(t)$ in a simple series circuit, as shown in Figure 1. The resistor has resistance $R = 8$ Ohms, the capacitor has capacitance $C = 1/16$ Farads and the inductor has inductance $L = 1$ Henrys. A battery or power source provides an impressed voltage of $V(t) = \sin(2t)$ volts at any given time. The rate of change of total charge, $Q(t)$ Coulombs, in the capacitor at time t, is thus governed by the linear, non-homogeneous second order differential equation,

$$\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + 8\frac{\mathrm{d}Q}{\mathrm{d}t} + 16Q = \sin(2t). \quad (1)$$

If initially, $Q(0) = 1$ and $Q'(0) = 0$, find the solution to the differential equation (1) which satisfies these initial conditions, and describe how the solution, $Q(t)$, evolves in time.

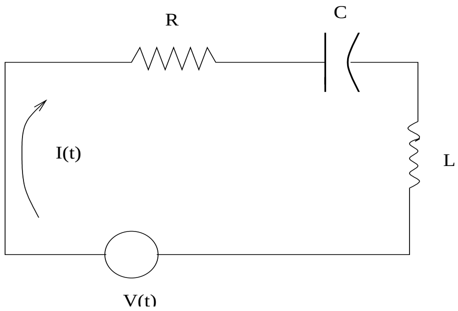

Figure 1: Simple LCR electrical circuit.

Question 3 (10 Marks)
Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 6\frac{\mathrm{d}y}{\mathrm{d}t} + 9y = \mathrm{e}^{3t}.$$

Question 4 (10 Marks)

(a) Find the Laplace transform of the function

$$f(t) = \mathrm{e}^{3t}\cos(5t)\,,$$

stating any formula from the Laplace transform table you have used.

(b) Find the inverse Laplace transform of the function

$$\overline{f}(s) = \frac{1}{s^2 + 2s + 2}\,.$$

Question 5 (15 Marks)

Consider the damped spring system shown in figure 2, which consists of a mass which slides on the horizontal surface, and is attached to a spring, which is fixed to a vertical wall at the other end. Suppose that the mass, initially at rest in the equilibrium position, is acted upon by an external forcing function $f(t)$, so that the initial value problem for the motion of the mass is

$$y'' + 4y' + 5y = f(t), \quad \text{with} \quad y(0) = 0,\ y'(0) = 0. \tag{2}$$

Use the Laplace transform to determine the solution to this initial value problem in the case when the external force is:

(a) $f(t) = 1$ for all $t \geq 0$, i.e. a constant unit force;

(b) $f(t) = \delta(t)$ for all $t \geq 0$, i.e. the mass is given a sharp hammer blow at time $t = 0$.

In both cases (a) and (b) sketch the behaviour of the solution for all $t \geq 0$.

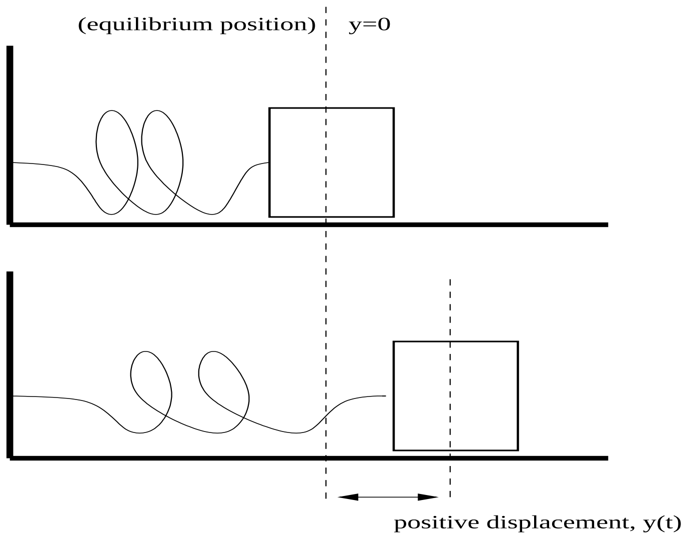

Figure 2: Simple damped, mass-spring system.

Question 6 (10 Marks)
Consider the electrical circuit in figure 3.
Kirchoff's loop rule, Ohm's law and Kirchoff's node rule applied to the left closed loop, the right closed loop, and at the nodes P and Q, respectively, tell us that the unknown currents, I_1, I_2 and I_3, in the electrical circuit in figure 3, satisfy the system of equations

$$\begin{aligned} 20I_1 + 10I_2 &= 80, \\ 10I_2 + 25I_3 &= 90, \\ I_1 - I_2 + I_3 &= 0, \\ -I_1 + I_2 - I_3 &= 0. \end{aligned}$$

Use Gaussian elimination to solve the system of equations and determine I_1, I_2 and I_3.

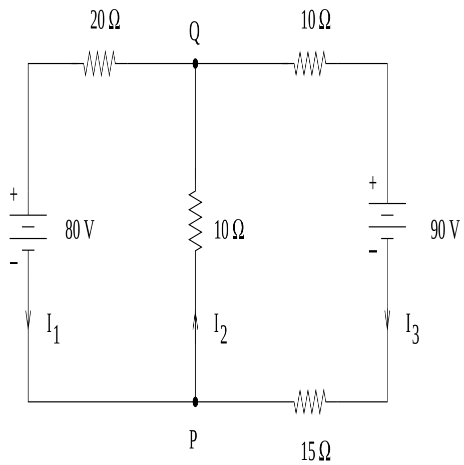

Figure 3: Complex electrical circuit in equilibrium.

Question 7 (10 Marks)
Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

In addition, find the eigenvector corresponding to the eigenvalue $\lambda = 2$.

Question 8 (15 Marks)

Two identical simple pendula oscillate in the plane as shown in Figure 4. Both pendula consist of light rods of length $\ell = 10$ and are suspended from the same ceiling a distance $L = 15$ apart, with equal masses $m = 1$ attached to their ends. The angles the pendula make to the downward vertical are θ_1 and θ_2, and they are coupled through the spring shown which has stiffness coefficient $k = 1$. The spring has unstretched length $L = 15$. You may also assume that the acceleration due to gravity $g = 10$.

(a) Assuming that the oscillations of the spring remain small in amplitude, so that $|\theta_1| \ll 1$ and $|\theta_2| \ll 1$, by applying Newton's second law and and Hooke's law, show that the coupled pendula system gives rise to the system of differential equations

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}t^2} = A\,\Theta, \quad \text{where} \quad A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \tag{4}$$

and

$$\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

is the vector of unknown angles for each of the pendula shown in Figure 4.

(b) By looking for a solution of the form $\Theta(t) = C\mathrm{e}^{\mathrm{i}\omega t}$ for a constant vector C, show that solving the system of differential equations (4) reduces to solving the eigenvalue problem

$$(A + \omega^2 I)\,C = \mathbf{0}\,. \tag{5}$$

(c) Solve the eigenvalue problem (5) in part (b) above, stating clearly the eigenvalues and associated eigenvectors.

(d) Hence enumerate the possible modes of oscillation of the masses corresponding to each eigenvalue-eigenvector pair.

(e) Finally, write down the general solution of the system of equations (4).

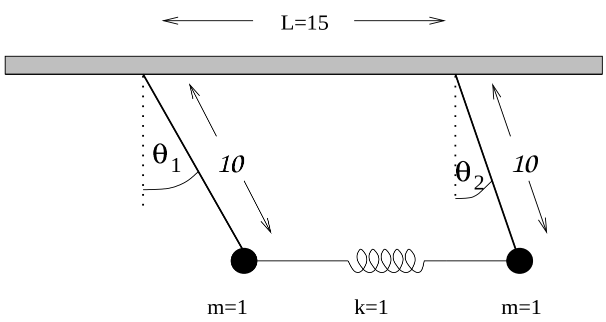

Figure 4: Simple coupled pendula system.

$f(t)$	$\int_0^\infty e^{-st} f(t) dt$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$af(t) + bg(t)$	$a\overline{f}(s) + b\overline{g}(s)$
$f'(t)$	$s\overline{f}(s) - f(0)$
$f''(t)$	$s^2\overline{f}(s) - sf(0) - f'(0)$
$e^{-at} f(t)$	$\overline{f}(s+a)$
$f(t) * g(t)$	$f(s) \cdot g(s)$
$\delta(t-a)$	e^{-sa}
$f(t) = \begin{cases} g(t-a), & t \geq a, \\ 0, & t < a, \end{cases}$	$e^{-sa}\overline{g}(s)$

Table 1: Table of Laplace transforms