# Non-local Currents, Quantum Groups and Discrete Holomorphicity


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CRM, 13/07/15

# Plan

1. Non-local Currents and Quantum Groups
2. Discrete Holomorphicity
3. The Chiral Potts Model
4. DH Relations and Perturbed CFT
5. Conclusions

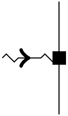
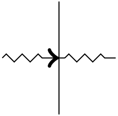
[*Ref:* Y. Ikhlef, RW, M. Wheeler and P. Zinn-Justin, J. Phys. A 46 (2013) arxiv:1302.4649; Y. Ikhlef and RW, J. Phys. A 48 (2015) arxiv:1502.04944]

# Non-local conserved currents and quantum groups

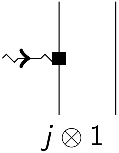
- Consider a quantum group (aka quasi-triangular Hopf algebra) $\mathcal{A}$ which includes elements $j, t$ with

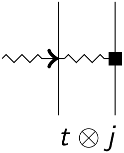
$$\Delta(j) = j \otimes 1 + t \otimes j, \quad \Delta(t) = t \otimes t$$

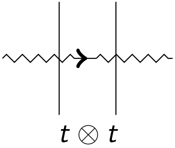
- Represent action on rep of $\mathcal{A}$ as

$j =$  $,\qquad t =$  $,$

- Acting on tensor product of reps

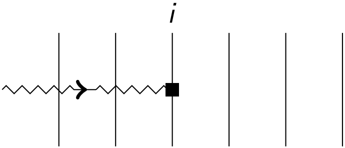
$\Delta(j) \;=$ 

$+$ 

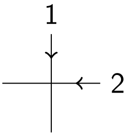
$\Delta(t) =$ 

,

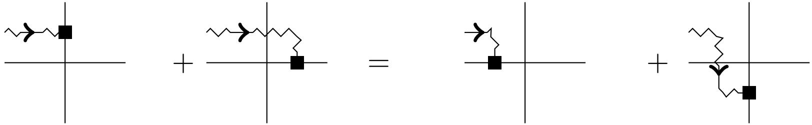
- with obvious extensions to $\Delta^{(N-1)}(j) \in \mathcal{A}^{\otimes N}$:

$\Delta^{(N-1)}(j) = \sum_i$ 

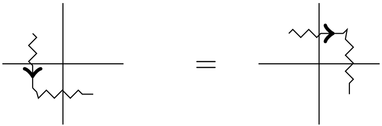
# Commutation with the R-matrix

- With $\check{R} : V_1 \otimes V_2 \to V_2 \otimes V_1$ 

, $\quad \check{R}\Delta(x) = \Delta(x)\check{R}$ is



$$\check{R}\,(j \otimes 1) \quad + \quad \check{R}\,(t \otimes j) \quad = \quad (j \otimes 1)\,\check{R} \quad + \quad (t \otimes j)\,\check{R}$$

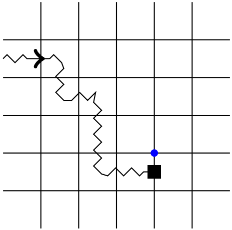


$$(t \otimes t)\,\check{R} \quad = \quad \check{R}\,(t \otimes t)$$

# Non-local currents

- A current $j(x,y)$ is an insertion of $j$ at posn $(x,y)$ with an attached $t$ 'tail' heading towards a fixed point on boundary.
- We have

$$\left\langle j(x, y-\frac{1}{2})\right\rangle =$$



- Commutation around vertex $(x,y)$ leads to

$$\left\langle j(x-\frac{1}{2},y)-j(x+\frac{1}{2},y)+j(x,y-\frac{1}{2})-j(x,y+\frac{1}{2})\right\rangle = 0$$

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  Let $f$ be a complex-valued fn defined at midpoint of edges

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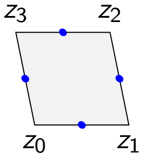
- $f$ said to be DH if it obeys lattice version of $\oint f(z)dz = 0$ around any cycle.

  Around elementary plaquette, we use:
  $f(z_{01})(z_1 - z_0) + f(z_{12})(z_2 - z_1) + f(z_{23})(z_3 - z_2) + f(z_{30})(z_0 - z_3) = 0$




$$z_{ij} = (z_i + z_j)/2$$

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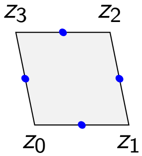


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- Can be written for this cycle as
  $$\frac{f(z_{23}) - f(z_{01})}{z_2 - z_1} = \frac{f(z_{12}) - f(z_{30})}{z_1 - z_0}, \quad \text{a discrete C-R reln } \bar{\partial} f = 0$$

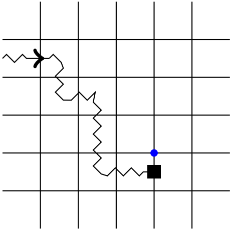
# What is use of DH in SM/CFT?

- For review see [S. Smirnov, Proc. ICM 2006, 2010]
- DH of observables has been as a key tool in rigorous proof of existence and uniqueness of scaling limit to particular conformal field theories, e.g.,
  - planar Ising model [S. Smirnov, C. Hongler D. Chelkak . . . , 2001-] - convergence of interfaces to SLE(3)
  - site percolation on triangular lattice - Cardy's crossing formula and reln to SLE(6) [S. Smirnov: 2001]
- We find DH condition also useful in identifying the particular integrable CFT perturbation to which SM lattice model corresponds

# DH and Integrability

- Observed by [Ikhlef, Cardy (09); de Gier, Lee, Rasmussen (09); Alam, Batchelor (12,14)] that candidate operators in various lattice models obey DH in the case when R-matrix obeys Yang-Baxter
- Our construction explains connection by building in terms of non-local conserved currents.

$$\left\langle j(x, y-\frac{1}{2})\right\rangle =$$



$$\left\langle j(x-\frac{1}{2},y)-j(x+\frac{1}{2},y)+j(x,y-\frac{1}{2})-j(x,y+\frac{1}{2})\right\rangle =0$$

# Examples

- Three examples considered:
  - Dense ($U_q(\widehat{sl}_2)$ ) and dilute loop models ($U_q(A_2^{(2)})$): [Ikhlef, RW, Wheeler, Zinn-Justin (13)]
  - Chiral Potts ($U_q(\widehat{sl}_2)$ ) : [Iklef, RW (15)]
- 4 term relns in massless case give DH relns - discrete version of $\partial_{\bar{z}}\Psi(z,\bar{z}) = 0$
- 4 terms relns in massive case are of form $\partial_{\bar{z}}\Psi(z,\bar{z}) = \sum_i \chi_i(z,\bar{z})$

  where in CFT

$$\Psi(z)\Phi_i^{pert}(w,\bar{w}) = \cdots + \frac{\chi_i(w,\bar{w})}{z-w} + \cdots$$

- The point:
  - Can identify the CFT perturbing fields
  - Obtain new parafermionic algebraic structures
  - Hopefully useful in rigorous proof of scaling limit

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- See [B. McCoy, Advanced Statistical Mech, OUP, 2010]

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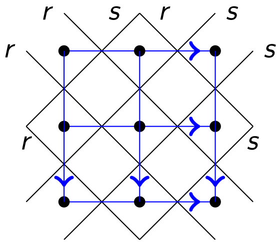
- Heights $a \in \{0, 1, \cdots, N-1\}$ on vertices:



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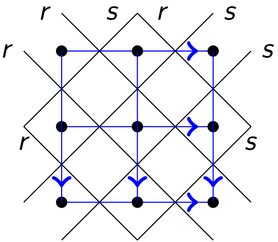
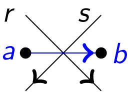
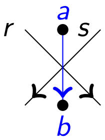



- Heights $a \in \{0, 1, \cdots, N-1\}$ on vertices:
- Boltzmann weights are

$W_{rs}(a-b) =$ 

, $\qquad \overline{W}_{rs}(a-b) =$ 

.

- Rapidities $r, s$ in $W_{rs}(a-b)$ are points on algebraic curve $\mathcal{C}_k$:

$$x^N + y^N = k(1 + x^N y^N), \quad \mu^N = \frac{k'}{1-kx^N} = \frac{1-ky^N}{k'}$$

# CP Representation Theory

- The CP models can be understood in terms of $N$ dim. cyclic representations $V_{rs}$ of $U_q(\widehat{\mathfrak{sl}}_2)$ at $q=-e^{i\pi/N}$, where $r,s\in\mathcal{C}_k$ [Bazhanov and Stroganov (90); Date, Jimbo, Miki, Miwa (1991)]

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- $\widetilde{U}_q(\widehat{\mathfrak{sl}}_2)$ has generators $e_i, f_i, t_i^{\pm 1}, z_i$, $(i=0,1)$

# CP Representation Theory. . .

- R-matrix $\check{R}(rr', ss') : V_{rr'} \otimes V_{ss'} \to V_{ss'} \otimes V_{rr'}$ is of form:

$$\check{R}(rr', ss')(v_a \otimes v_b) = \sum_{c,d} \check{R}(rr', ss')^{ab}_{cd}(v_d \otimes v_c),$$

where $\check{R}(rr', ss')^{ab}_{cd} = W_{r's}(d-c)\overline{W}_{r's'}(a-d)\overline{W}_{rs}(b-c)W_{rs'}(a-b)$.

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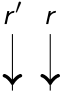
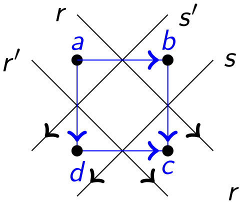
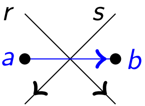
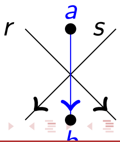
- Associating $V_{rr'}$ with 

, we can represent $\check{R}(rr', ss')^{ab}_{cd}$ by




where the CP weights are represented by

$W_{rs}(a-b) =$ 

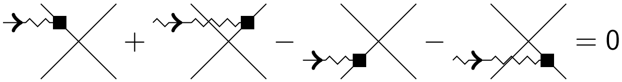
, $\overline{W}_{rs}(a-b) =$ 

# Non-local currents

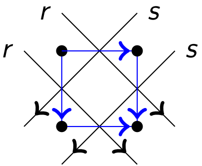
- The $\mathcal{A} = \widetilde{U}_q(\widehat{\mathfrak{sl}}_2)$ elements $j = \bar{e}_0 = t_0 f_0$ and $t = t_0 z_0^{-1}$ have the required:

$$\Delta(j) = j \otimes 1 + t \otimes j, \quad \Delta(t) = t \otimes t$$

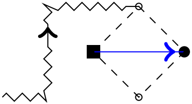
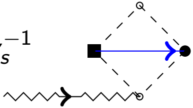
- The (diagonal) four term relation

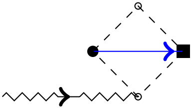
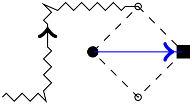


factorizes into relns around the four CP components



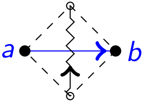
- Can express in terms of CP plaquette:

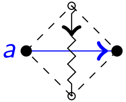
$-y_r^{-1}$  $+\, q^2 y_s^{-1}$ 

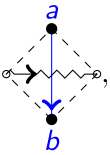
$-x_r^{-1}$  $+\, x_s^{-1}$  $= 0$

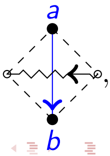
where:

$\sigma = \blacksquare$ returns value $e^{2\pi i \text{height}/N}$ and disorder operator $\mu$ is

$\frac{f_s}{f_r} W_{rs}(a-b-1) \quad =$ 

, $\quad \frac{f_r}{f_s} W_{rs}(a-b+1) =$ 


$\frac{1}{f_r f_s} \overline{W}_{rs}(a-b-1) \quad =$ 

, $\quad f_r f_s \overline{W}_{rs}(a-b+1) =$ 

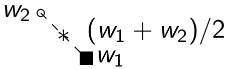
,

- Now define $\mathcal{O}(w)$ to be the operator

$$\mathcal{O}((w_1 + w_2)/2)) = \exp(-is\,\mathrm{Arg}(w_1 - w_2))\,T(\mu(w_2)\sigma(w_1))$$

where

  - $\sigma(w_1)$ is $X = \blacksquare$ at CP site $w_1$
  - $\mu(w_2)$ is disorder operator ending at dual CP site $w_2$




  - $T$ is time ordering (largest $Im(w_i)$ to right)
  - $\mathrm{Arg}(w)$ is principal argument of $w$
  - 'spin' $s = (1 - 1/N)$

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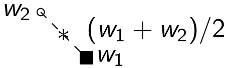


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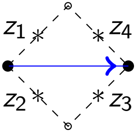
- There is a natural embedding such that 4 term reln becomes




$$e^{i\phi_r/N}\delta z_1\mathcal{O}(z_1) + e^{i\phi_s/N}\delta z_2\mathcal{O}(z_2) + e^{-i\phi_r/N}\delta z_3\mathcal{O}(z_3) + e^{-i\phi_s/N}\delta z_4\mathcal{O}(z_4) = 0$$

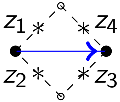
where $x/y = e^{i(2\phi-\pi)/N}$.

# CFT interpretation

- Want to interprete the 'twisted' DH cond

$$e^{i\phi_r/N}\delta z_1\mathcal{O}(z_1)+e^{i\phi_s/N}\delta z_2\mathcal{O}(z_2)+e^{-i\phi_r/N}\delta z_3\mathcal{O}(z_3)+e^{-i\phi_s/N}\delta z_4\mathcal{O}(z_4)=0$$

around



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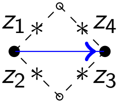
around 


1. Critical Fateev-Zamolodchikov case

- We have $\phi_r=\phi_s=k=0$ and $\mathcal{O}(z)$ is known $Z(N)$ F-Z lattice model parafermion with DH condition [Rajabpour & Cardy 07]

$$\delta z_1\mathcal{O}_1+\delta z_2\mathcal{O}_2+\delta z_3\mathcal{O}_3+\delta z_4\mathcal{O}_4=0$$

which is discrete version of $\bar{\partial}\mathcal{O}=0$

Described by CFT: $c=2(N-1)/(N+2)$, $\mathcal{O}=$ fund. spin $s=1-1/N$ parafermion.

# CFT interpretation . . .

## 2. General $N > 2$ Case

- Cardy (93), Watts (98) predict integrable CP identifiable as

$$S = S_{\mathrm{FZ}} + \int d^2 r \, [\delta_+ \Phi_+(z,\bar{z}) + \delta_- \Phi_-(z,\bar{z}) + \tau \varepsilon(z,\bar{z})]$$

  - spin 0 energy operator $\varepsilon$ has conf. dim. $(h_\varepsilon, h_\varepsilon)$ with $h_\varepsilon = 2/(N+2)$
  - spin $\pm 1$ $\Phi_\pm$ have conf. dim $(h_\varepsilon + 1, h_\varepsilon)$ and $(h_\varepsilon, h_\varepsilon + 1)$

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- CFT argument then implies

$$\bar{\partial}\mathcal{O}(z,\bar{z}) = \pi\Big(\delta_+\ \chi_+(z,\bar{z}) + \delta_-\ \chi_-(z,\bar{z}) + \tau\ \chi_0(z,\bar{z})\Big)$$

where

$$\mathcal{O}(z)\Phi_\pm(w,\bar{w}) = + \cdots \frac{\chi_\pm(w,\bar{w})}{z-w} + \cdots ;\ \mathrm{spin}(\chi_\pm) = s + 1 \mp 1$$

$$\mathcal{O}(z)\varepsilon(w,\bar{w}) = + \cdots \frac{\chi_0(w,\bar{w})}{z-w} + \cdots ;\ \mathrm{spin}(\chi_0) = s - 1$$

# CFT interpretation ...

- By expanding around FZ point our DH condition

$$e^{i\phi_r/N}\delta z_1\mathcal{O}(z_1)+e^{i\phi_s/N}\delta z_2\mathcal{O}(z_2)+e^{-i\phi_r/N}\delta z_3\mathcal{O}(z_3)+e^{-i\phi_s/N}\delta z_4\mathcal{O}(z_4)=0$$

can be described precisely in this way as discrete version of

$$\bar{\partial}\mathcal{O}(z,\bar{z})=\pi\Big(\delta_+\ \chi_+(z,\bar{z})+\delta_-\ \chi_-(z,\bar{z})+\tau\ \chi_0(z,\bar{z})\Big)$$

with $\chi_\pm$ and $\chi_0$ identified in terms of correct-spin lattice operators and parameters $(\delta_+,\delta_-,\tau)$ given in terms of $(r,s)$.

# CFT interpretation . . .

## 3.The Ising Case

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- Combining DH relations for $\bar{e}_0$ and $\bar{e}_1$ in Ising case gives a discrete version of

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- Combing DAH relations for $e_0$ and $e_1$ gives discrete version of

$$\partial\bar{\Psi} = im\Psi$$

- Together = Dirac eqn - seen in Ising by [Riva & Cardy (06)]

# Conclusions

- Quantum group currents give operators with 4 term relns which become condition

$$\delta z_1 \mathcal{O}_1 + \delta z_2 \mathcal{O}_2 + \delta z_3 \mathcal{O}_3 + \delta z_4 \mathcal{O}_4 = 0$$

  or a perturbation of it.

- Works for a range of models: dilute and dense loop models [IWWZ (13)] and CP [IW (15)]

- 4 term relns tell us about underlying CFT *and* the perturbations of CFT our lattice model corresponds to

- Hopefully useful in establishing rigourous scaling limits to CFT (i.e., the Smirnov programme) - but need missing half of DH conditions.